Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 201 (2008) 351–360
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

On the asymptotic solution for the Fourier–Bessel multiple scattering coefficients of an infinite grating of insulating dielectric circular cylinders at oblique incidence

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Abstract

The 'asymptotic solution' for the classical electromagnetic problem of the diffraction of obliquely incident plane E-polarized waves by an infinite array of infinitely long insulating dielectric circular cylinders is investigated. Exploiting the elementary function representations of 'Schlömilch series', which was originally developed by Twersky [V. Twersky, Elementary function representations of Schlömilch series. Arch. Ration. Mech. Anal. 8 (1961) 323–332.], we have obtained a 'new' set of equations describing the behavior of the 'Fourier–Bessel multiple scattering coefficients' of an infinite grating of circular dielectric cylinders for vertically polarized obliquely incident plane electromagnetic waves when the grating spacing '*d*' is small compare to a wavelength. In addition, we have achieved to acquire the 'asymptotic solution for the multiple scattering coefficients of the infinite grating at oblique incidence' as a function of the ratio of the cylinder radius '*a*' to grating spacing.

Keywords: Asymptotic expansions; Multiple scattering coefficients; E-polarized electromagnetic waves; Infinite grating; Oblique incidence

1. Introduction

Twersky [1,2] treated the classical electromagnetic problem of multiple scattering of waves by an infinite grating of dielectric circular cylinders at normal incidence and derived the equations describing the behavior of the multiple scattering coefficients of the infinite grating at normal incidence in terms of 'Schlömilch series' [3] and the 'scattering coefficients of an isolated cylinder at normal incidence' [4].

In the last decade, the classical electromagnetic problem of multiple scattering by gratings attained distinctive ascendances in contemporary applications such as modeling photonic crystal structures by Botten et al.

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2007.12.030

[5,6], and White et al. [7]. In the area of acoustical wave propagation, Cai and Williams [8,9] treated the multiple scattering of anti-plane shear waves in fiber-reinforced composite materials by extending an exact analytical solution for a general two-dimensional multiple scattering problem to a formulation called as the 'scatterer polymerization method' in order to reconstruct the solutions for abstract scatterers comprising lots of actual scatterers. In addition, Cai [10] investigated the 'layered multiple scattering techniques' for anti-plane shear wave scattering from multiple gratings consisting of parallel cylinders. In a more recent study, Cai et al. [11] investigated multiple scattering of acoustic waves in a planar horizontal waveguide by finite-length cylinders, in which the cylinder height equals the waveguide depth and both are vertically constrained by the pressure-release boundaries. Besides, Cai and Patil [12] performed large-scale deterministic simulations in order to observe the band gap formation in composite models having quasi-random fiber arrangements.

The diffraction by an infinite plane grating of perfectly conducting cylinders comprising 'multiple scattering effects' was first treated by Sivov [13] who determined the coefficients of reflection and transmission of an infinite grating in free space for the more generalized case of 'conical incidence'. The scattering by an arbitrary configuration of parallel, non-overlapping infinite cylinders was treated by Lee [14] who developed the expressions for the coefficients of the scattered wave potentials in terms of the independent scattering coefficients of an isolated cylinder and acquired the solution for the 'multiple scattering of an obliquely incident plane wave by a collection of closely-spaced, radially-stratified parallel cylinders with an arbitrary number of stratified layers' [15]. Recently, the problem of multiple scattering by a grating of finite number of cylinders at oblique incidence has been studied in a slightly different form in the context of diffraction from photonic crystal woodpile structures by Smith et al. [16,17].

The exact equations representing the behavior of the 'Fourier–Bessel multiple scattering coefficients' of a grating consisting of an infinite set of dielectric cylinders, which are aligned along the y-axis and parallel to the z-axis, for obliquely incident plane electromagnetic waves have been derived in Kavaklıoğlu [18–20], and the generalized representations associated with the reflected and transmitted fields have been formulated in terms of these multiple scattering coefficients in Kavaklıoğlu and Schneider [21] for obliquely incident plane H-polarized waves. Furthermore, the exact representations of the 'Fourier–Bessel multiple scattering coefficients' associated with both TE and TM polarizations have recently been acquired by the application of the 'direct Neumann iteration procedure' in Kavaklıoğlu and Schneider [22] in terms of 'Schlömilch series' and 'the scattering coefficients of an isolated dielectric circular cylinder at oblique incidence', which was originally derived by Wait [23].

The purpose of this article is to acquire an asymptotic solution for the 'Fourier–Bessel multiple scattering coefficients' of an infinite grating corresponding to the vertically polarized and obliquely incident plane electromagnetic waves, and present the solution of the scattering coefficients as a function of 'the ratio of the cylinder radius to the grating spacing' when the wavelength of the incident radiation is much greater than the grating spacing.

2. Description of the multiple scattering coefficients of the infinite grating at oblique incidence

In this section, we will present the exact systems of equations associated with the multiple scattering coefficients of the infinite grating of circular dielectric cylinders for obliquely incident and vertically polarized plane electromagnetic waves.

