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Floquet theory based on new periodicity concept for hybrid systems involving q-difference equations[☆]

Murat Adıvar[*], Halis Can Koyuncuoğlu

Department of Mathematics, Izmir University of Economics, 35330 Izmir, Turkey

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ABSTRACT

Using the new periodicity concept based on shifts, we construct a unified Floquet theory for homogeneous and nonhomogeneous hybrid periodic systems on domains having continuous, discrete or hybrid structure. New periodicity concept based on shifts enables the construction of Floquet theory on hybrid domains that are not necessarily additive periodic. This makes periodicity and stability analysis of hybrid periodic systems possible on non-additive domains. In particular, this new approach can be useful to know more about Floquet theory for linear q-difference systems defined on $\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ where $q > 1$. By constructing the solution of matrix exponential equation we establish a canonical Floquet decomposition theorem. Determining the relation between Floquet multipliers and Floquet exponents, we give a spectral mapping theorem on closed subsets of reals that are periodic in shifts. Finally, we show how the constructed theory can be utilized for the stability analysis of dynamic systems on periodic time scales in shifts.

1. Introduction

The theory of periodic systems has taken a prominent attention in the existing literature due to its tremendous application potential in engineering, biology, biomathematics, chemistry etc. Floquet theory is an important tool for the investigation of periodic solutions and stability analysis of dynamic systems. Floquet theory of differential and difference systems can be found in [22,23], respectively. Floquet theory of Volterra equation has been handled in [10]. An extension of the Floquet theory to the systems with memory has been studied in [11]. In [7], Floquet theory has been employed for stability analysis of nonlinear integro-differential equations. Moreover, a generalization of Floquet theory in continuous case is studied in [28].

Providing a wide perspective to discrete and continuous analysis, time scale calculus is a useful theory for the unification of differential and difference systems. For the sake of brevity, we suppose familiarity with fundamental theory of time scales. For a comprehensive review on time scale theory, we may refer readers to [12,13]. Unification of discrete and continuous dynamic systems under the theory of time scales avoids the separate studies for differential and difference systems by using the similar arguments. Motivated by unification and extension capabilities of time scale calculus, the researchers in recent years have been developing the time scale analogues of existing results for difference, q-difference, and differential equations. For instance in [9], the authors construct a Floquet theory for additive periodic time scales and focus on Putzer representations of matrix logarithms. We use the terminology "additive periodic time scale" to refer to an arbitrary, closed, non-empty subset $\mathbb{T}$ of reals satisfying the

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* Corresponding author. Tel.: +90 2324888377.
E-mail addresses: murat.adivar@ieu.edu.tr, murat.adivar@gmail.com (M. Adıvar), can.koyuncuoglu@ieu.edu.tr (H.C. Koyuncuoğlu).

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following property ([21]):

$$\text{there exists a fixed } P \in \mathbb{T} \text{ such that } t \pm P \in \mathbb{T} \text{ for all } t \in \mathbb{T}. \tag{1.1}$$

In [17], DaCunha unified Floquet theory for nonautonomous linear dynamic systems based on Lyapunov transformations by using matrix exponential on time scales (see [12, Section 5]). Afterward, DaCunha and Davis improved the results of [17] in [16]. Note that the results in [16,17] regarding Floquet theory are valid only on additive periodic time scales. This strong restriction prevents investigation of periodicity on very important particular time scales. For instance, the q-difference equations are established on the time scale

$$\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z}\} \cup \{0\}, \quad q > 1$$

which is not additive periodic. Hence, the existing unified Floquet theory does not cover the systems of q-difference equations. A q-difference equation is an equation including a q-derivative D_q, given by

$$D_q(f)(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad t \in q^{\mathbb{Z}},$$

of its unknown function. Observe that the q-derivative $D_q(f)$ of a function f turns into ordinary derivative f' if we let $q \to 1$. The theory of q-difference equations is a useful tool for the discretization of differential equations used for modeling continuous processes (see e.g. [19,24,25], and references therein). In [27] the author says "*in the p-adic context, q-difference equations are not simply a discretization of solutions of differential equations, but they are actually equal*". We may also refer to [8] for further discussion about the equivalence between q-difference equations and differential equations. There is a vast literature on the existence of periodic solutions of differential equations, unlike the existence of periodic solutions of q-difference equations. Thus, it is of importance to study the existence of periodic solutions of q-difference equations.

In recent years, the shift operators, denoted $\delta_{\pm}(s, t)$, are introduced to construct delay dynamic equations and a new periodicity concept on time scales (see [1,4,5]). We give a detailed information about the shift operators in further sections. We may also refer to the studies [2,3,5] for the basic definitions, properties and some applications of shift operators on time scales. In particular, we direct the readers to [1] for the construction of new periodicity concept on time scales. The motivation of new periodicity concept in [1] stems from the following ideas:

I.1. Addition is not always the only way to step forward and backward on a time scale, for instance, the operators $\delta_{\pm}(2, t) = 2^{\pm 1}t$ determine backward and forward shifts on the time scale $\{2^n : n \in \mathbb{Z}\} \cup \{0\}$.

I.2. We may use shift operators $\delta_{\pm}$ with certain properties to obtain a backward and forward motion on a general time scale. Similar to (1.1) a periodic time scale in shifts can be defined to be the one satisfying the following property:

$$\text{there exists a fixed } P \in \mathbb{T} \text{ such that } \delta_{\pm}(P, t) \in \mathbb{T} \text{ for all } t \in \mathbb{T}. \tag{1.2}$$

This approach enables the study of periodicity notion on a large class of time scales that are not necessarily additive periodic. For instance, the time scale $\overline{q^{\mathbb{Z}}}$ is periodic in shifts $\delta_{\pm}(s, t) = s^{\pm}t$ since

$$\delta_{\pm}(q, t) = q^{\pm}t \in \mathbb{T} \text{ for all } t \in \mathbb{T}.$$

Therefore, one may define a q^k-periodic function f on $\overline{q^{\mathbb{Z}}}$ as follows:

$$f\left(q^{\pm k}t\right) = f(t) \text{ for all } t \in \overline{q^{\mathbb{Z}}} \text{ and a fixed } k \in \{1, 2, \ldots\}.$$

More generally, a T-periodic function f on a P-periodic time scale $\mathbb{T}$ in shifts $\delta_{\pm}$ can be defined as follows

$$f(\delta_{\pm}(T, t)) = f(t) \text{ for all } t \in \mathbb{T} \text{ and a fixed } T \in [P, \infty) \cap \mathbb{T}.$$

In this paper, we use Lyapunov transformation (see [16, Definition 2.1]) and the new periodicity concept developed in [1] to construct a unified Floquet theory for hybrid systems on hybrid domains. As an alternative to the existing literature, our Floquet theory and stability results are valid on more time scales, such as $\overline{q^{\mathbb{Z}}}$ and

$$\cup_{k=1}^{\infty}[3^{\pm k}, 2.3^{\pm k}] \cup \{0\}$$

which cannot be covered by [16,17]. It should be mentioned that periodicity notion and Floquet theory on the time scale

$$q^{\mathbb{N}_0} = \{q^n : q > 1 \text{ and } n = 0, 1, 2, \ldots\}$$

have been studied in [14,15]. In [14,15] a ω-periodic function f on $q^{\mathbb{N}_0}$ is defined to be the one satisfying

$$f(q^{\omega}t) = \frac{1}{q^{\omega}} f(t) \text{ for all } t \in q^{\mathbb{N}_0} \text{ and a fixed } \omega \in \{1, 2, \ldots\}.$$

According to this periodicity definition the function $g(t) = 1/t$ is q-periodic over the time scale $q^{\mathbb{N}_0}$. Unlike the conventional periodic functions in the existing literature, the function $g(t) = 1/t$ does not repeat its values at each period t, $q^{\omega}t$, $(q^{\omega})^2t$,.... In parallel with conventional periodicity perception, we define a periodic function to be the one repeating its values at each forward/backward step on its domain with a certain size. For instance, according to our definition the function $h(t) = (-1)^{\frac{\ln t}{\ln q}}$ is a q^2-periodic function on $q^{\mathbb{Z}} = \{q > 1 : q^n, n \in \mathbb{Z}\}$ since

$$h(\delta_{\pm}(q^2, t)) = (-1)^{\frac{\ln t}{\ln q} \pm 2} = (-1)^{\frac{\ln t}{\ln q}} = h(t).$$

Obviously, the function $h(t)$ repeats the values -1 and 1 at each backward/forward step with the size q^2. Consequently, the use of new periodicity concept based on shifts $\delta_{\pm}$ in Floquet theory provides not only a generalization but also an alternative approach and new stability results to already existing literature in particular cases (e.g. [14,15]).

We organize the rest of the paper as follows. In Section 2, we introduce the basic concepts and in Section 3 we develop Floquet theory based on new periodicity concept on time scales. We end the paper by applying our results to stability analysis of linear systems.

2. Preliminaries

2.1. Matrix exponential

In this section we give some basic definitions and results that we require in our further analysis.

A time scale, denoted by $\mathbb{T}$, is an arbitrary, nonempty and closed subset of real numbers. A time scale may have a discrete or connected structure as well as a hybrid structure consisting of intervals and isolated points. The operator $\sigma: \mathbb{T} \to \mathbb{T}$ called forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T}, s > t\}$. The step size function $\mu : \mathbb{T} \to \mathbb{R}$ is given by $\mu(t) := \sigma(t) - t$. We say a point $t \in \mathbb{T}$ is right dense if $\mu(t) = 0$, and right scattered if $\mu(t) > 0$. Furthermore, a point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) := \sup\{s \in \mathbb{T}, s < t\} = t$ and left scattered if $\rho(t) < t$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd*-continuous if it is continuous at right dense points and its left sided limits exists at left dense points. The set $\mathbb{T}^k$ is defined in the following way: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Moreover, the delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^k$ is defined by

$$f^{\Delta}(t) := \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

Definition 1. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. We denote by $\mathcal{R}$ the set of all regressive functions.

Definition 2 (Exponential function)**.** Let $\varphi \in \mathcal{R}$ and $\mu(t) > 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_{\varphi}(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z)\varphi(z))\, \Delta z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma:

Lemma 1 ([12, Theorem 2.36])**.** *Let $p, q \in \mathcal{R}$. Then*

i. $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
ii. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
iii. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
iv. $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
v. $e_p(t, s)e_p(s, r) = e_p(t, r)$;
vi. $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Definition 3 (Matrix exponential)**.** [12, Definition 5.18] Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n \times n$ matrix-valued function. The unique matrix solution of the IVP

$$Y^{\Delta}(t) = A(t)Y, \quad Y(t_0) = I,$$

where I denotes as usual $n \times n$ identity matrix, is called the matrix exponential function, and is denoted by $e_A(., t_0)$.

