Construction and separability of nonlinear soliton integrable couplings

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ARTICLE INFO	ABSTRACT
	The paper is motivated by recent works of several authors, initiated by articles of Ma and Zhu [W. X. Ma, Z. N. Zhu, Constructing nonlinear discrete integrable Hamiltonian couplings, Comput. Math. Appl. 60 (2010) 2601] and Ma [W. X. Ma, Nonlinear continuous integrable Hamiltonian couplings, Appl. Math. Comput. 217 (2011) 7238], where new class of soliton systems, being nonlinear integrable couplings, was introduced. Here, we present a general construction of such class of systems and we develop the decoupling procedure, separating them into copies of underlying original equations.

1. Introduction

For a given nonlinear integrable dynamical system there usually exists many different integrable extensions which equations of motion take a triangular form. They are usually called triangular systems. For Liouville integrable nonlinear ODE's an interesting class of triangular systems is given in [1]. For soliton systems we know more examples. Triangular extensions of the KdV system was considered in [2]. In [3], the so-called 'dark equations', linear extensions of soliton systems which are also of the triangular form were constructed. Another class of linear extensions the so-called linear integrable couplings of soliton systems was introduced in [4] and then developed in [5] and many other papers. Recently, the theory of nonlinear integrable couplings of ordinary soliton systems was presented in [6–8] and further developed in [9–12]. Particulary noteworthy are the constructions of integrable couplings based on the non-semisimple Lie algebras, see for instance [6].

In the present paper, we introduce very natural triangular nonlinear couplings of integrable systems, including in particular those in [7–11], which are also integrable. The construction is made on the level of evolution equations by a modification of the algebra of dynamical fields. We also propose the decoupling procedure for the considered class of integrable couplings in the form of an appropriate change of variables.

In Section 2 we introduce the algebra of coupled scalars, which is the underlying algebra for the nonlinear integrable couplings defined in Section 3. We derive soliton integrable couplings of field and lattice type. In Section 4, the general form of solution of the coupled systems is obtained. An example of soliton solutions of the nonlinearly coupled KdV system is presented. In Section 5 we derive a matrix representation of the algebra of coupled scalars and in consequence the matrix Lax representations for nonlinear couplings constructed in previous sections. Finally, in Section 6, we prove that any member of the constructed family of coupled systems separates into copies of the original soliton systems. We also show the source of the decoupling procedure.

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2. The algebra of coupled scalars

Consider n-dimensional vector space over field of real numbers $\mathbb{R}$. We define an algebra structure by the following multiplication

$$\mathbf{e}_i \cdot \mathbf{e}_j := \mathbf{e}_{\max(i,j)}, \tag{1}$$

where $\mathbf{e}_i$ are the basis vectors. Let $\mathbf{a} = \sum_{i=1}^n a_i \mathbf{e}_i$, then

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{c},$$

where

$$c_i = a_i b_i + a_i \left(\sum_{k=1}^{i-1} b_k \right) + \left(\sum_{k=1}^{i-1} a_k \right) b_i.$$

We find that the value of the coefficient c_i is given by $a_i b_i$ plus terms depending on lower order elements, a_k, b_k with $k < i$. Therefore, we call this algebra as an algebra of coupled scalars. This algebra is unital, commutative and associative, and $\mathbf{e}_i$ are idempotent elements, what follows immediately from the definition (1). The unity element is $\mathbf{e}_1$.

For $n = 4$ we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 + a_2 b_1 + a_1 b_2 \\ a_3 b_3 + a_3(b_1 + b_2) + (a_1 + a_2)b_3 \\ a_4 b_4 + a_4(b_1 + b_2 + b_3) + (a_1 + a_2 + a_3)b_4 \end{pmatrix}.$$

In the algebra of coupled scalars, for product of m elements the following formula holds:

$$\mathbf{a}^1 \cdot \ldots \cdot \mathbf{a}^m = \begin{pmatrix} a_1^1 \\ \vdots \\ a_n^1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} a_1^m \\ \vdots \\ a_n^m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \mathbf{b}, \tag{2a}$$

where

$$b_k = \prod_{j=1}^m \left(\sum_{r=1}^k a_r^j \right) - \prod_{j=1}^m \left(\sum_{r=1}^{k-1} a_r^j \right). \tag{2b}$$