The exterior electric and magnetic fields of the infinite grating of dielectric circular cylinders, all having identical radii of 'a', excited by obliquely incident and vertically polarized plane electromagnetic waves, are expressed in [18,19] in the coordinate system of the sth cylinder located at r_s, in terms of the incident field plus a summation of cylindrical waves outgoing from each of the individual jth cylinder located at r_j, for $|r - r_j| \to \infty$ as

$$E_z^{(\mathrm{ext,TM})}(R_s,\phi_s,z) = \left\{ \mathrm{e}^{\mathrm{i}k_r sd\sin\psi_i} \sum_{n=-\infty}^{+\infty} \left[\left(E_n^i + \sum_{m=-\infty}^{\infty} A_m \mathscr{I}_{n-m}(k_r d) \right) J_n(k_r R_s) + A_n H_n^{(1)}(k_r R_s) \right] \mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)} \right\} \mathrm{e}^{-\mathrm{i}k_z z}, \tag{1a}$$

$$H_z^{(\mathrm{ext,TM})}(R_s,\phi_s,z)=\left\{\mathrm{e}^{\mathrm{i}k_r sd\sin\psi_i}\sum_{n=-\infty}^{+\infty}\left[\left(\sum_{m=-\infty}^{\infty}A_m^H\mathscr{I}_{n-m}(k_rd)\right)J_n(k_rR_s)\right.\right.$$
$$\left.\left.+A_n^H H_n^{(1)}(k_rR_s)\right]\mathrm{e}^{\mathrm{i}n(\phi_s+\pi/2)}\right\}\mathrm{e}^{-\mathrm{i}k_z z}. \tag{1b}$$

In the representation of the exterior electric and magnetic fields above, the centers of the cylinders of the infinite grating are located at positions r_0, r_1, $r_2, \ldots$, etc., separated by a distance 'd', and $\{A_n, A_n^H\}_{n=-\infty}^{\infty}$ denotes the set of all 'Fourier–Bessel multiple scattering coefficients' of the infinite grating corresponding to 'vertically polarized obliquely incident plane electromagnetic waves', associated with the exterior electric and magnetic fields, respectively. In the representation above, ϕ_i is the angle of incidence in x–y plane measured from x-axis in such a way that $\psi_i=\pi+\phi_i$ as it is indicated in Fig. 1, implying that the wave is arbitrarily incident in the first quadrant of the coordinate system and '$J_n(x)$' denotes Bessel function of order 'n'. In expression (1a) and (1b), we have:

$$k_r = k_0\sin\theta_i, \tag{2a}$$
$$k_z = k_0\cos\theta_i, \tag{2b}$$

k_0 stands for the free space wave number with $k_0 := 2\pi/\lambda_0$, where λ_0 denotes the wavelength of the incident radiation, and θ_i is the 'obliquity angle' made with z-axis. '$\mathrm{e}^{-\mathrm{i}\omega t}$' time dependence is suppressed throughout the article, where 'ω' represents the angular frequency of the incident wave in radians per second and 't' stands for time in seconds. In addition, we have:

$$E_n^i = \sin\theta_i E_{0v}\mathrm{e}^{-\mathrm{i}n\psi_i}, \tag{3a}$$
$$\mathscr{I}_n(2\pi\Delta)=\sum_{p=1}^{+\infty}H_n^{(1)}(2\pi p\Delta)[\mathrm{e}^{2\pi\mathrm{i}p\Delta\sin\psi_i}(-1)^n+\mathrm{e}^{-2\pi\mathrm{i}p\Delta\sin\psi_i}], \tag{3b}$$

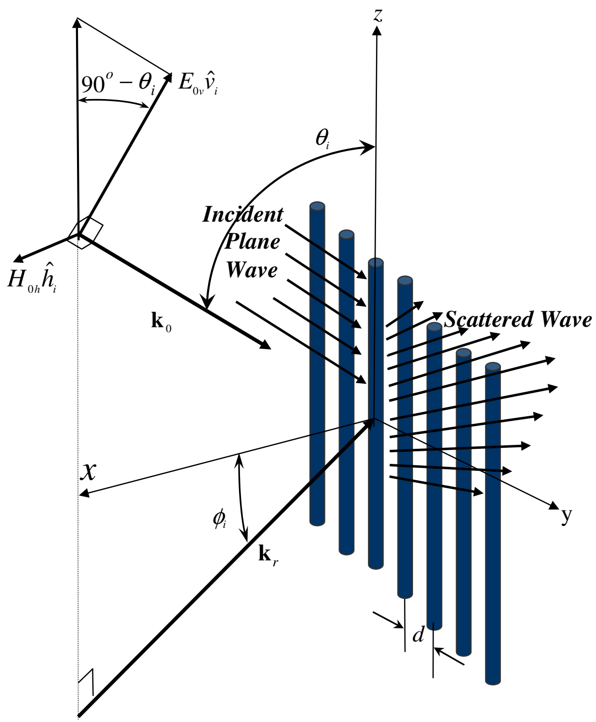

Fig. 1. The geometry of the infinite grating at oblique incidence.

where $\Delta \equiv \frac{k_r d}{2\pi}$ and '$H_n^{(1)}(x)$' denotes the nth order Hankel function of first kind. The series in expression (3b) is the generalized form of the 'Schlömilch series [3] for obliquely incident waves $\mathscr{I}_{n-m}(k_r d)$' [20] and convergent provided that $k_r d(1 \pm \sin\psi_i)/2\pi$ does not equal integers.