Theorem 1 ([12, Theorem 5.21])**.** *Let $A, B \in \mathcal{R}$ be $n \times n$ matrix-valued functions on time scale $\mathbb{T}$, then we have*

1. $e_0(t, s) \equiv I$ *and* $e_A(t, t) \equiv I$, *where 0 and I indicate the zero matrix and the identity matrix, respectively;*
2. $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$;
3. $e_A(t, s) = e_A^{-1}(s, t)$;
4. $e_A(t, s)e_A(s, r) = e_A(t, r)$;
5. $e_A(t, s)e_B(t, s) = e_{A\oplus B}(t, s)$, *where*

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t).$$

Theorem 2 ([12, Theorem 5.24] (variation of constants))**.** *Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^{\Delta} = A(t)y + f(t), \quad y(t_0) = y_0$$

Table 1
Shift operators on some time scales.

$\mathbb{T}$	t_0	$\mathbb{T}^*$	$\delta_-(s,t)$	$\delta+(s,t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t-s$	$t+s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t-s$	$t+s$
$q^{\mathbb{Z}}\cup\{0\}$	1	$q^{\mathbb{Z}}$	$\frac{t}{s}$	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$(t^2-s^2)^{1/2}$	$(t^2+s^2)^{1/2}$

has a unique solution $y:\mathbb{T}\to\mathbb{R}^n$. Moreover, this solution is given by

$$y(t)=e_A(t,t_0)y_0+\int_{t_0}^{t}e_A(t,\sigma(\tau))f(\tau)\Delta\tau.$$

2.2. Shift operators and new periodicity concept based on shift operators

In this section, we aim to introduce basic definitions and properties of shift operators. The following definitions, lemmas and examples can be found in [1–3,5].

Definition 4 (Shift operators)**.** Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0\in\mathbb{T}^*$ such that there exists operators $\delta_\pm:[t_0,\infty)_{\mathbb{T}}\times\mathbb{T}^*\to\mathbb{T}^*$ satisfying the following properties:

1. The functions $\delta_\pm$ are strictly increasing with respect to their second arguments, if

   $$(T_0,t),(T_0,u)\in\mathcal{D}_\pm:=\{(s,t)\in[t_0,\infty)_{\mathbb{T}}\times\mathbb{T}^*:\delta_\pm(s,t)\in\mathbb{T}^*\},$$

   then

   $$T_0\le t<u \text{ implies } \delta_\pm(T_0,t)<\delta_\pm(T_0,u).$$

2. If $(T_1,u),(T_2,u)\in\mathcal{D}_-$ with $T_1<T_2$, then $\delta_-(T_1,u)>\delta_-(T_2,u)$ and if $(T_1,u),(T_2,u)\in\mathcal{D}_+$ with $T_1<T_2$, then $\delta_+(T_1,u)<\delta_+(T_2,u)$.
3. If $t\in[t_0,\infty)_{\mathbb{T}}$, then $(t,t_0)\in\mathcal{D}_+$ and $\delta_+(t,t_0)=t$. Moreover, if $t\in\mathbb{T}^*$, then $(t_0,t)\in\mathcal{D}_+$ and $\delta_+(t_0,t)=t$.
4. (a) If $(s,t)\in\mathcal{D}_+$, then $(s,\delta_+(s,t))\in\mathcal{D}_-$ and $\delta_-(s,\delta_+(s,t))=t$;
   (b) If $(s,t)\in\mathcal{D}_-$, then $(s,\delta_-(s,t))\in\mathcal{D}_+$ and $\delta_+(s,\delta_-(s,t))=t$.
5. (a) If $(s,t)\in\mathcal{D}_+$ and $(u,\delta_+(s,t))\in\mathcal{D}_-$, then $(s,\delta_-(u,t))\in\mathcal{D}_+$ and $\delta_-(u,\delta_+(s,t))=\delta_+(s,\delta_-(u,t))$;
   (b) If $(s,t)\in\mathcal{D}_-$ and $(u,\delta_-(s,t))\in\mathcal{D}_+$, then $(s,\delta_+(u,t))\in\mathcal{D}_-$ and $\delta_+(u,\delta_-(s,t))=\delta_-(s,\delta_+(u,t))$.

Then the operators δ_+ and δ_- are called forward and backward shift operators associated with the initial point t_0 on $\mathbb{T}^*$ and the sets $\mathcal{D}_+$ and $\mathcal{D}_-$ are domain of the operators, respectively.

Example 1. Table 1 shows the shift operators $\delta_\pm(s,t)$ on several time scales.

Lemma 2. *Let $\delta_\pm$ be the shift operators associated with the initial point t_0. Then we have the following:*

1. $\delta_-(t,t)=t_0$ *for all* $t\in[t_0,\infty)_{\mathbb{T}}$;
2. $\delta_-(t_0,t)=t$ *for all* $t\in\mathbb{T}^*$;
3. *If* $(s,t)\in\mathcal{D}_+$, *then* $\delta_+(s,t)=u$ *implies* $\delta_-(s,u)=t$ *and if* $(s,u)\in\mathcal{D}_-$, *then* $\delta_-(s,u)=t$ *implies* $\delta_+(s,t)=u$;
4. $\delta_+(t,\delta_-(s,t_0))=\delta_-(s,t)$ *for all* $(s,t)\in\mathcal{D}_+$ *with* $t\ge t_0$;
5. $\delta_+(u,t)=\delta_+(t,u)$ *for all* $(u,t)\in([t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}})\cap\mathcal{D}_+$;
6. $\delta_+(s,t)\in[t_0,\infty)_{\mathbb{T}}$ *for all* $(s,t)\in\mathcal{D}_+$ *with* $t\ge t_0$;
7. $\delta_-(s,t)\in[t_0,\infty)_{\mathbb{T}}$ *for all* $(s,t)\in([t_0,\infty)_{\mathbb{T}}\times[s,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$;
8. *If* $\delta_+(s,.)$ *is* Δ*-differentiable in its second variable, then* $\delta_+^{\Delta_t}(s,.)>0$;
9. $\delta_+(\delta_-(u,s),\delta_-(s,v))=\delta_-(u,v)$ *for all* $(s,v)\in([t_0,\infty)_{\mathbb{T}}\times[s,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$ *and* $(u,s)\in([t_0,\infty)_{\mathbb{T}}\times[u,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$;
10. *If* $(s,t)\in\mathcal{D}_-$ *and* $\delta_-(s,t)=t_0$, *then* $s=t$.

Definition 5 (Periodicity in shifts)**.** Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0\in\mathbb{T}^*$, then $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$, if there exists a $p\in(t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t)\in\mathcal{D}_\mp$ for all $t\in\mathbb{T}^*$. P is called the period of $\mathbb{T}$ if

$$P=\inf\{p\in(t_0,\infty)_{\mathbb{T}^*}:(p,t)\in\mathcal{D}_\mp \text{ for all } t\in\mathbb{T}^*\}>t_0.$$

Observe that an additive periodic time scale must be unbounded. The following example indicates that a time scale, periodic in shifts, may be bounded.

Example 2. The following time scales are not additive periodic but periodic in shifts $\delta_\pm$.

1. $\mathbb{T}_1=\{\pm n^2:n\in\mathbb{Z}\}$, $\delta_\pm(P,t)=\begin{cases}(\sqrt{t}\pm\sqrt{P})^2 & \text{if } t>0\\ \pm P & \text{if } t=0,\\ -(\sqrt{-t}\pm\sqrt{P})^2 & \text{if } t<0\end{cases}$ $P=1,\ t_0=0,$

2. $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = q$, $t_0 = 1$,
3. $\mathbb{T}_3 = \cup_{n\in\mathbb{Z}}[2^{2n}, 2^{2n+1}]$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = 4$, $t_0 = 1$,
4. $\mathbb{T}_4 = \{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\} \cup \{0, 1\}$,

$$\delta_\pm(P,t) = \frac{q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}{1 + q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}, \quad P = \frac{q}{1+q}, \quad t_0 = \frac{1}{2}.$$

Note that the time scale $\mathbb{T}_4$ in Example 2 is bounded above and below and

$$\mathbb{T}_4^* = \left\{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\right\}.$$

Corollary 1. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Then we have*

$$\delta_\pm(P, \sigma(t)) = \sigma(\delta_\pm(P,t)) \text{ for all } t \in \mathbb{T}^*. \tag{2.1}$$

Example 3. The time scale $\widetilde{\mathbb{T}} = (-\infty, 0] \cup [1, \infty)$ cannot be periodic in shifts $\delta_\pm$. Because if there was a $p \in (t_0, \infty)_{\widetilde{\mathbb{T}}^*}$ such that $\delta_\pm(p,t) \in \widetilde{\mathbb{T}}^*$, then the point $\delta_-(p, 0)$ would be right scattered due to (2.1). However, we have $\delta_-(p, 0) < 0$ by (i) of Definition 4. This leads to a contradiction since every point less than 0 is right dense.

Definition 6 (Periodic function in shifts $\delta_\pm$)**.** Let $\mathbb{T}$ be a time scale that is P-periodic in shifts $\delta_\pm$. We say that a real valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T,t) \in \mathcal{D}_\pm \text{ and } f\left(\delta_\pm^T(t)\right) = f(t) \text{ for all } t \in \mathbb{T}^*, \tag{2.2}$$

where $\delta_\pm^T(t) = \delta_\pm(T,t)$. The number T is called the period of f, if it is the smallest number satisfying (2.2).

Example 4. Let $\mathbb{T} = \mathbb{R}$ with initial point $t_0 = 1$, the function

$$f(t) = \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi\right), \quad t \in \mathbb{R}^* := \mathbb{R} - \{0\}$$

is four-periodic in shifts $\delta_\pm$ since

$$\begin{aligned} f(\delta_\pm(4,t)) &= \begin{cases} f(t4^{\pm 1}) \text{ if } t \ge 0 \\ f(t/4^{\pm 1}) \text{ if } t < 0 \end{cases} \\ &= \sin\left(\frac{\ln|t| \pm 2\ln(1/2)}{\ln(1/2)}\pi\right) \\ &= \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi \pm 2\pi\right) \\ &= \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi\right) \\ &= f(t). \end{aligned}$$

Definition 7 (Δ-periodic function in shifts $\delta_\pm$)**.** Let $\mathbb{T}$ be a time scale P-periodic in shifts. A real valued function f defined on $\mathbb{T}^*$ is Δ-periodic function in shifts if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T,t) \in \mathcal{D}_\pm \text{ for all } t \in \mathbb{T}^* \tag{2.3}$$

$$\text{the shifts } \delta_\pm^T \text{ are } \Delta\text{-differentiable with rd-continuous derivatives} \tag{2.4}$$

and

$$f\left(\delta_\pm^T(t)\right)\delta_\pm^{\Delta T}(t) = f(t) \tag{2.5}$$

for all $t \in \mathbb{T}^*$, where $\delta_\pm^T(t) = \delta_\pm(T,t)$. The smallest number T satisfying (2.3–2.5) is called period of f.