3. Nonlinear couplings of soliton systems

Consider a commutative and associative algebra, with respect to the ordinary dot multiplication, of smooth functions on $\mathbb{R}^m$ with m derivations $\frac{\partial}{\partial x_i} : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)$. Let us construct its coupled counterpart $C_d^\infty(\mathbb{R}^m)$, that is an algebra of functions

$$\mathbf{f}(x) = f_1(x)\mathbf{e}_1 + \cdots + f_n(x)\mathbf{e}_n = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix},$$

where $x = (x_1, \ldots, x_m)$, taking values in the algebra of coupled scalars. So, it is a commutative and associative algebra with respect to the multiplication (1) and the derivations in $C_d^\infty(\mathbb{R}^m)$ can be defined by

$$\frac{\partial}{\partial x_i} := \frac{\partial}{\partial x_i} \mathbf{e}_1 \quad i = 1, \ldots, m.$$

These derivations are well-defined as

$$\frac{\partial}{\partial x_i} \mathbf{e}_1 \cdot (f \cdot g) = \left(\frac{\partial}{\partial x_i} \mathbf{e}_1 \cdot f \right) \cdot g + f \cdot \left(\frac{\partial}{\partial x_i} \mathbf{e}_1 \cdot g \right)$$

and

$$\mathbf{f}_{x_i} := \frac{\partial}{\partial x_i} \mathbf{e}_1 \cdot \mathbf{f} = \begin{pmatrix} (f_1)_{x_i} \\ \vdots \\ (f_n)_{x_i} \end{pmatrix}.$$

Here we will concentrate on the coupled extension of nonlinear integrable PDE's in $(1+1)$-dimension in the evolutionary form. Consider a one-field soliton system

$$u_t = K[u] \equiv K(u, u_x, u_{xx,\dots}). \tag{3}$$

Its extension to the system of coupled PDE's takes the form

$$\mathbf{u}_t = K[\mathbf{u}] \equiv K(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots), \tag{4}$$

where

$$\mathbf{u} = u_1 \mathbf{e}_1 + \cdots + u_n \mathbf{e}_n = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \quad u_1 = u$$

and in (4) the ordinary dot multiplication is replaced by (1).

The system of coupled Eqs. (4) takes the form

$$\begin{aligned} (u_1)_t &= K[u_1] \\ (u_k)_t &= K\left[\sum_{i=1}^{k} u_i\right] - K\left[\sum_{i=1}^{k-1} u_i\right] \quad k = 2, \dots, n, \end{aligned} \tag{5}$$

where $K[\cdot]$ is the same as in (3). To show that the form (5) holds one can use the power series expansion of $K[u]$ and the relation (2). In fact, it is sufficient to consider homogenous terms as

$$(\mathbf{u}_{i_1 x} \cdot \mathbf{u}_{i_2 x} \cdot \dots \cdot \mathbf{u}_{i_m x})_k = \prod_{s=1}^{m} \left(\sum_{r=1}^{k} u_r\right)_{i_s x} - \prod_{s=1}^{m} \left(\sum_{r=1}^{k-1} u_r\right)_{i_s x} \quad m > 0,$$

where $(\cdot)_k$ means kth coefficient of $\mathbf{u}_{i_1 x} \cdot \dots \cdot \mathbf{u}_{i_m x}$ and i_s are arbitrary nonnegative integers.

Note that the Eq. (5) reconstructs the basic equation for u_1. Moreover, for linear PDE's the procedure is trivial as all related coupled systems are just copies of the basic equation. The extension to the multi-field systems is obvious.

As an instructive example let us consider the n-coupled KdV (nc-KdV):

$$\begin{aligned} \mathbf{u}_t &= \mathbf{u}_{xxx} + 6\mathbf{u} \cdot \mathbf{u}_x \\ &\Updownarrow \\ (u_1)_t &= (u_1)_{xxx} + 6u_1(u_1)_x \\ (u_2)_t &= (u_2)_{xxx} + 6u_2(u_2)_x + 6(u_1 u_2)_x \\ &\vdots \\ (u_n)_t &= (u_n)_{xxx} + 6u_n(u_n)_x + 6\sum_{k=1}^{n-1} (u_k u_n)_x. \end{aligned} \tag{6}$$

It is a particular case of the triangular systems. Obviously in dispersionless limit we get n-coupled dispersionless KdV (nc-dKdV).