The multiple scattering coefficients associated with the exterior electric and magnetic fields of the infinite grating of dielectric circular cylinder corresponding to an obliquely incident vertically polarized plane wave have been acquired by the application of the separation-of-variables technique in Kavaklıoğlu [18] as

$$b_n^{\mu}\left\{A_n + c_n\left[E_n^i + \sum_{m=-\infty}^{+\infty} A_m \mathscr{I}_{n-m}(k_r d)\right]\right\} = -\left[A_n^H + a_n^{\mu}\sum_{m=-\infty}^{+\infty} A_m^H \mathscr{I}_{n-m}(k_r d)\right] \tag{4a}$$

$\forall n \ni n \in Z$, and

$$b_n^{\varepsilon}\left[A_n^H + c_n \sum_{m=-\infty}^{+\infty} A_m^H \mathscr{I}_{n-m}(k_r d)\right] = A_n + a_n^{\varepsilon}\left[E_n^i + \sum_{m=-\infty}^{+\infty} A_m \mathscr{I}_{n-m}(k_r d)\right] \tag{4b}$$

$\forall n \ni n \in Z$. Throughout the paper, 'N' stands for the set of all natural numbers, 'Z' represents the set of all integers, and $Z_+ = \{0, 1, 2, 3, \ldots\}$. In these infinite set of equations, A_n and A_n^H represent the 'Fourier–Bessel multiple scattering coefficients' corresponding to the electric and magnetic field intensities associated with obliquely incident vertically polarized plane electromagnetic waves, respectively. The coefficients appearing in the infinite set of linear algebraic equations above are defined as

$$c_n := \frac{J_n(k_r a)}{H_n^{(1)}(k_r a)} \tag{5}$$

$\forall n \ni n \in Z$. Two sets of constants a_n^{ζ} and b_n^{ζ} appearing in the Eqs. (4a) and (4b), in which $\zeta_r \in \{\varepsilon_r, \mu_r\}$ stands for the dielectric constant and relative permeability of the dielectric cylinders, respectively, are given as

$$a_n^{\zeta} = \left[\frac{J_n(k_1 a)J_n'(k_r a) - \zeta_r\left(\frac{k_r}{k_1}\right)J_n(k_r a)J_n'(k_1 a)}{J_n(k_1 a)H_n'^{(1)}(k_r a) - \zeta_r\left(\frac{k_r}{k_1}\right)H_n^{(1)}(k_r a)J_n'(k_1 a)}\right], \tag{6}$$

$$b_n^{\zeta} = \sqrt{\frac{\varepsilon_0\mu_0}{\zeta_0^2}}\left[\frac{J_n(k_1 a)H_n^{(1)}(k_r a)}{J_n(k_1 a)H_n'^{(1)}(k_r a) - \zeta_r\left(\frac{k_r}{k_1}\right)H_n^{(1)}(k_r a)J_n'(k_1 a)}\right]\left(\frac{\mathrm{i}nF}{k_r a}\right) \tag{7}$$

for $\zeta \in \{\varepsilon, \mu\}$, where k_1 is defined as $k_1 = k_0\sqrt{\varepsilon_r\mu_r - \cos^2\theta_i}$, and '$F$' in the expression (7) above is given as

$$F = \frac{(\mu_r\varepsilon_r - 1)\cos\theta_i}{\mu_r\varepsilon_r - \cos^2\theta_i} \tag{8}$$

$\forall n \ni n \in Z$. In (7), ε_0 and μ_0 stands for the permittivity and permeability of the free space, respectively.

3. Formulation of the asymptotic matrix equations for the scattering coefficients of the infinite grating at oblique incidence

In this section, we present the 'asymptotic equations associated with the multiple scattering coefficients of the electric and magnetic fields of an infinite grating of dielectric circular cylinders for obliquely incident and vertically polarized plane electromagnetic waves in matrix form' under the assumption that the 'wavelength of the incident radiation is much larger than the grating spacing'. The behavior of the multiple scattering coefficients associated with obliquely incident and vertically polarized waves can be estimated when the wavelength of the incident field is much greater than the grating spacing, namely $(k_r a) \ll 1$ and $\left(\frac{k_r a}{k_r d}\right) \equiv \xi < \frac{1}{2}$, as

$$A_{\pm(2n-1)} \cong A_{\pm(2n-1),0}(k_r a)^{2n}, \tag{9a}$$
$$A^H_{\pm(2n-1)} \cong A^H_{\pm(2n-1),0}(k_r a)^{2n} \tag{9b}$$

$\forall n \ni n \in N$, for odd coefficients, and

$$A_{\pm 2n} \cong A_{\pm 2n,0}(k_r a)^{2n+2}, \tag{9c}$$
$$A^H_{\pm 2n} \cong A^H_{\pm 2n,0}(k_r a)^{2n+2} \tag{9d}$$

$\forall n \ni n \in Z_+$, for even coefficients. The conclusion declared above is achieved as a result of the detailed investigation of the behavior of the 'multiple scattering coefficients at oblique incidence' and comparing their 'asymptotic behavior' with those originally derived by Twersky [2] for the 'normal incidence'. Defining a new (2×1) vector $\underline{\varpi}_p$ for the pth multiple scattering coefficients of the infinite grating at oblique incidence as

$$\underline{\varpi}_p \equiv \begin{bmatrix} A_{p,0} \\ A^H_{p,0} \end{bmatrix} \quad \forall p \ni p \in Z, \tag{10}$$

we have acquired the 'asymptotic matrix system of equations for the multiple scattering coefficients of an infinite grating of circular dielectric cylinders for obliquely incident and vertically polarized plane electromagnetic waves' corresponding to 'odd' subscripts as