Example 5. The function $f(t) = 1/t$ is Δ-periodic function on $q^{\mathbb{Z}}$ with the period $T = q$.

The following result is useful for integration of functions which are Δ-periodic in shifts.

Theorem 3. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with period $P \in (t_0, \infty)_{\mathbb{T}^*}$ and f a Δ-periodic function in shifts $\delta_\pm$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_\pm^T(t_0)}^{\delta_\pm^T(t)} f(s)\Delta s.$$

For more examples of periodic time scales, periodic functions and Δ-periodic functions in shifts, we may direct readers to [1].

3. Floquet theory based on new periodicity concept

In this section we use Lyapunov transformation and construct a unified Floquet theory based on new periodicity concept to give necessary and sufficient conditions for existence of periodic solutions of homogenous and nonhomogeneous dynamic equations on time scales.

Hereafter, we suppose that $\mathbb{T}$ is a periodic time scale in shifts $\delta_{\pm}$ and that the shift operators $\delta_{\pm}$ are Δ-differentiable with *rd*-continuous derivatives. For brevity, we use the term "periodic in shifts" to mean periodicity in shifts $\delta_{\pm}$. Throughout the paper, we use the notation $\delta_{\pm}^{T}(t)$ to indicate the shifts $\delta_{\pm}(T, t)$. Furthermore, we denote by $\delta_{\pm}^{(k)}(T,t)$, $k \in \mathbb{N}$, the k-times composition of shifts of $\delta_{\pm}^{T}$ with itself, namely,

$$\delta_{\pm}^{(k)}(T,t) := \underbrace{\delta_{\pm}^{T} \circ \delta_{\pm}^{T} \circ \ldots \circ \delta_{\pm}^{T}}_{k\text{-times}}(t).$$

Observe that, the period of a function f does not have to be equal to period of the time scale on which f is determined. However, for simplicity of our results we set the period of time scale $\mathbb{T}$ to be equal to period of the all functions defined on $\mathbb{T}$.

Definition 8. [16, Definition 2.1] A Lyapunov transformation is an invertible matrix $L(t) \in C_{rd}^{1}(\mathbb{T}, \mathbb{R}^{n\times n})$ satisfying

$$\left\| L(t) \right\| \le \rho \text{ and } \left| \det L(t) \right| \ge \eta \text{ for all } t \in \mathbb{T}$$

where ρ and η are arbitrary positive reals.

3.1. Homogenous case

In this section we consider the regressive time varying linear dynamic initial value problem

$$x^{\Delta}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{3.1}$$

where $A : \mathbb{T}^{*} \to \mathbb{R}^{n\times n}$ is Δ-periodic in shifts with period T. Note that if the time scale is additive periodic, then $\delta_{\pm}^{\Delta}(T,t) = 1$ and Δ-periodicity in shifts becomes the same as the periodicity in shifts. Hence, the homogeneous system we consider in this section is more general than that of [16,17].

In [18], the solution of the system (3.1) (for an arbitrary matrix A) is expressed by the equality

$$x(t) = \Phi_A(t, t_0)x_0,$$

where $\Phi_A(t, t_0)$, called the transition matrix for the system (3.1), is given by

$$\begin{aligned}\Phi_A(t,t_0) &= I + \int_{t_0}^{t} A(\tau_1)\Delta\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\Delta\tau_2\Delta\tau_1 + \ldots \\ &\quad + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\ldots\int_{t_0}^{\tau_{i-1}} A(\tau_i)\Delta\tau_i\ldots\Delta\tau_1 + \ldots.\end{aligned} \tag{3.2}$$

As mentioned in [16] the matrix exponential $e_A(t, t_0)$ is not always identical to $\Phi_A(t, t_0)$ since

$$A(t)e_A(t,t_0) = e_A(t,t_0)A(t)$$

is always true but the equality

$$A(t)\Phi_A(t,t_0) = \Phi_A(t,t_0)A(t)$$

is not. It can be seen from (3.9) that one has $e_A(t, t_0) \equiv \Phi_A(t, t_0)$ only if the matrix A satisfies

$$A(t)\int_{s}^{t} A(\tau)\Delta\tau = \int_{s}^{t} A(\tau)\Delta\tau A(t).$$

In preparation for the next result we define the set

$$P(t_0) := \left\{\delta_{+}^{(k)}(T,t_0), k = 0,1,2,\ldots\right\} \tag{3.3}$$

and the function

$$\Theta(t) := \sum_{j=1}^{m(t)} \delta_{-}\left(\delta_{+}^{(j-1)}(T,t_0), \delta_{+}^{(j)}(T,t_0)\right) + G(t), \tag{3.4}$$

where

$$m(t) := \min\left\{k \in \mathbb{N} : \delta_{+}^{(k)}(T,t_0) \ge t\right\} \tag{3.5}$$

and

$$G(t) := \begin{cases} 0 & \text{if } t \in P(t_0) \\ -\delta_{-}\left(t, \delta_{+}^{(m(t))}(T,t_0)\right) & \text{if } t \notin P(t_0)\end{cases}. \tag{3.6}$$

Remark 1. For an additive periodic time scale we always have $\Theta(t)=t-t_0$.

For the construction of matrix R, a solution of the matrix exponential equation, it is necessary to define the real power of a matrix.

Definition 9 (Real power of a matrix [16, Definition A.5])**.** Given an $n\times n$ nonsingular matrix M with elementary divisors $\{(\lambda-\lambda_i)^{m_i}\}_{i=1}^{k}$ and any $r\in\mathbb{R}$, the real power of the matrix M is given by

$$M^r := \sum_{i=1}^{k} P_i(M)\lambda_i^r\left[\sum_{j=0}^{m_i-1}\frac{\Gamma(r+1)}{j!\,\Gamma(r-j+1)}\left(\frac{M-\lambda_i I}{\lambda_i}\right)^j\right], \tag{3.7}$$

where

$$P_i(\lambda) := a_i(\lambda)b_i(\lambda),$$
$$b_i(\lambda) := \Pi_{\substack{j\neq i\\ j=1}}^{k}(\lambda-\lambda_j),$$
$$\frac{1}{p(\lambda)} = \sum_{i=1}^{k}\frac{a_i(\lambda)}{(\lambda-\lambda_i)^{m_i}},$$

and $p(\lambda)$ is the characteristic polynomial of M.

It has been deduced by [16, Proposition A.3] that the set $\{P_i(M)\}_{i=1}^{k}$ is orthogonal. That is, for any $r,s\in\mathbb{R}$ we have $M^{s+r}=M^sM^r$. In the following theorem we construct the matrix R as a solution of matrix exponential equation.

Theorem 4. *Let M be a nonsingular $n\times n$ constant matrix. Then a solution $R:\mathbb{T}\to\mathbb{C}^{n\times n}$ of the matrix exponential equation*

$$e_R\left(\delta_+^T(t_0),t_0\right)=M$$

can be given by

$$R(t)=\lim_{s\to t}\frac{M^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]}-I}{\sigma(t)-s}, \tag{3.8}$$

where I is the $n\times n$ identity matrix and Θ is as in (3.4).

Proof. Let's construct the matrix exponential function $e_R(t,t_0)$ as follows

$$e_R(t,t_0) := M^{\frac{1}{T}\Theta(t)} \text{ for } t\geq t_0, \tag{3.9}$$

where Θ is given by (3.4) and real power of a nonsingular matrix M is given by (3.7). To show that the function $e_R(t,t_0)$ constructed in (3.9) is the matrix exponential we first observe that

$$e_R(t_0,t_0)=M^{\frac{1}{T}\Theta(t_0)}=I,$$

where we use (3.9) along with $\Theta(t_0)=G(t_0)=0$. Second, differentiating (3.9) we obtain

$$e_R^{\Delta}(t,t_0)=R(t)e_R(t,t_0).$$

To see this, first suppose that t is right-scattered. Then, we have

$$\begin{aligned}
e_R^{\Delta}(t,t_0) &= \frac{e_R(\sigma(t),t_0)-e_R(t,t_0)}{\sigma(t)-t}\\
&= \frac{M^{\frac{1}{T}\Theta(\sigma(t))}-M^{\frac{1}{T}\Theta(t)}}{\sigma(t)-t}\\
&= \frac{M^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(t)]}-I}{\sigma(t)-t}M^{\frac{1}{T}\Theta(t)}\\
&= R(t)e_R(t,t_0).
\end{aligned}$$

If t is right dense, then $\sigma(t)=t$. Setting $s=t+h$ in (3.4) and using (3.9) we get

$$\begin{aligned}
e_R^{\Delta}(t,t_0) &= \lim_{h\to0}\frac{e_R(t+h,t_0)-e_R(t,t_0)}{h}\\
&= \lim_{h\to0}\frac{M^{\frac{1}{T}\Theta(t+h)}-M^{\frac{1}{T}\Theta(t)}}{h}\\
&= \lim_{h\to0}\frac{M^{\frac{1}{T}[\Theta(t+h)-\Theta(t)]}-I}{h}M^{\frac{1}{T}\Theta(t)}\\
&= R(t)e_R(t,t_0).
\end{aligned}$$

In any case, we have $e_R^{\Delta}(t,t_0) = R(t)e_R(t,t_0)$. Finally, it follows from Lemma 2 that

$$\Theta\left(\delta_+^T(t_0)\right) = \delta_-\left(t_0, \delta_+^T(t_0)\right) = \delta_+^T(t_0) = T,$$

and therefore,

$$e_R\left(\delta_+^T(t_0), t_0\right) = M^{\frac{1}{T}\Theta\left(\delta_+^T(t_0)\right)} = M.$$

The proof is complete. □

Corollary 2. *The matrices $R(t)$ and M have identical eigenvectors.*

Proof. For any eigenpairs $\{\lambda_i, v_i\}$, $i = 1, 2, \ldots, n$ of M, we get by using $Mv_i = \lambda_i v_i$ that

$$\lim_{s\to t} M^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]} v_i = \lim_{s\to t} \lambda_i^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]} v_i.$$

This implies

$$R(t)v_i = \lim_{s\to t}\left(\frac{\lambda_i^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]} - 1}{\sigma(t) - s}\right) v_i. \tag{3.10}$$

Substituting $\gamma_i(t) = \lim_{s\to t}\left(\frac{\lambda_i^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]}-1}{\sigma(t)-s}\right)$ into (3.10) we conclude that $R(t)$ has the eigenpairs $\{\gamma_i(t), v_i\}_{i=1}^n$. □

Lemma 3. *Let $\mathbb{T}$ be a time scale and $P \in \mathcal{R}(\mathbb{T}^*, \mathbb{R}^{n\times n})$ a Δ-periodic matrix valued function in shifts with period T, i.e.*

$$P(t) = P\left(\delta_\pm^T(t)\right)\delta_\pm^{\Delta T}(t).$$

Then the solution of the dynamic matrix initial value problem

$$Y^{\Delta}(t) = P(t)Y(t), \quad Y(t_0) = Y_0, \tag{3.11}$$

is unique up to a period T in shifts. That is

$$\Phi_P(t,t_0) = \Phi_P\left(\delta_+^T(t), \delta_+^T(t_0)\right) \tag{3.12}$$

for all $t \in \mathbb{T}^*$.