The presented construction is also valid for differential-difference systems. If, for instance, instead of the derivative $\frac{\partial}{\partial x_i} \mathbf{e}_1$ one defines on $C_d^\infty(\mathbb{R}^m)$ a shift operator $T_k \mathbf{e}_1 := \left(\exp \frac{\partial}{\partial x_k}\right) \mathbf{e}_1$, then

$$T_k \mathbf{e}_1 \cdot \mathbf{f} = \begin{pmatrix} f_1(x_1, \dots, x_k + 1, \dots, x_m) \\ \vdots \\ f_n(x_1, \dots, x_k + 1, \dots, x_m) \end{pmatrix}.$$

As an example we consider nc-Voltera differential-difference equation:

$$\mathbf{v}(x)_t = \mathbf{v}(x) \cdot [\mathbf{v}(x+1) - \mathbf{v}(x-1)]$$
$$\Updownarrow$$
$$v_1(x)_t = v_1(x)[v_1(x+1) - v_1(x-1)]$$
$$v_2(x)_t = v_2(x)[v_2(x+1) - v_2(x-1)] + v_2(x)[v_1(x+1) - v_1(x-1)]$$
$$\qquad + v_1(x)[v_2(x+1) - v_2(x-1)]$$
$$\vdots \tag{7}$$
$$v_n(x)_t = v_n(x)[v_n(x+1) - v_n(x-1)] + v_n(x)\left[\sum_{i=1}^{n-1} v_i(x+1) - \sum_{i=1}^{n-1} v_i(x-1)\right]$$
$$\qquad + \left(\sum_{i=1}^{n-1} v_i(x)\right)[v_n(x+1) - v_n(x-1)].$$

The two field case, that is 2c-Voltera, is equivalent to the lattice integrable coupling system in [7].

Each coupled soliton system have a Lax and zero-curvature representation, that can be obtained by replacing an underlying algebra by its tensor product with the algebra of coupled scalars. For the above examples, $\mathbf{L}_t = [\mathbf{L}, \mathbf{A}]$, where the right-hand side is the usual commutator, are equations in the algebras of pseudo-differential and shift operators, respectively, over algebra of coupled functions. Actually, for the nc-KdV (6) the Lax pair is given by

$$\mathbf{L} = \partial_x^2 \mathbf{e}_1 + \mathbf{u}, \quad \mathbf{A} = 4\partial_x^3 \mathbf{e}_1 + 6\mathbf{u} \cdot \partial_x \mathbf{e}_1 + 3\mathbf{u}_x \tag{8}$$

and for nc-Voltera (7) by

$$\mathbf{L} = T\mathbf{e}_1 + \mathbf{v}(x) \cdot T^{-1}\mathbf{e}_1, \quad \mathbf{A} = T^2\mathbf{e}_1 + \mathbf{v}(x+1) + \mathbf{v}(x). \tag{9}$$

As an example of coupled extension of two-field system let us consider 2-coupled AKNS (2c-AKNS), $n = 2$:

$$\mathbf{p}_t = -\frac{1}{2}\mathbf{p}_{xx} + \mathbf{p} \cdot \mathbf{p} \cdot \mathbf{q}, \quad \mathbf{q}_t = \frac{1}{2}\mathbf{q}_{xx} - \mathbf{p} \cdot \mathbf{q} \cdot \mathbf{q}$$
$$\Updownarrow$$
$$p_t = -\frac{1}{2}p_{xx} + p^2 q, \quad q_t = \frac{1}{2}q_{xx} - pq^2 \tag{10}$$
$$r_t = -\frac{1}{2}r_{xx} + p^2 s + 2pqr + 2prs + qr^2 + r^2 s$$
$$s_t = \frac{1}{2}s_{xx} - ps^2 - 2pqs - 2qrs - q^2 r - rs^2,$$

where

$$\mathbf{p} = \begin{pmatrix} p \\ r \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} q \\ s \end{pmatrix}.$$

In fact the system (10) was obtained in [9] in a different way and is equivalent to the one from [8]. Its zero-curvature representation will be presented in Section 5.