$$\left[\begin{array}{cc} -\underline{\mathbf{I}} & \begin{matrix} \vdots & & \vdots & & \vdots \\ \left(\frac{a}{d}\right)^{8} h_8 \underline{S}_{+7} & \left(\frac{a}{d}\right)^{10} h_{10} \underline{S}_{+7} & \left(\frac{a}{d}\right)^{12} h_{12} \underline{S}_{+7} & \left(\frac{a}{d}\right)^{14} h_{14} \underline{S}_{+7} & \cdots \\ \left(\frac{a}{d}\right)^{6} h_6 \underline{S}_{+5} & \left(\frac{a}{d}\right)^{8} h_8 \underline{S}_{+5} & \left(\frac{a}{d}\right)^{10} h_{10} \underline{S}_{+5} & \left(\frac{a}{d}\right)^{12} h_{12} \underline{S}_{+5} & \\ \left(\frac{a}{d}\right)^{4} h_4 \underline{S}_{+3} & \left(\frac{a}{d}\right)^{6} h_6 \underline{S}_{+3} & \left(\frac{a}{d}\right)^{8} h_8 \underline{S}_{+3} & \left(\frac{a}{d}\right)^{10} h_{10} \underline{S}_{+3} & \cdots \\ \left(\frac{a}{d}\right)^{2} h_2 \underline{S}_{+1} & \left(\frac{a}{d}\right)^{4} h_4 \underline{S}_{+1} & \left(\frac{a}{d}\right)^{6} h_6 \underline{S}_{+1} & \left(\frac{a}{d}\right)^{8} h_8 \underline{S}_{+1} & \end{matrix} \\ \begin{matrix} \cdots & \left(\frac{a}{d}\right)^{8} h_8 \underline{S}_{-1} & \left(\frac{a}{d}\right)^{6} h_6 \underline{S}_{-1} & \left(\frac{a}{d}\right)^{4} h_4 \underline{S}_{-1} & \left(\frac{a}{d}\right)^{2} h_2 \underline{S}_{-1} \\ & \left(\frac{a}{d}\right)^{10} h_{10} \underline{S}_{-3} & \left(\frac{a}{d}\right)^{8} h_8 \underline{S}_{-3} & \left(\frac{a}{d}\right)^{6} h_6 \underline{S}_{-3} & \left(\frac{a}{d}\right)^{4} h_4 \underline{S}_{-3} \\ \cdots & \left(\frac{a}{d}\right)^{12} h_{12} \underline{S}_{-5} & \left(\frac{a}{d}\right)^{10} h_{10} \underline{S}_{-5} & \left(\frac{a}{d}\right)^{8} h_8 \underline{S}_{-5} & \left(\frac{a}{d}\right)^{6} h_6 \underline{S}_{-5} \\ & \left(\frac{a}{d}\right)^{14} h_{14} \underline{S}_{-7} & \left(\frac{a}{d}\right)^{12} h_{12} \underline{S}_{-7} & \left(\frac{a}{d}\right)^{10} h_{10} \underline{S}_{-7} & \left(\frac{a}{d}\right)^{8} h_8 \underline{S}_{-7} \\ \cdots & \vdots & & \vdots & \end{matrix} & -\underline{\mathbf{I}} \end{array}\right] \begin{bmatrix} \vdots \\ \vdots \\ \underline{\varpi}_{+7} \\ \underline{\varpi}_{+5} \\ \underline{\varpi}_{+3} \\ \underline{\varpi}_{+1} \\ \underline{\varpi}_{-1} \\ \underline{\varpi}_{-3} \\ \underline{\varpi}_{-5} \\ \underline{\varpi}_{-7} \\ \vdots \\ \vdots \end{bmatrix} = - \begin{bmatrix} \vdots \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ s_1^{\varepsilon\mu} E^i_{+1} \\ s_1^{\eta} E^i_{+1} \\ s_1^{\varepsilon\mu} E^i_{-1} \\ s_{-1}^{\eta} E^i_{-1} \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ \vdots \end{bmatrix}. \tag{11}$$

The 'asymptotic system matrix' described by the Eq. (11) above is of the order of $(\infty \times \infty)$ and the unknown is an ∞-dimensional vector corresponding to the 'odd orders' of the multiple scattering coefficients of the infinite grating. Similarly, we have acquired the 'asymptotic matrix system of equations for the even orders of multiple scattering coefficients associated with the exterior electric and magnetic fields at oblique incidence' when the 'grating spacing' is small compare to a 'wavelength', i.e., $(k_r d) \ll 1$ and $\left(\frac{k_r a}{k_r d}\right) \equiv \xi < \frac{1}{2}$, as