Proof. By [18, Theorem 3.2], the unique solution to (3.11) is $Y(t) = \Phi_P(t,t_0)Y_0$. Observe that

$$Y^{\Delta}(t) = \Phi_P^{\Delta}(t,t_0)Y_0 = P(t)\Phi_P(t,t_0)Y_0$$

and

$$Y(t_0) = \Phi_P(t_0,t_0)Y_0 = Y_0.$$

To verify (3.12) we first need to show that $\Phi_P\left(\delta_+^T(t), \delta_+^T(t_0)\right)Y_0$ is also solution for (3.11). Since the shift operator δ_+ is strictly increasing, the chain rule ([12, Theorem 1.93]) yields

$$\begin{aligned}\left[\Phi_P\left(\delta_+^T(t), \delta_+^T(t_0)\right)Y_0\right]^{\Delta} &= P\left(\delta_\pm^T(t)\right)\delta_\pm^{\Delta T}(t)\Phi_P\left(\delta_+^T(t), \delta_+^T(t_0)\right)Y_0 \\ &= P(t)\Phi_P\left(\delta_+^T(t), \delta_+^T(t_0)\right)Y_0.\end{aligned}$$

On the other hand, we have

$$\Phi_P\left(\delta_+^T(t), \delta_+^T(t_0)\right)_{t=t_0} Y_0 = \Phi_P\left(\delta_+^T(t_0), \delta_+^T(t_0)\right)Y_0 = Y_0.$$

This means $\Phi_P\left(\delta_+^T(t), \delta_+^T(t_0)\right)Y_0$ solves (3.11). From the uniqueness of solution of (3.11), we get (3.12). □

One may similarly prove the next result.

Corollary 3. *Let $\mathbb{T}$ be a time scale and $P \in \mathcal{R}(\mathbb{T}^*, \mathbb{R}^{n\times n})$ be a Δ-periodic matrix valued function in shifts, i.e.*

$$P(t) = P\left(\delta_\pm^T(t)\right)\delta_\pm^{\Delta T}(t).$$

Then

$$e_P(t,t_0) = e_P\left(\delta_+^T(t), \delta_+^T(t_0)\right). \tag{3.13}$$

Theorem 5 (Floquet decomposition)**.** *Let A be a matrix valued function that is Δ-periodic in shifts with period T. The transition matrix for A can be given in the form*

$$\Phi_A(t,\tau) = L(t)e_R(t,\tau)L^{-1}(\tau), \text{ for all } t,\tau \in \mathbb{T}^*, \tag{3.14}$$

where $R : \mathbb{T} \to \mathbb{C}^{n\times n}$ is Δ-periodic function in shifts and $L(t) \in C_{rd}^1(\mathbb{T}^*, \mathbb{R}^{n\times n})$ is periodic in shifts with the same period T.

Proof. Setting $M := \Phi_A\big(\delta_+^T(t_0), t_0\big)$ define the matrix R as in Theorem 4. Then we have

$$e_R\big(\delta_+^T(t_0), t_0\big) = \Phi_A\big(\delta_+^T(t_0), t_0\big).$$

Define the matrix $L(t)$ by

$$L(t) := \Phi_A(t, t_0) e_R^{-1}(t, t_0). \tag{3.15}$$

Obviously, $L(t) \in C_{rd}^1(\mathbb{T}^*, \mathbb{R}^{n\times n})$ and L is invertible. The equality

$$\Phi_A(t, t_0) = L(t) e_R(t, t_0), \tag{3.16}$$

along with (3.16) implies

$$\begin{aligned}\Phi_A(t_0, t) &= e_R^{-1}(t, t_0) L^{-1}(t) \\ &= e_R(t_0, t) L^{-1}(t).\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we obtain (3.14). To show periodicity of L in shifts we use (3.12)–(3.13) to get

$$\begin{aligned}L\big(\delta_+^T(t)\big) &= \Phi_A\big(\delta_+^T(t), t_0\big) e_R^{-1}\big(\delta_+^T(t), t_0\big) \\ &= \Phi_A\big(\delta_+^T(t), \delta_+^T(t_0)\big)\Phi_A\big(\delta_+^T(t_0), t_0\big) e_R\big(t_0, \delta_+^T(t)\big) \\ &= \Phi_A\big(\delta_+^T(t), \delta_+^T(t_0)\big)\Phi_A\big(\delta_+^T(t_0), t_0\big) e_R\big(t_0, \delta_+^T(t_0)\big) e_R\big(\delta_+^T(t_0), \delta_+^T(t)\big) \\ &= \Phi_A\big(\delta_+^T(t), \delta_+^T(t_0)\big) e_R\big(\delta_+^T(t_0), \delta_+^T(t)\big) \\ &= \Phi_A\big(\delta_+^T(t), \delta_+^T(t_0)\big) e_R^{-1}\big(\delta_+^T(t), \delta_+^T(t_0)\big) \\ &= \Phi_A(t, t_0) e_R^{-1}(t, t_0) \\ &= L(t).\end{aligned}$$

This completes the proof. □

Hereafter, we shall refer to (3.14) as the *Floquet decomposition* for Φ_A. The following result can be proven similar to [16, Theorem 3.7].

Theorem 6. *Let $\Phi_A(t, t_0) = L(t) e_R(t, t_0)$ be a Floquet decomposition for Φ_A. Then, $x(t) = \Phi_A(t, t_0)x_0$ is a solution of the T-periodic system (3.1) if and only if $z(t) = L^{-1}(t)x(t)$ is a solution of the system*

$$z^{\Delta}(t) = R(t) z(t), \quad z(t_0) = x_0.$$

Theorem 7. *There exists an initial state $x(t_0) = x_0 \neq 0$ such that the solution of (3.1) is T-periodic in shifts if and only if one of the eigenvalues of*

$$e_R\big(\delta_+^T(t_0), t_0\big) = \Phi_A\big(\delta_+^T(t_0), t_0\big)$$

is 1.

Proof. Suppose that $x(t_0) = x_0$ and $x(t)$ is a solution of (3.1) which is T-periodic in shifts. By Theorem 5, the Floquet decomposition of x is given by

$$x(t) = \Phi_A(t, t_0)x_0 = L(t) e_R(t, t_0) L^{-1}(t_0) x_0,$$

which also yields

$$x\big(\delta_+^T(t)\big) = L\big(\delta_+^T(t)\big) e_R\big(\delta_+^T(t), t_0\big) L^{-1}(t_0) x_0.$$

By T-periodicity of x and L in shifts, we have

$$e_R(t, t_0) L^{-1}(t_0) x_0 = e_R\big(\delta_+^T(t), t_0\big) L^{-1}(t_0) x_0,$$

and therefore,

$$e_R(t, t_0) L^{-1}(t_0) x_0 = e_R\big(\delta_+^T(t), \delta_+^T(t_0)\big) e_R\big(\delta_+^T(t_0), t_0\big) L^{-1}(t_0) x_0.$$

Since $e_R\big(\delta_+^T(t), \delta_+^T(t_0)\big) = e_R(t, t_0)$ the last equality implies

$$e_R(t, t_0) L^{-1}(t_0) x_0 = e_R(t, t_0) e_R\big(\delta_+^T(t_0), t_0\big) L^{-1}(t_0) x_0$$

and thus

$$L^{-1}(t_0) x_0 = e_R\big(\delta_+^T(t_0), t_0\big) L^{-1}(t_0) x_0.$$

Since $L^{-1}(t_0)x_0 \neq 0$, we see that $L^{-1}(t_0)x_0$ is an eigenvector of the matrix $e_R\big(\delta_+^T(t_0), t_0\big)$ corresponding to an eigenvalue of 1.

Conversely, let us assume that 1 is an eigenvalue of $e_R\big(\delta_+^T(t_0), t_0\big)$ with corresponding eigenvector z_0. This means z_0 is real valued and nonzero. Using $e_R(t, t_0) = e_R\big(\delta_+^T(t), \delta_+^T(t_0)\big)$, we arrive at the following equality

$$\begin{aligned} z\big(\delta_+^T(t)\big) &= e_R\big(\delta_+^T(t), t_0\big)z_0 \\ &= e_R\big(\delta_+^T(t), \delta_+^T(t_0)\big)e_R\big(\delta_+^T(t_0), t_0\big)z_0 \\ &= e_R\big(\delta_+^T(t), \delta_+^T(t_0)\big)z_0 \\ &= e_R(t, t_0)z_0 \\ &= z(t), \end{aligned}$$

which shows that $z(t) = e_R(t, t_0)z_0$ is T-periodic in shifts. Applying the Floquet decomposition and setting $x_0 := L(t_0)z_0$, we obtain the nontrivial solution x of (3.1) as follows

$$x(t) = \Phi_A(t, t_0)x_0 = L(t)e_R(t, t_0)L^{-1}(t_0)x_0 = L(t)e_R(t, t_0)z_0 = L(t)z(t),$$

which is T-periodic in shifts since L and z are T-periodic in shifts. □

3.2. Nonhomogeneous case

Let us focus on the nonhomogeneous regressive time varying linear dynamic initial value problem

$$x^\Delta(t) = A(t)x(t) + F(t), \quad x(t_0) = x_0, \tag{3.18}$$

where $A : \mathbb{T}^* \to \mathbb{R}^{n\times n}$, $F \in C_{rd}(\mathbb{T}^*, \mathbb{R}^n) \cap \mathcal{R}(\mathbb{T}^*, \mathbb{R}^n)$. Hereafter, we suppose both A and F are Δ-periodic in shifts with the period T.