4. Solutions of coupling systems

The solutions of the coupled systems (4) are completely determined by solutions of the basic equations.

We will show now that, assuming $S^1, \ldots, S^n$ to be n arbitrary different solutions of the basic equation $u_t = K[u]$, the solution of coupled system (4) is given by

$$u_1 = S^1, \quad u_k = S^k - S^{k-1}, \quad k = 2, \ldots, n. \tag{11}$$

Moreover, any solution of the coupled system is of the form (11).

After plugging (11) to (5),

$$(S^k - S^{k-1})_t - (K[S^k] - K[S^{k-1}]) = 0,$$

it is evident that (11) solves (4). On the other hand, if the solution of the basic equation is $u_1 = S^1$, then the first coupled equation is

$$(u_2)_t = K[u_2 + S^1] - K[S^1] \quad \Longleftrightarrow \quad (u_2 + S^1)_t = K[u_2 + S^1],$$

thus $u_2 + S^1 = S^2$, where S^2 is another solution of the basic equation. Thus, by the rest of the induction, it is obvious that any solution of (4) must be in the form (11). The consequence of this fact is that for the coupled PDE's any n different solutions of the basic equation build up an appropriate solution of the coupled system.
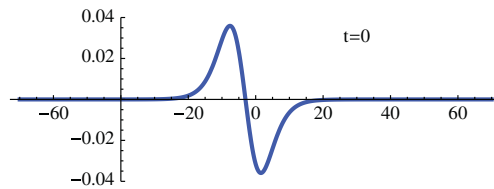
Fig. 1. Coupled 1-soliton, $u_i = S^i - S^{i-1}$ ($i \neq 1$).

Let us illustrate the whole construction on the example of solitons of nc-KdV (6) in the Hirota form. Let us start from 1-soliton solution. Consider n one-soliton solutions of the KdV with the same wave velocities and different phases γ_j:

$$S^j = 2\partial_x^2 \log(1 + f^j), \quad f^j = \exp(kx + k^3 t + \gamma_j), \quad k, \gamma_j = const, \quad j = 1, \ldots, n.$$

Then, 1-soliton solution of (6) is given in the form

$$u_1 = S^1 = 2\partial_x^2 \log(1 + f^1), \quad u_j = S^j - S^{j-1} = 2\partial_x^2 \log\left(\frac{1 + f^j}{1 + f^{j-1}}\right), \quad j = 2, \ldots, n.$$

Obviously, u_1 is the ordinary KdV soliton, while remaining u_i are coupled solitons (illustrated in Fig. 1), that differ among themselves by phase arguments.

The coupled 2-soliton solution of nc-KdV (6) is of the form

$$u_1 = S^1 = 2\partial_x^2 \log(1 + f_1^1 + f_2^1 + af_1^1 f_2^1),$$

$$u_j = S^j - S^{j-1} = 2\partial_x^2 \log\left(\frac{1 + f_1^j + f_2^j + af_1^j f_2^j}{1 + f_1^{j-1} + f_2^{j-1} + af_1^{j-1} f_2^{j-1}}\right), \quad j = 2, \ldots, n,$$

where

$$f_1^j = \exp(k_1 x + k_1^3 t + \gamma_{1j}), \quad f_2^j = \exp(k_2 x + k_2^3 t + \gamma_{2j}), \quad j = 1, \ldots, n$$

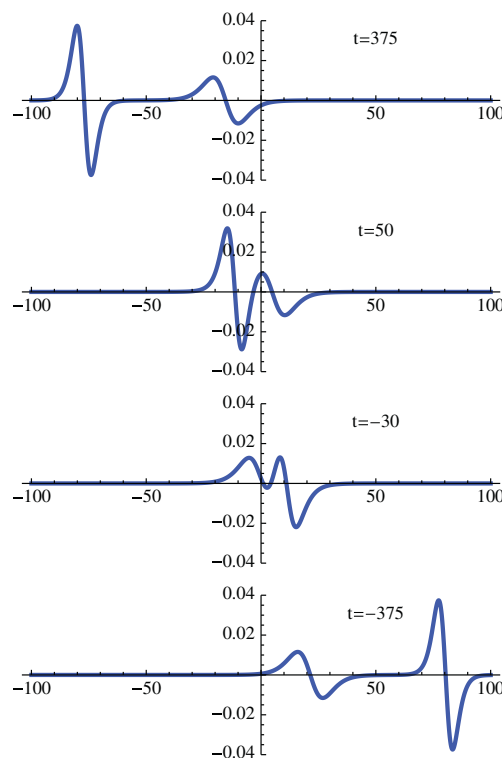**Fig. 2.** Coupled 2-soliton, $u_i = S^i - S^{i-1}$ ($i \neq 1$).