$$
\left[\begin{array}{cc}
-\underline{\mathbf{I}} & \begin{array}{cccc} & \bullet & & \bullet \\
\left(\frac{a}{d}\right)^{10} h_{10}\underline{\underline{S}}_{+8} & \left(\frac{a}{d}\right)^{12} h_{12}\underline{\underline{S}}_{+8} & \left(\frac{a}{d}\right)^{14} h_{14}\underline{\underline{S}}_{+8} & \left(\frac{a}{d}\right)^{16} h_{16}\underline{\underline{S}}_{+8} \\
\left(\frac{a}{d}\right)^{8} h_{8}\underline{\underline{S}}_{+6} & \left(\frac{a}{d}\right)^{10} h_{10}\underline{\underline{S}}_{+6} & \left(\frac{a}{d}\right)^{12} h_{12}\underline{\underline{S}}_{+6} & \left(\frac{a}{d}\right)^{14} h_{14}\underline{\underline{S}}_{+6} \\
\left(\frac{a}{d}\right)^{6} h_{6}\underline{\underline{S}}_{+4} & \left(\frac{a}{d}\right)^{8} h_{8}\underline{\underline{S}}_{+4} & \left(\frac{a}{d}\right)^{10} h_{10}\underline{\underline{S}}_{+4} & \left(\frac{a}{d}\right)^{12} h_{12}\underline{\underline{S}}_{+4} \\
\left(\frac{a}{d}\right)^{4} h_{4}\underline{\underline{S}}_{+2} & \left(\frac{a}{d}\right)^{6} h_{6}\underline{\underline{S}}_{+2} & \left(\frac{a}{d}\right)^{8} h_{8}\underline{\underline{S}}_{+2} & \left(\frac{a}{d}\right)^{10} h_{10}\underline{\underline{S}}_{+2}
\end{array} \\
\begin{array}{cccc}
\left(\frac{a}{d}\right)^{10} h_{10}\underline{\underline{S}}_{-2} & \left(\frac{a}{d}\right)^{8} h_{8}\underline{\underline{S}}_{-2} & \left(\frac{a}{d}\right)^{6} h_{6}\underline{\underline{S}}_{-2} & \left(\frac{a}{d}\right)^{4} h_{4}\underline{\underline{S}}_{-2} \\
\left(\frac{a}{d}\right)^{12} h_{12}\underline{\underline{S}}_{-4} & \left(\frac{a}{d}\right)^{10} h_{10}\underline{\underline{S}}_{-4} & \left(\frac{a}{d}\right)^{8} h_{8}\underline{\underline{S}}_{-4} & \left(\frac{a}{d}\right)^{6} h_{6}\underline{\underline{S}}_{-4} \\
\left(\frac{a}{d}\right)^{14} h_{14}\underline{\underline{S}}_{-6} & \left(\frac{a}{d}\right)^{12} h_{12}\underline{\underline{S}}_{-6} & \left(\frac{a}{d}\right)^{10} h_{10}\underline{\underline{S}}_{-6} & \left(\frac{a}{d}\right)^{8} h_{8}\underline{\underline{S}}_{-6} \\
\left(\frac{a}{d}\right)^{16} h_{16}\underline{\underline{S}}_{-8} & \left(\frac{a}{d}\right)^{14} h_{14}\underline{\underline{S}}_{-8} & \left(\frac{a}{d}\right)^{12} h_{12}\underline{\underline{S}}_{-8} & \left(\frac{a}{d}\right)^{10} h_{10}\underline{\underline{S}}_{-8} \\
\bullet & \bullet & \bullet &
\end{array} & -\underline{\mathbf{I}}
\end{array}\right]
\begin{bmatrix} \bullet \\ \bullet \\ \underline{\varpi}_{+8} \\ \underline{\varpi}_{+6} \\ \underline{\varpi}_{+4} \\ \underline{\varpi}_{+2} \\ \underline{\varpi}_{-2} \\ \underline{\varpi}_{-4} \\ \underline{\varpi}_{-6} \\ \underline{\varpi}_{-8} \\ \bullet \\ \bullet \end{bmatrix}
$$

$$
= -\left[\begin{array}{cc}
\underline{\mathbf{0}} & \begin{array}{cccc} & \bullet & & \bullet \\
\left(\frac{a}{d}\right)^{8} h_{+9}\underline{\underline{S}}_{+8} & \left(\frac{a}{d}\right)^{10} h_{+11}\underline{\underline{S}}_{+8} & \left(\frac{a}{d}\right)^{12} h_{+13}\underline{\underline{S}}_{+8} & \left(\frac{a}{d}\right)^{14} h_{+15}\underline{\underline{S}}_{+8} \\
\left(\frac{a}{d}\right)^{6} h_{+7}\underline{\underline{S}}_{+6} & \left(\frac{a}{d}\right)^{8} h_{+9}\underline{\underline{S}}_{+6} & \left(\frac{a}{d}\right)^{10} h_{+11}\underline{\underline{S}}_{+6} & \left(\frac{a}{d}\right)^{12} h_{+13}\underline{\underline{S}}_{+6} \\
\left(\frac{a}{d}\right)^{4} h_{+5}\underline{\underline{S}}_{+4} & \left(\frac{a}{d}\right)^{6} h_{+7}\underline{\underline{S}}_{+4} & \left(\frac{a}{d}\right)^{8} h_{+9}\underline{\underline{S}}_{+4} & \left(\frac{a}{d}\right)^{10} h_{+11}\underline{\underline{S}}_{+4} \\
\left(\frac{a}{d}\right)^{2} h_{+3}\underline{\underline{S}}_{+2} & \left(\frac{a}{d}\right)^{4} h_{+5}\underline{\underline{S}}_{+2} & \left(\frac{a}{d}\right)^{6} h_{+7}\underline{\underline{S}}_{+2} & \left(\frac{a}{d}\right)^{8} h_{+9}\underline{\underline{S}}_{+2}
\end{array} \\
\begin{array}{cccc}
\left(\frac{a}{d}\right)^{8} h_{-9}\underline{\underline{S}}_{-2} & \left(\frac{a}{d}\right)^{6} h_{-7}\underline{\underline{S}}_{-2} & \left(\frac{a}{d}\right)^{4} h_{-5}\underline{\underline{S}}_{-2} & \left(\frac{a}{d}\right)^{2} h_{-3}\underline{\underline{S}}_{-2} \\
\left(\frac{a}{d}\right)^{10} h_{-11}\underline{\underline{S}}_{-4} & \left(\frac{a}{d}\right)^{8} h_{-9}\underline{\underline{S}}_{-4} & \left(\frac{a}{d}\right)^{6} h_{-7}\underline{\underline{S}}_{-4} & \left(\frac{a}{d}\right)^{4} h_{-5}\underline{\underline{S}}_{-4} \\
\left(\frac{a}{d}\right)^{12} h_{-13}\underline{\underline{S}}_{-6} & \left(\frac{a}{d}\right)^{10} h_{-11}\underline{\underline{S}}_{-6} & \left(\frac{a}{d}\right)^{8} h_{-9}\underline{\underline{S}}_{-6} & \left(\frac{a}{d}\right)^{6} h_{-7}\underline{\underline{S}}_{-6} \\
\left(\frac{a}{d}\right)^{14} h_{-15}\underline{\underline{S}}_{-8} & \left(\frac{a}{d}\right)^{12} h_{-13}\underline{\underline{S}}_{-8} & \left(\frac{a}{d}\right)^{10} h_{-11}\underline{\underline{S}}_{-8} & \left(\frac{a}{d}\right)^{8} h_{-9}\underline{\underline{S}}_{-8} \\
\bullet & \bullet & \bullet &
\end{array} & \underline{\mathbf{0}}
\end{array}\right]
\begin{bmatrix} \bullet \\ \bullet \\ \underline{\varpi}_{+7} \\ \underline{\varpi}_{+5} \\ \underline{\varpi}_{+3} \\ \underline{\varpi}_{+1} \\ \underline{\varpi}_{-1} \\ \underline{\varpi}_{-3} \\ \underline{\varpi}_{-5} \\ \underline{\varpi}_{-7} \\ \bullet \\ \bullet \end{bmatrix}.
$$