Lemma 4. *A solution $x(t)$ of (3.18) is T-periodic in shifts if and only if $x\big(\delta_+^T(t)\big) = x(t)$ for all $t \in \mathbb{T}^*$.*

Proof. Suppose that $x(t)$ is T-periodic in shifts. Let us define $z(t)$ as

$$z(t) = x\big(\delta_+^T(t)\big) - x(t). \tag{3.19}$$

Obviously $z(t_0) = 0$. Moreover, if we take delta derivative of both sides of (3.19), we have the following:

$$\begin{aligned} z^\Delta(t) &= \Big[x\big(\delta_+^T(t)\big) - x(t)\Big]^\Delta \\ &= x^\Delta\big(\delta_+^T(t)\big) - x^\Delta(t) \\ &= x^\Delta\big(\delta_+^T(t)\big)\delta_+^{\Delta T}(t) - x^\Delta(t) \\ &= A\big(\delta_+^T(t)\big)x\big(\delta_+^T(t)\big)\delta_+^{\Delta T}(t) + F\big(\delta_+^T(t)\big)\delta_+^{\Delta T}(t) - A(t)x(t) - F(t). \end{aligned}$$

Since A and F are both Δ-periodic in shifts with the period T, we have

$$\begin{aligned} z^\Delta(t) &= A(t)x\big(\delta_+^T(t)\big) + F(t) - A(t)x(t) - F(t) \\ &= A(t)\Big[x\big(\delta_+^T(t)\big) - x(t)\Big] \\ &= A(t)z(t). \end{aligned}$$

By uniqueness of solutions, we can conclude that $z(t) \equiv 0$ and that $x\big(\delta_+^T(t)\big) = x(t)$ for all $t \in \mathbb{T}^*$. □

Theorem 8. *For any initial point $t_0 \in \mathbb{T}^*$ and for any function F, Δ-periodic in shifts with period T, there exists an initial state $x(t_0) = x_0$ such that the solution of (3.18) is T-periodic in shifts if and only if there is no a nonzero $z(t_0) = z_0$ and $t_0 \in \mathbb{T}^*$ such that the homogeneous initial value problem*

$$z^\Delta(t) = A(t)z(t), \quad z(t_0) = z_0, \tag{3.20}$$

(where A is Δ-periodic in shifts with period T) has a T-periodic solution in shifts.

Proof. In [6], the following representation for the solution of (3.18) is given

$$x(t) = X(t)X^{-1}(\tau)x_0 + \int_\tau^t X(t)X^{-1}(\sigma(s))F(s)\Delta s,$$

where $X(t)$ is a fundamental matrix solution of the homogenous system (3.1) with respect to initial condition $x(\tau) = x_0$. As it is done in [6], we can express $x(t)$ as follows

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^t \Phi_A(t, \sigma(s))F(s)\Delta s.$$

By the previous lemma we know that $x(t)$ is T-periodic in shifts if and only if $x(\delta_+^T(t_0)) = x_0$ or equivalently

$$\Big[I - \Phi_A\big(\delta_+^T(t_0), t_0\big)\Big]x_0 = \int_{t_0}^{\delta_+^T(t_0)} \Phi_A\big(\delta_+^T(t_0), \sigma(s)\big)F(s)\Delta s. \tag{3.21}$$

By guidance of Theorem 7, we have to show that (3.18) has a solution with respect to initial condition $x(t_0) = x_0$ if and only if $e_R\left(\delta_+^T(t_0), t_0\right)$ has no eigenvalues equal to 1.

Let $e_R\left(\delta_+^T(\eta), \eta\right) = \Phi_A\left(\delta_+^T(\eta), \eta\right)$, for some $\eta \in \mathbb{T}^*$, has no eigenvalues equal to 1. That is,

$$\det\left[I - \Phi_A\left(\delta_+^T(\eta), \eta\right)\right] \neq 0.$$

Invertibility and periodicity of Φ_A imply

$$\begin{aligned} 0 &\neq \det\left[\Phi_A\left(\delta_+^T(t_0), \delta_+^T(\eta)\right)\left(I - \Phi_A\left(\delta_+^T(\eta), \eta\right)\right)\Phi_A(\eta, t_0)\right] \\ &= \det\left[\Phi_A\left(\delta_+^T(t_0), \delta_+^T(\eta)\right)\Phi_A(\eta, t_0) - \Phi_A\left(\delta_+^T(t_0), t_0\right)\right]. \end{aligned} \tag{3.22}$$

By periodicity of Φ_A, the invertibility of $[I - \Phi_A(\delta_+^T(t_0), t_0)]$ is equivalent to (3.22) for any $t_0 \in \mathbb{T}^*$. Thus, (3.21) has a solution

$$x_0 = \left[I - \Phi_A\left(\delta_+^T(t_0), t_0\right)\right]^{-1} \int_{t_0}^{\delta_+^T(t_0)} \Phi_A\left(\delta_+^T(t_0), \sigma(s)\right) F(s) \Delta s$$

for any $t_0 \in \mathbb{T}^*$ and for any Δ-periodic function F in shifts with period T.

Suppose that (3.21) has a solution for every $t_0 \in \mathbb{T}^*$ and every Δ-periodic function F in shifts with period T. Let us define the set $P_-(t)$ as

$$P_-(t) = \{k \in \mathbb{Z} : \delta_-^{(k)}(T, t)\}.$$

It is clear that, $P_-(t) = P_-(\delta_+^T(t))$. Additionally, let the function ξ be defined by

$$\begin{aligned} \xi(t) &:= \prod_{s \in P_-(t) \cap [t_0, t)} \left(\delta_+^{\Delta T}(s)\right)^{-1} \\ &= \left(\delta_+^{\Delta T}(\delta_-(T, t))\right)^{-1} \times \left(\delta_+^{\Delta T}\left(\delta_-^{(2)}(T, t)\right)\right)^{-1} \times \ldots \times \left(\delta_+^{\Delta T}\left(\delta_-^{(m^-(t))}(T, t)\right)\right)^{-1}, \end{aligned}$$

where $m^-(t) = \max\{k \in \mathbb{Z} : \delta_-^{(k)}(T, t) \geq t_0\}$. By definition of ξ, we have

$$\begin{aligned} \xi\left(\delta_+^T(t)\right) &= \prod_{s \in P_-\left(\delta_+^T(t)\right) \cap \left[t_0, \delta_+^T(t)\right)} \left(\delta_+^{\Delta T}(s)\right)^{-1} \\ &= \prod_{s \in P_-(t) \cap \left[t_0, \delta_+^T(t)\right)} \left(\delta_+^{\Delta T}(s)\right)^{-1} \\ &= \left(\delta_+^{\Delta T}(t)\right)^{-1} \prod_{s \in P_-(t) \cap [t_0, t)} \left(\delta_+^{\Delta T}(s)\right)^{-1} \\ &= \left(\delta_+^{\Delta T}(t)\right)^{-1} \xi(t), \end{aligned}$$

which shows that ξ is Δ-periodic in shifts with period T. For an arbitrary t_0 and corresponding F_0, we can define a regressive and Δ-periodic function F in shifts as follows

$$F(t) := \Phi_A\left(\sigma(t), \delta_+^T(t_0)\right)\xi(t) F_0, \quad t \in \left[t_0, \delta_+^T(t_0)\right) \cap \mathbb{T}. \tag{3.23}$$

Then, we have

$$\int_{t_0}^{\delta_+^T(t_0)} \Phi_A\left(\delta_+^T(t_0), \sigma(s)\right) F(s) \Delta s = F_0 \int_{t_0}^{\delta_+^T(t_0)} \xi(s) \Delta s. \tag{3.24}$$

Thus, (3.21) can be rewritten as follows

$$\left[I - \Phi_A\left(\delta_+^T(t_0), t_0\right)\right] x_0 = \int_{t_0}^{\delta_+^T(t_0)} \xi(s) \Delta s. \tag{3.25}$$

For any F that is constructed in (3.23), and hence for any corresponding F_0, (3.25) has a solution for x_0 by assumption. Therefore,

$$\det\left[I - \Phi_A\left(\delta_+^T(t_0), t_0\right)\right] \neq 0.$$

Consequently, $e_R\left(\delta_+^T(t_0), t_0\right) = \Phi_A\left(\delta_+^T(t_0), t_0\right)$ has no eigenvalue 1. Then, we can conclude by Theorem 7, (3.20) has no periodic solution in shifts. The proof is complete. □

Example 6. Consider the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ that is q-periodic in shifts $\delta_\pm(s, t) = s^{\pm 1} t$ associated with the initial point $t_0 = 1$. Let us define the matrix function $A(t) : \mathbb{T}^* \to \mathbb{R}^{n \times n}$ as follows

$$A(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix}.$$

Then

$$A\left(\delta_+^q(t)\right)\delta_+^{\Delta q}(t) = \begin{bmatrix} \frac{1}{qt} & 0 \\ 0 & \frac{1}{qt} \end{bmatrix} \times q = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix} = A(t),$$

which shows that A is Δ-periodic in shifts with period q.

Consider the system

$$x^\Delta(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix} x(t),$$

with the transition matrix $\Phi_A(t, 1)$ given by

$$\Phi_A(t, 1) = \begin{bmatrix} e_{1/t}(t, 1) & 0 \\ 0 & e_{1/t}(t, 1) \end{bmatrix},$$

where q-exponential function defined as

$$e_p(t, t_0) = \prod_{s\in[t_0,t)} [1 + (q-1)sp(s)].$$

By (3.12), we get

$$\Phi_A\left(\delta_+^q(t), \delta_+^q(1)\right) = \Phi_A(t, 1)$$

and

$$\Phi_A\left(\delta_+^q(1), 1\right) = \Phi_A(q, 1) = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}.$$

Now, as in Theorem 4 we have

$$e_R(q, 1) = \Phi_A(q, 1) = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} = M.$$

Then $R(t)$ in the Floquet decomposition is given by

$$\begin{aligned} R(t) &= \frac{1}{qt - t}[M^{\frac{1}{q}(\Theta(qt)-\Theta(t))} - I] \\ &= \frac{1}{(q-1)t}[M^{\frac{1}{q}\times q} - I] \\ &= \frac{1}{(q-1)t}[M - I] \\ &= \begin{bmatrix} \frac{q-1}{(q-1)t} & 0 \\ 0 & \frac{q-1}{(q-1)t} \end{bmatrix} = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix}. \end{aligned}$$

By (3.9), we have

$$\begin{aligned} e_R(t, 1) &= M^{\frac{1}{q}\Theta(t)} \\ &= M^{\frac{1}{q}[\delta_-(1,q)+\ldots+\delta_-\left(t_{m(t)-1},t_{m(t)}\right)]} \\ &= M^{\frac{1}{q}qm(t)} = M^{m(t)}. \end{aligned}$$

Then, the matrix function L which is q-periodic in shifts is obtained as follows:

$$\begin{aligned} L(t) &= \Phi_A(t, 1)e_R^{-1}(t, 1) \\ &= \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}\begin{bmatrix} q^{-m(t)} & 0 \\ 0 & q^{-m(t)} \end{bmatrix} \\ &= \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}\begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix} = I \end{aligned}$$

since $q^{-m(t)} = q^{-n} = t^{-1}$ for $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