and $a = (k_1 - k_2)^2/(k_1 + k_2)^2$. Notice that u_1 is the ordinary 2-soliton solution of the KdV, while remaining u_i are coupled 2-soliton solutions, which interaction is presented in Fig. 2.

5. Matrix representation of the algebra of coupled scalars

Consider n quadratic matrices $\mathbf{E}_k$ of dimension $n \times n$:

$$(\mathbf{E}_k)_{ij} = \begin{cases} 1 & \text{if} \quad j = k, \quad i \leqslant k, \\ 1 & \text{if} \quad j = i, \quad i > k, \\ 0 & \text{otherwise}, \end{cases}$$

where $k = 1, \ldots, n$. The matrices $\mathbf{E}_i$ constitute generating elements of commutative and associative sub-algebra of triangular matrices. Let us call the matrix algebra spanned by $\mathbf{E}_k$ as a pseudo-scalar algebra of matrices, $ps(n)$.

The algebra $ps(n)$ is a matrix representation of the algebra of coupled scalars defined by the multiplication (1), which follows immediately by showing that

$$\mathbf{E}_i\mathbf{E}_j = \mathbf{E}_j\mathbf{E}_i = \mathbf{E}_{\max(i,j)}$$

for $i,j = 1, \ldots, n$. For example if $n = 4$:

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{E}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, a typical element for $n = 4$ is

$$\mathbf{A} = \sum_{i=1}^n a_i \mathbf{E}_i = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 + a_2 & a_3 & a_4 \\ 0 & 0 & a_1 + a_2 + a_3 & a_4 \\ 0 & 0 & 0 & a_1 + a_2 + a_3 + a_4 \end{pmatrix}. \tag{12}$$

The Lax representation $\mathbf{L}_t = [\mathbf{L}, \mathbf{A}]$ of nc-KdV (8) and nc-Voltera (9) in the matrix algebra $ps(n)$ are respectively given by

$$\mathbf{L} = \partial_x^2 \mathbf{E}_1 + U, \quad \mathbf{A} = 4\partial_x^3 \mathbf{E}_1 + 6U\partial_x \mathbf{E}_1 + 3U_x,$$

where

$$U = u_1 \mathbf{E}_1 + u_2 \mathbf{E}_2 + \cdots + u_n \mathbf{E}_n$$

and

$$\mathbf{L} = T\mathbf{E}_1 + V(x)T^{-1}\mathbf{E}_1, \quad A = T^2\mathbf{E}_1 + V(x+1) + V(x),$$

where

$$V(x) = v_1(x)\mathbf{E}_1 + v_2(x)\mathbf{E}_2 + \cdots + v_n(x)\mathbf{E}_n.$$

Taking the tensor product of some Lie algebras, like in the above examples of pseudo-differential and shift operators, with the pseudo-scalar algebra $ps(n)$ one can derive several examples of the coupled extensions of known integrable systems.

This is in fact the case of the coupled AKNS systems (10), where its zero-curvature equation,

$$\mathbf{L}_t - \mathbf{W}_x + [\mathbf{L}, \mathbf{W}] = 0 \tag{13}$$

is from the Lie algebra being tensor product of $ps(2)$ with the loop algebra of $sl(2)$.