$$
+\begin{bmatrix}
\left[\left(\frac{a}{d}\right)^{8} h_8 s_0^{\varepsilon\mu} E_0^i\right]\begin{pmatrix} s_8^{\varepsilon\mu} \\ s_8^{\eta} \end{pmatrix} \\
\left[\left(\frac{a}{d}\right)^{6} h_6 s_0^{\varepsilon\mu} E_0^i\right]\begin{pmatrix} s_6^{\varepsilon\mu} \\ s_6^{\eta} \end{pmatrix} \\
\left[\left(\frac{a}{d}\right)^{4} h_4 s_0^{\varepsilon\mu} E_0^i\right]\begin{pmatrix} s_4^{\varepsilon\mu} \\ s_4^{\eta} \end{pmatrix} \\
\left[E_{+2}^i + \left(\frac{a}{d}\right)^{2} h_2 s_0^{\varepsilon\mu} E_0^i\right]\begin{pmatrix} s_2^{\varepsilon\mu} \\ s_2^{\eta} \end{pmatrix} \\
\left[E_{-2}^i + \left(\frac{a}{d}\right)^{2} h_2 s_0^{\varepsilon\mu} E_0^i\right]\begin{pmatrix} s_2^{\varepsilon\mu} \\ s_{-2}^{\eta} \end{pmatrix} \\
\left[\left(\frac{a}{d}\right)^{4} h_4 s_0^{\varepsilon\mu} E_0^i\right]\begin{pmatrix} s_4^{\varepsilon\mu} \\ s_{-4}^{\eta} \end{pmatrix} \\
\left[\left(\frac{a}{d}\right)^{6} h_6 s_0^{\varepsilon\mu} E_0^i\right]\begin{pmatrix} s_6^{\varepsilon\mu} \\ s_{-6}^{\eta} \end{pmatrix} \\
\left[\left(\frac{a}{d}\right)^{8} h_8 s_0^{\varepsilon\mu} E_0^i\right]\begin{pmatrix} s_8^{\varepsilon\mu} \\ s_{-8}^{\eta} \end{pmatrix}
\end{bmatrix} \tag{12}
$$

The ‘asymptotic system matrix’ in Eq. (12) is of the order of $(\infty \times \infty)$ and the unknown is an ∞-dimensional vector corresponding to the ‘even orders’ of the multiple scattering coefficients of the electric and magnetic fields associated with vertically polarized obliquely incident waves. In the Eqs. (11) and (12), $\underline{\underline{S}}_n$ is a (2×2) matrix defined as

$$\underline{\underline{S}}_{\pm n} := \frac{1}{D}\left(\frac{\mathrm{i}n\pi}{(2^n n!)^2}\right)\begin{pmatrix} s_{\varepsilon\mu} & s_{\pm\xi} \\ s_{\pm\eta} & s_{\mu\varepsilon} \end{pmatrix}(k_r a)^{2n}. \tag{13}$$

In the above, we have:

$$D = \left[1 + \varepsilon_r\left(\frac{k_r}{k_1}\right)^2\right]\left[1 + \mu_r\left(\frac{k_r}{k_1}\right)^2\right] - F^2. \tag{14}$$