Example 7. Suppose that $\mathbb{T} = \cup_{k=0}^{\infty}[3^{\pm k}, 2.3^{\pm k}] \cup \{0\}$. Then, $\mathbb{T}$ is three-periodic in shifts $\delta_{\pm}(s,t) = s^{\pm 1}t$. If we set $A(t) = 1/t$, then we get

$$A(\delta_{\pm}(3,t))\delta_{\pm}^{\Delta}(3,t) = A(3t)3 = \frac{1}{t} = A(t)$$

which shows that A is Δ-periodic in shifts with the period 3. Consider the system

$$x^{\Delta}(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix} x(t)$$

whose transition matrix is given by

$$\Phi_A(t,1) = \begin{bmatrix} e_{1/t}(t,1) & 0 \\ 0 & e_{1/t}(t,1) \end{bmatrix}.$$

Then

$$\Phi_A\left(\delta_+^3(1),1\right) = \Phi_A(3,1) = \begin{bmatrix} e_{1/3}(3,1) & 0 \\ 0 & e_{1/3}(3,1) \end{bmatrix}.$$

As in Theorem 4, we can write that

$$e_R(3,1) = \Phi_A(3,1) = \begin{bmatrix} e_{1/3}(3,1) & 0 \\ 0 & e_{1/3}(3,1) \end{bmatrix} = M.$$

On the other hand, by (3.8) and (3.9) we have

$$\begin{aligned} e_R(t,1) &= M^{\frac{1}{3}\Theta(t)} \\ &= \begin{cases} M^{\frac{1}{3}[3m(t) - 3^{m(t)}/t]} & \text{if } t \notin P(1) \\ M^{\frac{1}{3}m(t)} & \text{if } t \in P(1) \end{cases}, \end{aligned}$$

and

$$\begin{aligned} R(t) &= \lim_{s \to t} \frac{M^{\frac{1}{3}[\Theta(\sigma(t)) - \Theta(s)]} - I}{\sigma(t) - s} \\ &= \begin{cases} \frac{2}{t}\left(M^{\frac{1}{3}[\Theta(\frac{3}{2}t) - \Theta(t)]} - I\right) & \text{if } \sigma(t) > t \\ \frac{1}{3}\log[M] & \text{if } \sigma(t) = t \end{cases}, \end{aligned}$$

where $P(t)$ and $m(t)$ are defined by (3.3) and (3.5), respectively. Then we obtain the matrix function $L(t)$ which is three-periodic in shifts as follows:

$$\begin{aligned} L(t) &= \Phi_A(t,1)e_R^{-1}(t,1) \\ &= \begin{bmatrix} e_{1/t}(t,1) & 0 \\ 0 & e_{1/t}(t,1) \end{bmatrix} \begin{bmatrix} e_{1/3}(3,1) & 0 \\ 0 & e_{1/3}(3,1) \end{bmatrix}^{-\frac{1}{3}\Theta(t)}. \end{aligned}$$

Example 8. Consider the time scale $\mathbb{T} = \mathbb{R}$ that is periodic in shifts $\delta_{\pm}(s,t) = s^{\pm 1}t$ associated with the initial point $t_0 = 1$. Let us define the matrix function $A(t) : \mathbb{T}^* \to \mathbb{R}^{n \times n}$ as follows

$$A(t) = \begin{bmatrix} \frac{1}{t}\sin\left(\pi \frac{\ln t}{\ln 2}\right) & 0 \\ 0 & \frac{1}{t}\sin\left(\pi \frac{\ln t}{\ln 2}\right) \end{bmatrix}.$$

Then $A(t)$ is Δ-periodic in shifts with the period 4. The following system

$$x^{\Delta}(t) = \begin{bmatrix} \frac{1}{t}\sin\left(\pi \frac{\ln t}{\ln 2}\right) & 0 \\ 0 & \frac{1}{t}\sin\left(\pi \frac{\ln t}{\ln 2}\right) \end{bmatrix} x(t)$$

has the transition matrix

$$\Phi_A(t,1)=\begin{bmatrix} e_{u(t)}(t,1) & 0 \\ 0 & e_{u(t)}(t,1)\end{bmatrix},$$

where $u(t)=\frac{1}{t}\sin\left(\pi\frac{\ln t}{\ln 2}\right)$. Moreover,

$$\Phi_A\left(\delta_+^4(1),1\right)=\Phi_A(4,1)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=M.$$

Thus, $R(t)$ is 2×2 zero matrix, and hence, $e_R(t,1)=I$. Finally, the matrix function $L(t)$ which is four-periodic in shifts is obtained as follows:

$$\begin{aligned}L(t) &= \Phi_A(t,1)e_R^{-1}(t,1)\\ &= \Phi_A(t,1).\end{aligned}$$

3.3. Floquet multipliers and Floquet exponents

In this section we investigate Floquet multipliers and exponents for the system (3.1). Let $\Phi_A(t,t_0)$ be the transition matrix and $\Phi(t)$ the fundamental matrix at $t=\tau$ (i.e. $\Phi(\tau)=I$) for the system (3.1). Then, we can write any fundamental matrix $\Psi(t)$ as follows

$$\Psi(t)=\Phi(t)\Psi(\tau)\text{ or }\Psi(t)=\Phi_A(t,t_0)\Psi(t_0). \tag{3.26}$$

Definition 10. Let $x_0\in\mathbb{R}^n$ be a nonzero vector and $\Psi(t)$ be any fundamental matrix for the linear dynamic system (3.1). The vector solution of the system with initial condition $x(t_0)=x_0$ is given by $\Phi_A(t,t_0)x_0$. We define the monodromy operator $M:\mathbb{R}^n\to\mathbb{R}^n$ as follows:

$$M(x_0):=\Phi_A\left(\delta_+^T(t_0),t_0\right)x_0=\Psi\left(\delta_+^T(t_0)\right)\Psi^{-1}(t_0)x_0. \tag{3.27}$$

The eigenvalues of the monodromy operator are called Floquet multipliers of the linear system (3.1).

Similar to [16, Theorem 5.2 (i)] we can give the following result.

Remark 2. The monodromy operator of the linear system (3.1) is invertible. In particular, every characteristic multiplier is nonzero.

Theorem 9. *The monodromy operator M corresponding to different fundamental matrices of the system (3.1) is unique.*

Proof. Suppose that M_1 and M_2 are the monodromy operators corresponding to fundamental matrices $\Psi_1(t)$ and $\Psi_2(t)$, respectively. By using Definition 10, we can express the monodromy operator $M_2(x_0)$ corresponding to $\Psi_2(t)$ as

$$M_2(x_0)=\Psi_2\left(\delta_+^T(t_0)\right)\Psi_2^{-1}(t_0)x_0.$$

Using (3.26), we get

$$\begin{aligned}M_2(x_0) &= \Psi_2\left(\delta_+^T(t_0)\right)\Psi_2^{-1}(t_0)x_0\\ &= \Psi_1\left(\delta_+^T(t_0)\right)\Psi_2(\tau)\Psi_2^{-1}(\tau)\Psi_1^{-1}(t_0)x_0\\ &= \Psi_1\left(\delta_+^T(t_0)\right)\Psi_1^{-1}(t_0)x_0\\ &= M_1(x_0).\end{aligned}$$

The proof is complete. □

By using Theorem 5, (3.26) and (3.27), we obtain

$$\Phi_A(t,t_0)=\Psi_1(t)\Psi_1^{-1}(t_0)=L(t)e_R(t,t_0)L^{-1}(t_0) \tag{3.28}$$

and

$$M(x_0)=\Phi_A\left(\delta_+^T(t_0),t_0\right)x_0=\Psi_1\left(\delta_+^T(t_0)\right)\Psi_1^{-1}(t_0)x_0. \tag{3.29}$$

If we combine (3.28) and (3.29), we get

$$\Phi_A\left(\delta_+^T(t_0),t_0\right)=\Psi_1\left(\delta_+^T(t_0)\right)\Psi_1^{-1}(t_0)=L\left(\delta_+^T(t_0)\right)e_R\left(\delta_+^T(t_0),t_0\right)L^{-1}\left(\delta_+^T(t_0)\right).$$

By using the periodicity in shifts of L, we have

$$\Phi_A\left(\delta_+^T(t_0),t_0\right)=L(t_0)e_R\left(\delta_+^T(t_0),t_0\right)L^{-1}(t_0). \tag{3.30}$$

Hence, we arrive at the next result:

Corollary 4. *The Floquet multipliers of the system (3.1) are the eigenvalues of the matrix* $e_R(\delta_+^T(t_0),t_0)$.

Definition 11 (Floquet exponent)**.** The Floquet exponent of the system (3.1) is the function $\gamma(t)$ satisfying the equation

$$e_{\gamma}\left(\delta_{+}^{T}(t_0), t_0\right) = \lambda,$$

where λ is the Floquet multiplier of the system.

Definition 12 ([12, Definition 2.4])**.** Let $\frac{-\pi}{h} < \omega \le \frac{\pi}{h}$. Then Hilger purely imaginary number $\overset{\circ}{\imath}\omega$ is defined by $\overset{\circ}{\imath}\omega = \frac{e^{iwh}-1}{h}$. For $z \in \mathbb{C}_h$, we have $\overset{\circ}{\imath}\mathrm{Im}_h(z) \in \mathbb{I}_h$. Also $\overset{\circ}{\imath}\omega = i\omega$ provided $h = 0$.

Theorem 10. *Suppose that $\gamma(t) \in \mathcal{R}$ is a Floquet exponent of the system (3.1) satisfying $e_{\gamma}\left(\delta_{+}^{T}(t_0), t_0\right) = \lambda$, where λ is corresponding Floquet multiplier of the T-periodic system. Then $\gamma(t) \oplus \overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}$ is also a Floquet exponent for (3.1) for all $k \in \mathbb{Z}$.*

Proof. For all $k \in \mathbb{Z}$ and any $t_0 \in \mathbb{T}^*$ we have

$$\begin{aligned}
e_{\gamma \oplus \overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}\left(\delta_{+}^{T}(t_0), t_0\right) &= e_{\gamma}\left(\delta_{+}^{T}(t_0), t_0\right) e_{\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}\left(\delta_{+}^{T}(t_0), t_0\right) \\
&= e_{\gamma}\left(\delta_{+}^{T}(t_0), t_0\right) \exp\left(\int_{t_0}^{\delta_{+}^{T}(t_0)} \frac{\log\left(1+\mu(\tau)\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}\right)}{\mu(\tau)}\Delta\tau\right) \\
&= e_{\gamma}\left(\delta_{+}^{T}(t_0), t_0\right) \exp\left(\int_{t_0}^{\delta_{+}^{T}(t_0)} \frac{\log\left(\exp\left(i\frac{2\pi k\mu(\tau)}{\delta_{+}^{T}(t_0)-t_0}\right)\right)}{\mu(\tau)}\Delta\tau\right) \\
&= e_{\gamma}\left(\delta_{+}^{T}(t_0), t_0\right) \exp\left(\int_{t_0}^{\delta_{+}^{T}(t_0)} \frac{i2\pi k}{\delta_{+}^{T}(t_0)-t_0}\Delta\tau\right) \\
&= e_{\gamma}\left(\delta_{+}^{T}(t_0), t_0\right) e^{i2\pi k} \\
&= e_{\gamma}\left(\delta_{+}^{T}(t_0), t_0\right),
\end{aligned}$$

which gives the desired result. □

The next result can be proven similar to [16, Theorem 5.3].