The respective generating operators are given by

$$\mathbf{L} = \mathbf{E}_1 \otimes U_1 + \mathbf{E}_2 \otimes U_2 = \begin{pmatrix} U_1 & U_2 \\ 0 & U_1 + U_2 \end{pmatrix}, \tag{14}$$

where

$$U_1 = \begin{pmatrix} -\lambda & p \\ q & \lambda \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix}$$

and

$$\mathbf{W} = \mathbf{E}_1 \otimes W_1 + \mathbf{E}_2 \otimes W_2 = \begin{pmatrix} W_1 & W_2 \\ 0 & W_1 + W_2 \end{pmatrix},$$

where

$$W_1 = \begin{pmatrix} -\lambda^2 + \frac{1}{2}pq & p\lambda - \frac{1}{2}p_x \\ q\lambda + \frac{1}{2}q_x & \lambda^2 - \frac{1}{2}pq \end{pmatrix}, \quad W_2 = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}$$

with

$$d = \alpha\lambda^2 + \frac{1}{2}(1-\alpha)(ps+qr+rs) - \frac{1}{2}\alpha pq,$$

$$e = ((1-\alpha)r - \alpha p)\lambda + \frac{1}{2}\alpha p_x - \frac{1}{2}(1-\alpha)r_x,$$

$$f = ((1-\alpha)s - \alpha q)\lambda - \frac{1}{2}\alpha q_x + \frac{1}{2}(1-\alpha)s_x.$$

We have left here some freedom related to the parameter α being integration constant from the computation of $\mathbf{W}$. For various details on derivations of integrable systems we send the reader to [13] and the references therein. Then from the zero-curvature Eq. (13) we get the following system

$$p_t = -\frac{1}{2}p_{xx} + p^2q, \quad q_t = \frac{1}{2}q_{xx} - pq^2,$$

$$r_t = \frac{1}{2}\alpha p_{xx} - \alpha p^2q + (1-\alpha)\left(-\frac{1}{2}r_{2x} + 2pqr + 2prs + qr^2 + p^2s + r^2s\right), \tag{15}$$

$$s_t = -\frac{1}{2}\alpha q_{xx} + \alpha pq^2 + (\alpha-1)\left(-\frac{1}{2}s_{2x} + 2pqs + 2qrs + q^2r + ps^2 + rs^2\right).$$

The 2c-AKNS (10) (also derived in [9]) one obtains for $\alpha = 0$. Whereas, the coupled AKNS obtained in [8] is generated for $\alpha = -1$. This system is in fact a linear composition of the symmetry (10) (the case of $\alpha = 0$) and the one obtained from (15) for $\alpha = 1$.

In the papers [7–11] there are also other examples of integrable couplings for field and lattice soliton systems of the same class. All of them are generated by the spectral Lax operators, from the Lie algebra $ps(2) \otimes sl(2)[[\lambda, \lambda^{-1}]]$, being in the form (14), however with different entries of U_1 and U_2.

6. Decoupling procedure

Consider the transformation of the pseudo-scalar algebra $ps(n)$ in the form of a similarity relation

$$T(\mathbf{A}) := S^{-1}\mathbf{A}S,$$

where

$$S_{ij} = \begin{cases} 1 & \text{for} \quad i \leqslant j, \\ 0 & \text{for} \quad i > j, \end{cases} \quad (S^{-1})_{ij} = \begin{cases} 1 & \text{for} \quad j = i, \\ -1 & \text{for} \quad j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, this transformation is apparently an isomorphism of matrix algebras. One finds that

$$T(\mathbf{E}_k) = \begin{cases} 1 & \text{for} \quad j = i, \quad j \geqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $n = 4$

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and for $\mathbf{A}$ given by (12)

$$T(\mathbf{A}) := \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 + a_2 & 0 & 0 \\ 0 & 0 & a_1 + a_2 + a_3 & 0 \\ 0 & 0 & 0 & a_1 + a_2 + a_3 + a_4 \end{pmatrix}.$$

The above transformation naturally extends to any tensor product of some algebra $\mathfrak{g}$ with $ps(n)$, that is $ps(n) \otimes \mathfrak{g}$. Let

$$\mathbf{L} = \sum_{k=1}^{n} \mathbf{E}_k \otimes A_k, \tag{16}$$

where A_k belong to $\mathfrak{g}$. Then, the transformation extends to elements of the form (16) by the formula

$$T(\mathbf{L}) := \sum_{k=1}^{n} T(\mathbf{E}_k) \otimes A_k. \tag{17}$$

Since this transformation is an algebra isomorphism it preserves the Lie algebraic construction of the coupled systems. Combining (17) with the transformation of components:

$$\widetilde{A}_k = \sum_{i=1}^{k} A_i \quad k = 1, \ldots, n, \tag{18}$$

we decouple the whole construction into n independent copies, that is $ps(n) \otimes \mathfrak{g}$ decouples into the direct product of n copies of $\mathfrak{g}$.