The constants appearing in the matrix of expression (13) are defined as

$$s_{\varepsilon\mu} = \left[1 - \varepsilon_r\left(\frac{k_r}{k_1}\right)^2\right]\left[1 + \mu_r\left(\frac{k_r}{k_1}\right)^2\right] + F^2, \tag{15a}$$

$$s_{\mu\varepsilon} = \left[1 - \mu_r\left(\frac{k_r}{k_1}\right)^2\right]\left[1 + \varepsilon_r\left(\frac{k_r}{k_1}\right)^2\right] + F^2, \tag{15b}$$

$$s_{\pm\xi} = \pm 2\mathrm{i}\xi_0 F, \tag{15c}$$

$$s_{\pm\eta} = \mp 2\mathrm{i}\eta_0 F. \tag{15d}$$

The h_n's arising in the asymptotic equations (11) and (12) represents the leading asymptotic terms of the 'Schlömilch series', which are expressed by Twersky [2,3] for normal incidence and Kavaklıoğlu [20] for oblique incidence, and are given as

$$h_0 \equiv 2\sec\phi_0, \tag{16a}$$

$$h_1 \equiv -2\mathrm{i}\tan\phi_0, \tag{16b}$$

$$h_2 \equiv \frac{4\pi}{3\mathrm{i}}, \tag{16c}$$

$$h_3 \equiv -\frac{16\pi\sin\phi_0}{3}, \tag{16d}$$

$$h_4 \equiv \frac{2^5\pi^3}{15\mathrm{i}}, \tag{16e}$$

$$h_5 \equiv -\frac{2^8\pi^3\sin\phi_0}{15}, \tag{16f}$$

h_{2n}'s and h_{2n+1}'s for large n are given as

$$h_{2n} \to \frac{\mathrm{i}}{n}(-1)^n 2^{4n-1}\pi^{2n-1}B_{2n}(0), \tag{16g}$$

and

$$h_{2n+1} \to \frac{\mathrm{i}}{n}(-1)^n 2^{4n+1}\pi^{2n-1}B_{2n}(0)\sin\phi_0 \equiv -4\mathrm{i}nh_{2n}\sin\phi_0. \tag{16h}$$

In the expressions above, $B_{2\xi}(x)$ and B_ξ represent the 'Bernoulli polynomial' and the 'Bernoulli numbers', respectively. The relationship between them is given as $B_{2\xi}(0) \equiv (-1)^{\xi-1}B_\xi$.

4. Solution for the asymptotic matrix system of equations for the scattering coefficients of the infinite grating at oblique incidence

The solution for the 'asymptotic matrix system' of equations, given in (11) and (12), can then be obtained by expanding '$\underline{\varpi}_n$' in the form of an 'infinite asymptotic series expansion'. For this purpose, we have introduced:

$$\underline{\varpi}_n = \sum_{m=0}^{\infty}\underline{\varpi}_n^{(m)}\left(\frac{a}{d}\right)^m \tag{17}$$

$\forall n \ni n \in N$, into Eqs. (11) and (12) and determined the coefficients in the 'asymptotic series expansion' in (17), namely, $\varpi_n^{(m)}$ for '$m = 0, 1, 2, 3, \ldots$'. After some algebraic work, the multiple scattering coefficients corresponding to the fundamental mode are obtained as

$$A_{0,0} \cong \sin\theta_i s_0^{\varepsilon\mu}, \tag{18a}$$

for the scattered electric field at oblique incidence, and

$$A_{0,0}^H \equiv 0, \tag{18b}$$

for the scattered magnetic field at oblique incidence, where $s_0^{\varepsilon\mu}$ in (18a) is given by

$$s_0^{\varepsilon\mu} = \frac{\mathrm{i}\pi}{4}(\varepsilon_r - 1). \tag{18c}$$

We have obtained for $n = 1$:

$$\begin{aligned} A_{\pm1,0} \cong \sin\theta_i\left(\frac{\mathrm{i}\pi}{4D}\right)\Bigg\{ & s_{\varepsilon\mu}\mathrm{e}^{\mp\mathrm{i}\psi_i} + \left(\frac{a}{d}\right)^2 h_2(s_{\varepsilon\mu}^2 - 4F^2)\left(\frac{\mathrm{i}\pi}{4D}\right)\mathrm{e}^{\pm\mathrm{i}\psi_i} \\ & + \left(\frac{a}{d}\right)^4 h_2^2\left[s_{\varepsilon\mu}(s_{\varepsilon\mu}^2 - 4F^2) + 8F^2(\varepsilon_r - \mu_r)\left(\frac{k_r}{k_1}\right)^2\right]\left(\frac{\mathrm{i}\pi}{4D}\right)^2\mathrm{e}^{\mp\mathrm{i}\psi_i} + \mathrm{O}\left(\left(\frac{a}{d}\right)^6\right)\Bigg\}, \end{aligned} \tag{19a}$$

for the scattered electric field at oblique incidence, and

$$\begin{aligned} A_{\pm1,0}^H \cong \mp 2\mathrm{i}\eta_0 F\sin\theta_i\left(\frac{\mathrm{i}\pi}{4D}\right)\Bigg\{ & \mathrm{e}^{\mp\mathrm{i}\psi_i} + \left(\frac{a}{d}\right)^2 h_2 2(\mu_r - \varepsilon_r)\left(\frac{k_r}{k_1}\right)^2\left(\frac{\mathrm{i}\pi}{4D}\right)\mathrm{e}^{\pm\mathrm{i}\psi_i} \\ & + \left(\frac{a}{d}\right)^4 h_2^2\left[(s_{\mu\varepsilon}^2 - 4F^2) + 2(\mu_r - \varepsilon_r)\left(\frac{k_r}{k_1}\right)^2 s_{\varepsilon\mu}\right]\left(\frac{\mathrm{i}\pi}{4D}\right)^2\mathrm{e}^{\mp\mathrm{i}\psi_i} + \mathrm{O}\left(\left(\frac{a}{d}\right)^6\right)\Bigg\}, \end{aligned} \tag{19b}$$