Theorem 11. *Let $R(t)$ be a matrix function as in Theorem 4, with eigenvalues $\gamma_1(t), \ldots, \gamma_n(t)$ repeated according to multiplicities. Then $\gamma_1^k(t), \ldots, \gamma_n^k(t)$ are the eigenvalues of $R^k(t)$ and eigenvalues of e_R are $e_{\gamma_1}, \ldots, e_{\gamma_n}$.*

Lemma 5. *Let $\mathbb{T}$ be a time scale that is p-periodic in shifts $\delta_{\pm}$ associated with the initial point t_0 and $k \in \mathbb{Z}$. If $\frac{\delta_{+}^{p}(t)-t}{\delta_{+}^{p}(t_0)-t_0} \in \mathbb{Z}$, then the functions $e_{\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}$ and $e_{\ominus\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}$ are p periodic in shifts.*

Proof. If $\frac{\delta_{+}^{p}(t)-t}{\delta_{+}^{p}(t_0)-t_0} \in \mathbb{Z}$, then we have

$$\begin{aligned}
e_{\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}\left(\delta_{+}^{p}(t), t_0\right) &= \exp\left(\int_{t_0}^{\delta_{+}^{p}(t)} \frac{i2\pi k}{\delta_{+}^{p}(t_0)-t_0}\Delta\tau\right) \\
&= \exp\left(\int_{t}^{\delta_{+}^{p}(t)} \frac{i2\pi k}{\delta_{+}^{p}(t_0)-t_0}\Delta\tau\right)\exp\left(\int_{t_0}^{t} \frac{i2\pi k}{\delta_{+}^{p}(t_0)-t_0}\Delta\tau\right) \\
&= \exp\left(i2\pi k\frac{\delta_{+}^{p}(t)-t}{\delta_{+}^{p}(t_0)-t_0}\right)\exp\left(\int_{t_0}^{t} \frac{i2\pi k}{\delta_{+}^{p}(t_0)-t_0}\Delta\tau\right) \\
&= \exp\left(\int_{t_0}^{t} \frac{i2\pi k}{\delta_{+}^{p}(t_0)-t_0}\Delta\tau\right) = e_{\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}(t, t_0)
\end{aligned}$$

which proves the periodicity of $e_{\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}$. The periodicity of $e_{\ominus\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}$ can be proven by using the periodicity of $e_{\overset{\circ}{\imath}\frac{2\pi k}{\delta_{+}^{T}(t_0)-t_0}}$ and the identity $e_{\ominus\alpha} = 1/e_{\alpha}$. □

Remark 3. Note that the condition $\frac{\delta_{+}^{p}(t)-t}{\delta_{+}^{p}(t_0)-t_0} \in \mathbb{Z}$ holds not only for all additive periodic time scales but also for the many time scales that are periodic in shifts. For example for the two-periodic time scales $\overline{2^{\mathbb{Z}}}$ and $\cup_{k=0}^{\infty}[2^{\pm k}, 2^{\pm(k+1)}] \cup \{0\}$ in shifts $\delta_{\pm}(s,t) = s^{\pm 1}t$ associated with the initial point $t_0 = 1$, the condition $\frac{\delta_{+}^{p}(t)-t}{\delta_{+}^{p}(t_0)-t_0} \in \mathbb{Z}$ is always satisfied.

Theorem 12. *If $\gamma(t)$ is a Floquet exponent for the system (3.1) and $\Phi_A(t, t_0)$ is the associated transition matrix, then there exists a Floquet decomposition of the form*

$$\Phi_A(t,t_0) = L(t)e_R(t,t_0)$$

such that $\gamma(t)$ is an eigenvalue of $R(t)$.

Proof. Consider the Floquet decomposition $\Phi_A(t,t_0) = \widetilde{L}(t)e_{\widetilde{R}}(t,t_0)$. By Definition 11, there exists a characteristic multiplier λ such that $e_\gamma\left(\delta_+^T(t_0), t_0\right) = \lambda$. Moreover, there is an eigenvalue $\tilde{\gamma}(t)$ of $\tilde{R}(t)$ so that $e_{\tilde{\gamma}}\left(\delta_+^T(t_0), t_0\right) = \lambda$, where $\tilde{\gamma}(t)$ can be defined as

$$\tilde{\gamma}(t) := \gamma(t) \oplus \mathring{i}\frac{2\pi k}{\delta_+^T(t_0) - t_0}$$

by Theorem 10. If we set

$$R(t) := \widetilde{R}(t) \ominus \mathring{i}\frac{2\pi k}{\delta_+^T(t_0) - t_0}I$$

and

$$L(t) := \tilde{L}(t)e_{\mathring{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}I}(t,t_0),$$

then we can write

$$\widetilde{R}(t) := R(t) \oplus \mathring{i}\frac{2\pi k}{\delta_+^T(t_0) - t_0}I,$$

and hence,

$$L(t)e_R(t,t_0) = \tilde{L}(t)e_{\mathring{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}I}(t,t_0)e_R(t,t_0) = \tilde{L}(t)e_{\mathring{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}I\oplus R}(t,t_0) = \tilde{L}(t)e_{\widetilde{R}}(t,t_0).$$

This means $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ is another Floquet decomposition where $\gamma(t)$ is an eigenvalue of $R(t)$. □

Theorem 13. *Suppose that λ is a characteristic multiplier of the system (3.1) and that $\gamma(t)$ is the corresponding Floquet exponent. Then, (3.1) has a nontrivial solution of the form*

$$x(t) = e_\gamma(t,t_0)q(t) \tag{3.31}$$

satisfying

$$x\left(\delta_+^T(t)\right) = \lambda x(t),$$

where q is a T-periodic function in shifts.

Proof. Let $\Phi_A(t, t_0)$ be the transition matrix of (3.1) and $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ is Floquet decomposition such that $\gamma(t)$ is an eigenvalue of $R(t)$. There exists a nonzero vector $u \neq 0$ such that $R(t)u = \gamma(t)u$, and therefore, $e_R(t,t_0)u = e_\gamma(t,t_0)u$. Then, we can represent the solution $x(t) := \Phi_A(t, t_0)u$ as follows

$$x(t) = L(t)e_R(t,t_0)u = e_\gamma(t,t_0)L(t)u.$$

If we set $q(t) = L(t)u$, the last equality implies (3.31). Thus, the first part of the theorem is proven.

The second part is proven by the following equality.

$$\begin{aligned}
x\left(\delta_+^T(t)\right) &= e_\gamma\left(\delta_+^T(t), t_0\right)q\left(\delta_+^T(t)\right)\\
&= e_\gamma\left(\delta_+^T(t), \delta_+^T(t_0)\right)e_\gamma\left(\delta_+^T(t_0), t_0\right)q(t)\\
&= e_\gamma\left(\delta_+^T(t_0), t_0\right)e_\gamma(t,t_0)L(t)u\\
&= e_\gamma\left(\delta_+^T(t_0), t_0\right)x(t)\\
&= \lambda x(t).
\end{aligned}$$

□

The preceding theorem provides a procedure for the construction of a solution to the system (3.1) when a characteristic multiplier is given. In the following theorem, we show that two solutions corresponding to two distinct characteristic multipliers are linearly independent.

Theorem 14. *Let λ_1 and λ_2 be the characteristic multipliers of the system (3.1) and γ_1 and γ_2 are Floquet exponents such that*

$$e_{\gamma_i}(\delta_+^T(t_0), t_0) = \lambda_i, \quad i = 1, 2.$$

If $\lambda_1 \neq \lambda_2$, then there exist T-periodic functions q_1 and q_2 in shifts such that

$$x_i(t) = e_{\gamma_i}(t,t_0)q_i(t), \quad i = 1, 2$$

are linearly independent solutions of (3.1).

Proof. Let $\Phi_A(t,t_0)=L(t)e_R(t,t_0)$ and $\gamma_1(t)$ be an eigenvalue of $R(t)$ corresponding to nonzero eigenvector υ_1. Since λ_2 is an eigenvalue of $\Phi_A(\delta_+^T(t_0),t_0)$, by Theorem 11 there is an eigenvalue $\gamma(t)$ of $R(t)$ satisfying

$$e_\gamma\left(\delta_+^T(t_0),t_0\right)=\lambda_2=e_{\gamma_2}\left(\delta_+^T(t_0),t_0\right).$$

Hence, for some $k\in\mathbb{Z}$ we have $\gamma_2(t)=\gamma(t)\oplus\mathring{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}$. Furthermore, $\lambda_1\neq\lambda_2$ implies that $\gamma(t)\neq\gamma_1(t)$. If υ_2 is a nonzero eigenvector of $R(t)$ corresponding to eigenvalue $\gamma(t)$, then the eigenvectors υ_1 and υ_2 are linearly independent. Similar to the related part in the proof of Theorem 13, we can state the solutions of the system (3.1) as follows:

$$x_1(t)=e_{\gamma_1}(t,t_0)L(t)\upsilon_1 \tag{3.32}$$

and

$$x_2(t)=e_{\gamma}(t,t_0)L(t)\upsilon_2.$$

Since $x_1(t_0)=L(t_0)\upsilon_1$ and $x_2(t_0)=L(t_0)\upsilon_2$, the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Moreover, the solution x_2 can be rewritten in the following form

$$\begin{aligned}x_2(t)&=e_{\gamma_2}(t,t_0)e_{\gamma\ominus\gamma_2}(t,t_0)L(t)\upsilon_2\\&=e_{\gamma_2}(t,t_0)e_{\ominus\mathring{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}}(t,t_0)L(t)\upsilon_2.\end{aligned}\tag{3.33}$$

Letting $q_1(t)=L(t)\upsilon_1$ and $q_2(t)=e_{\ominus\mathring{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}}(t,t_0)L(t)\upsilon_2$ in (3.32) and (3.33), respectively, we complete the proof. □

4. Floquet theory and stability

In this section, we employ the unified Floquet theory that we established in previous sections to investigate the stability characteristics of the regressive periodic system

$$x^\Delta(t)=A(t)x(t),\quad x(t_0)=x_0. \tag{4.1}$$

We know by Theorem 4 that the matrix R in the Floquet decomposition of Φ_A is given by

$$R(t)=\lim_{s\to t}\frac{\Phi_A\left(\delta_+^T(t_0),t_0\right)^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]}-I}{\sigma(t)-s}. \tag{4.2}$$

Also, Theorem 6 concludes that the solution $z(t)$ of the regressive system

$$z^\Delta(t)=R(t)z(t),\quad z(t_0)=x_0 \tag{4.3}$$

can be expressed in terms of the solution $x(t)$ of the system (4.1) as follows: $z(t)=L^{-1}(t)x(t)$, where $L(t)$ is the Lyapunov transformation given by (3.15).