As a consequence of the above decoupling procedure, the linear transformation to new field variables

$$\widetilde{u}_k = \sum_{i=1}^{k} u_i \quad k = 1, \ldots, n,$$

separates the coupled Eqs. (4) and (5) to n copies of the basic Eq. (3) in new fields $\widetilde{u}_k$, that is

$$(\widetilde{u}_k)_t = K[\widetilde{u}_k] \quad k = 1, \ldots, n.$$

In particular this is the case of nc-KdV (6) and for nc-Voltera (7) systems.

In the case of the coupled AKNS system (10) or (15) generated by the Lax operator (14) the decoupling transformation (18) takes the form

$$\begin{aligned} \widetilde{U}_1 &= U_1 \\ \widetilde{U}_2 &= U_1 + U_2 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \widetilde{p}_1 &= p \quad \widetilde{q}_1 = q, \\ \widetilde{p}_2 &= p + r, \\ \widetilde{q}_2 &= q + s, \end{aligned}$$

where

$$\widetilde{U}_i = \begin{pmatrix} -\lambda & \widetilde{p}_i \\ \widetilde{q}_i & \lambda \end{pmatrix} \quad i = 1, 2.$$

As result the coupled AKNS separates into two copies of the standard AKNS system.

All coupled triangular systems considered in previous sections as well as these from [7–11] (as their construction is based on the Lie algebra $ps(2) \otimes sl(2)[[\lambda, \lambda^{-1}]]$) separate (decouple) into copies of the basic equations in the same manner. In fact the same situation is in the general case, when the underlying Lie algebra has the form $ps(n) \otimes \mathfrak{g}$. This follows from the fact that the above transformation is in the form of similarity relation which naturally preserves Lie commutator, trace form, Lax and zero-curvature representations, Hamiltonian structure and other related structures.

References

[1] K. Marciniak, S. Rauch-Wojciechowski, Separable systems of coordinates for triangular Newton equations, Stud. Appl. Math. 118 (2007) 45.
[2] M. Gurses, A. Karasu, Integrable coupled KdV systems, J. Math. Phys. 39 (1998) 2103.
[3] B.A. Kupershmidt, Dark equations, J. Nonlinear. Math. Phys. 8 (2001) 363.
[4] W.X. Ma, B. Fuchssteiner, Integrable theory of the perturbation equations, Chaos Solitons Fract. 7 (1996) 1227.
[5] W.X. Ma, Integrable couplings of soliton equations by perturbations, Methods Appl. Anal. 7 (2000) 21.
[6] W.X. Ma, Variational identities and Hamiltonian structures, in: Nonlinear and modern mathematical physics, AIP Conf. Proc., 1212, The American Institute of Physics, Melville, NY, 2010, pp. 1–27.
[7] W.X. Ma, Z.N. Zhu, Constructing nonlinear discrete integrable Hamiltonian couplings, Comput. Math. Appl. 60 (2010) 2601.
[8] W.X. Ma, Nonlinear continuous integrable Hamiltonian couplings, Appl. Math. Comput. 217 (2011) 7238.
[9] F. Yu, A real nonlinear integrable couplings of continuous soliton hierarchy and its Hamiltonian structure, Phys. Lett. A 375 (2011) 1504.
[10] Y.F. Zhang, Lie algebras for constructing nonlinear integrable couplings, Commun. Theor. Phys. 56 (2011) 805.
[11] T. Chen, L.L. Zhu, L. Zhang, The generalized Broer–Kaup–Kupershmidt system and its Hamiltonian extension, Appl. Math. Sci. 5 (2011) 3767.
[12] W.X. Ma, Loop algebras and bi-integrable couplings, Chin. Ann. Math. B 33 (2012) 207.
[13] M. Błaszak, B.M. Szablikowski, Classical R-matrix theory for bi-Hamiltonian field systems, J. Phys. A Math. Theor. 42 (2009) 404002.