for the scattered magnetic field at oblique incidence, respectively. Similarly, for $n = 2$, we have obtained:

$$\begin{aligned} A_{\pm2,0} \cong \sin\theta_i\left(\frac{\mathrm{i}\pi}{32D}\right)\Bigg\{ & s_{\varepsilon\mu}\mathrm{e}^{\mp2\mathrm{i}\psi_i} + \left(\frac{a}{d}\right)^2\left(h_2 s_0^{\varepsilon\mu}s_{\varepsilon\mu} \pm h_3(s_{\varepsilon\mu}^2 - 4F^2)\left(\frac{\mathrm{i}\pi}{4D}\right)\mathrm{e}^{\pm\mathrm{i}\psi_i}\right) \\ & + \left(\frac{a}{d}\right)^4\left[h_4\left(\frac{\mathrm{i}\pi}{32D}\right)(s_{\varepsilon\mu}^2 - 4F^2)\mathrm{e}^{\pm2\mathrm{i}\psi_i} \pm h_5 h_2\left(\frac{\mathrm{i}\pi}{4D}\right)^2\left(s_{\varepsilon\mu}(s_{\varepsilon\mu}^2 - 4F^2) + 8F^2(\varepsilon_r - \mu_r)\left(\frac{k_r}{k_1}\right)^2\right)\mathrm{e}^{\mp\mathrm{i}\psi_i}\right] \\ & + \mathrm{O}\left(\left(\frac{a}{d}\right)^6\right)\Bigg\}, \end{aligned} \tag{20a}$$

for the scattered electric field at oblique incidence, and

$$\begin{aligned} A_{\pm2,0}^H \cong \mp 2\mathrm{i}\eta_0 F\sin\theta_i\left(\frac{\mathrm{i}\pi}{32D}\right)\Bigg\{ & \mathrm{e}^{\mp2\mathrm{i}\psi_i} + \left(\frac{a}{d}\right)^2\left(h_2 s_0^{\varepsilon\mu} \pm h_3\left(\frac{\mathrm{i}\pi}{4D}\right)2(\mu_r - \varepsilon_r)\left(\frac{k_r}{k_1}\right)^2\mathrm{e}^{\pm\mathrm{i}\psi_i}\right) \\ & + \left(\frac{a}{d}\right)^4\left[h_4\left(\frac{\mathrm{i}\pi}{32D}\right)2(\mu_r - \varepsilon_r)\left(\frac{k_r}{k_1}\right)^2\mathrm{e}^{\pm2\mathrm{i}\psi_i} \pm h_5 h_2\left(\frac{\mathrm{i}\pi}{4D}\right)^2\left((s_{\varepsilon\mu}^2 - 4F^2) + 2(\varepsilon_r - \mu_r)\left(\frac{k_r}{k_1}\right)^2 s_{\mu\varepsilon}\right)\mathrm{e}^{\mp\mathrm{i}\psi_i}\right] \\ & + \mathrm{O}\left(\left(\frac{a}{d}\right)^6\right)\Bigg\}, \end{aligned} \tag{20b}$$

for the scattered magnetic field at oblique incidence, respectively. Finally, for $n = 3$, we have acquired:

$$A_{\pm3,0} \cong \sin\theta_i\left(\frac{\mathrm{i}\pi}{3.2^8 D}\right)\left[\left(\frac{a}{d}\right)^4 h_4(s_{\varepsilon\mu}^2 - 4F^2)\left(\frac{\mathrm{i}\pi}{4D}\right)\mathrm{e}^{\pm\mathrm{i}\psi_i} + \mathrm{O}\left(\left(\frac{a}{d}\right)^6\right)\right], \tag{21a}$$

for the scattered electric field at oblique incidence, and

$$A^{H}_{\pm 3,0} \cong \mp 2\mathrm{i}\eta_0 F \sin\theta_{\mathrm{i}}\left(\frac{i\pi}{3.2^8 D}\right)\left[\left(\frac{a}{d}\right)^4 h_4 2(\mu_r - \varepsilon_r)\left(\frac{k_r}{k_1}\right)^2\left(\frac{\mathrm{i}\pi}{4D}\right)\mathrm{e}^{\pm\mathrm{i}\psi_i} + \mathrm{O}\left(\left(\frac{a}{d}\right)^6\right)\right], \tag{21b}$$

for the scattered magnetic field at oblique incidence, respectively.

5. Conclusion

We have presented the ‘asymptotic matrix system of equations of the multiple scattering coefficients’ of an infinite grating of circular dielectric cylinders for obliquely incident and vertically polarized plane electromagnetic waves associated with the exterior electric and magnetic field intensities. Furthermore, we have acquired the ‘asymptotic solution’ for the multiple scattering coefficients up to and including third order as a function of the cylinder radius to grating spacing when the grating spacing ‘d’ is small compare to a wavelength, i.e., $(d\sin\theta_i)(1 \pm \sin\psi_i) << \lambda_0 \equiv \frac{2\pi}{k_0}$ and $\left(\frac{k_r a}{k_r d}\right) \equiv \xi < \frac{1}{2}$.

Acknowledgements

We are indebted to deceased Professor Emeritus Victor Twersky of the Department of Mathematics of the University of Illinois at Chicago for the kind concern he spared to our work. The first author takes this opportunity to express his sincere thanks to Professor Dr Roger Henry Lang for valuable discussions.

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