In preparation for the main result we can give the following definitions and results which can be found in [16].

Definition 13 (Stability)**.** The time varying linear dynamic Eq. (4.1) is uniformly stable if there exists a positive constant α such that for any t_0 the corresponding solution $x(t)$ satisfies

$$\|x(t)\|\leq\alpha\|x(t_0)\|,\quad t\geq t_0.$$

Theorem 15. *The time varying linear dynamic Eq. (4.1) is uniformly stable if and only if there exists a $\alpha>0$ such that the transition matrix Φ_A satisfies*

$$\|\Phi_A(t,t_0)\|\leq\alpha,\quad t\geq t_0.$$

Definition 14 (Exponential stability)**.** The time varying linear dynamic Eq. (4.1) is uniformly exponentially stable if there exist positive constants α,β with $-\beta\in\mathcal{R}^+$ such that for any t_0 the corresponding solution $x(t)$ satisfies

$$\|x(t)\|\leq\|x(t_0)\|\alpha e_{-\beta}(t,t_0),\quad t\geq t_0.$$

Moreover, necessary and sufficient conditions for exponential stability can be stated as the following:

Theorem 16. *The time varying linear dynamic Eq. (4.1) is uniformly exponentially stable if and only if there exist $\alpha,\beta>0$ with $-\beta\in\mathcal{R}^+$ such that the transition matrix Φ_A satisfies*

$$\|\Phi_A(t,t_0)\|\leq\alpha e_{-\beta}(t,t_0),\quad t\geq t_0.$$

Definition 15 (Asymptotical stability)**.** The system (4.1) is said to be uniformly asymptotically stable if it is uniformly stable and given any $c>0$, there exists a $K>0$ so that for any t_0 and $x(t_0)$, the corresponding solution $x(t)$ satisfies

$$\|x(t)\|\leq c\|x(t_0)\|,\quad t\geq t_0+K.$$

Given a constant $n \times n$ matrix M, let S be a nonsingular matrix that transforms M into its Jordan canonical form

$$J := S^{-1}MS = \mathrm{diag}[J_{m_1}(\lambda_1), \ldots, J_{m_k}(\lambda_k)],$$

where $k \le n$, $\sum_{i=1}^{k} m_i = n$, λ_i are the eigenvalues of M, and $J_m(\lambda)$ is an $m \times m$ Jordan block given by

$$J_m(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}.$$

Definition 16 ([26] See also [16, Definition 7.1]). The scalar function $\gamma : \mathbb{T} \to \mathbb{C}$ is uniformly regressive if there exists a constant $\theta > 0$ such that $0 < \theta^{-1} \le \left|1 + \mu(t)\gamma(t)\right|$, for all $t \in \mathbb{T}^{\kappa}$.

Lemma 6. *Each eigenvalue of the matrix $R(t)$ in (4.3) is uniformly regressive.*

Proof. Define $\Lambda(t, s)$ by

$$\Lambda(t, s) := \Theta(\sigma(t)) - \Theta(s). \tag{4.4}$$

As we did in Corollary 2, let

$$\gamma_i(t) = \lim_{s \to t} \left(\frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)} - 1}{\sigma(t) - s} \right), \quad i = 1, 2, \ldots, k$$

be any of the $k \le n$ distinct eigenvalues of $R(t)$. Now, there are two cases:

1. If $|\lambda_i| \ge 1$, then

$$|1 + \mu(t)\gamma_i(t)| = \lim_{s \to t} \left| 1 + \mu(s) \frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)} - 1}{\sigma(t) - s} \right| = \lim_{s \to t} \left| \lambda_i^{\frac{1}{T}\Lambda(t,s)} \right| > 1.$$

2. If $0 \le |\lambda_i| < 1$, then,

$$\left|1 + \mu(t)\gamma_i(t)\right| = \lim_{s \to t} \left| 1 + \mu(s) \frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)} - 1}{\sigma(t) - s} \right| = \lim_{s \to t} \left| \lambda_i^{\frac{1}{T}\Lambda(t,s)} \right| \ge |\lambda_i|.$$

If we set $\theta^{-1} := \min\{1, |\lambda_1|, \ldots, |\lambda_k|\}$, then we obtain

$$0 < \theta^{-1} < \left|1 + \mu(t)\gamma_i(t)\right|,$$

where we used Remark 2 to get $0 < \theta^{-1}$. □

Definition 17 ([16, Definition 7.3]). A nonzero, delta differentiable vector $w(t)$ is said to be a dynamic eigenvector of a matrix $H(t)$ associated with the dynamic eigenvalue $\xi(t)$ if the pair satisfies the dynamic eigenvalue problem

$$w^{\Delta}(t) = H(t)w(t) - \xi(t)w^{\sigma}(t), \quad t \in \mathbb{T}^k. \tag{4.5}$$

We call $\{\xi(t), w(t)\}$ a dynamic eigenpair. Also, the nonzero, delta differentiable vector

$$\chi_i := e_{\xi_i}(t, t_0)w_i(t), \tag{4.6}$$

is called the mode vector of $M(t)$ associated with the dynamic eigenpair $\{\xi_i(t), w_i(t)\}$.

Now, we can give the following results similar to [16, Lemma 7.4, Theorem 7.5]:

Lemma 7. *Given the $n \times n$ regressive matrix K, there always exists a set of n dynamic eigenpairs with linearly independent eigenvectors. Each of the eigenpairs satisfies the vector dynamic eigenvalue problem (4.5) associated with H. Furthermore, when the n vectors form the columns of $W(t)$, then $W(t)$ satisfies the equivalent matrix dynamic eigenvalue problem*

$$W^{\Delta}(t) = H(t)W(t) - W^{\sigma}(t)\Xi(t), \text{ where } \Xi(t) := diag[\xi_1(t), \ldots, \xi_n(t)]. \tag{4.7}$$

Theorem 17. *Solutions to the uniformly regressive (but not necessarily periodic) time varying linear dynamic system (4.1) are:*

1. *stable if and only if there exists a $\gamma > 0$ such that every mode vector $\chi_i(t)$ of $A(t)$ satisfies $\|\chi_i(t)\| \le \gamma < \infty$, $t > t_0$, for all $1 \le i \le n$;*
2. *asymptotically stable if and only if, in addition to (1), $\|\chi_i(t)\| \to 0$, $t > t_0$, for all $1 \le i \le n$,*
3. *exponentially stable if and only if there exists γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ such that $\|\chi_i(t)\| \le \gamma e_{\lambda}(t, t_0)$, $t > t_0$, for all $1 \le i \le n$.*

Definition 18. For each $k \in \mathbb{N}_0$ the mappings $h_k : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}^+$, recursively defined by

$$h_0(t,t_0) :\equiv 1, \quad h_{k+1}(t,t_0) = \int_{t_0}^{t} \left(\lim_{s\to\tau} \frac{\Lambda(\tau,s)}{\sigma(\tau)-s} \right) h_k(\tau,t_0)\Delta\tau \text{ for } n \in \mathbb{N}_0, \tag{4.8}$$

are called monomials, where $\Lambda(t,s)$ is given by (4.4).

Remark 4. For an additive periodic time scale we always have $\Theta(t) = t - t_0$, and hence, $\Lambda(t,s) = \sigma(t) - s$.

Lemma 8. Let $\mathbb{T}$ be a time scale which is unbounded above and $\gamma(t)$ be an eigenvalue of $R(t)$. If there exists a constant $H \geq t_0$ such that

$$\inf_{t\in[H,\infty)_{\mathbb{T}}} \left[-\left(\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) \right)^{-1} \mathrm{Re}_\mu \gamma(t) \right] > 0 \tag{4.9}$$

holds, then

$$\lim_{t\to\infty} h_k(t,t_0) e_\gamma(t,t_0) = 0, \quad k \in \mathbb{N}_0. \tag{4.10}$$

Proof. It suffices to show that $\lim_{t\to\infty} h_k(t,t_0) e_{\mathrm{Re}_\mu \gamma(t)}(t,t_0) = 0$ (see [20, Theorem 7.4]). We proceed by mathematical induction. For $k = 0$, we know that $h_0(t,t_0) = 1$ and by [26], we have

$$\lim_{t\to\infty} e_{\mathrm{Re}_\mu \gamma_i(t)}(t,t_0) = 0 \text{ for } t_0 \in \mathbb{T}.$$

Suppose that it is true for a fixed $k \in \mathbb{N}$ and focus on the $(k+1)$th step.

$$\begin{aligned}
&\lim_{t\to\infty} h_{k+1}(t,t_0) e_{\mathrm{Re}_\mu \gamma(t)}(t,t_0) \\
&= \lim_{t\to\infty} \left[\int_{t_0}^{t} \mathfrak{R} \lim_{s\to\tau} \left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s} \right) h_k(\tau,t_0)\Delta\tau + \int_{t_0}^{t} \dot{\mathfrak{I}} \lim_{s\to\tau} \left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s} \right) h_k(\tau,t_0)\Delta\tau \right] e_{\ominus \mathrm{Re}_\mu \gamma(t)}(t,t_0)^{-1} \\
&= \lim_{t\to\infty} \left[\mathfrak{R} \lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) h_k(t,t_0) + \dot{\mathfrak{I}} \lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) h_k(t,t_0) \right] \frac{e_{\mathrm{Re}_\mu \gamma(t)}(t,t_0)}{\ominus \mathrm{Re}_\mu \gamma(t)} \\
&= \lim_{t\to\infty} \left[\frac{\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) h_k(t,t_0) e_{\mathrm{Re}_\mu \gamma(t)}(t,t_0)}{\ominus \mathrm{Re}_\mu \gamma(t)} \right],
\end{aligned} \tag{4.11}$$

where we used (4.9) together with [12, Theorem 1.120] to obtain the second equality. Since

$$\ominus \mathrm{Re}_\mu \gamma_i(t) = \frac{-\mathrm{Re}_\mu \gamma(t)}{1+\mu(t)\mathrm{Re}_\mu \gamma(t)},$$

the last term in (4.11) can be written as

$$\begin{aligned}
&\lim_{t\to\infty} \left[\frac{\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) h_k(t,t_0) e_{\mathrm{Re}_\mu \gamma(t)}(t,t_0)}{\ominus \mathrm{Re}_\mu \gamma(t)} \right] \\
&= \lim_{t\to\infty} \left[\frac{(1+\mu(t)\mathrm{Re}_\mu \gamma(t)) h_k(t,t_0) e_{\mathrm{Re}_\mu \gamma(t)}(t,t_0)}{-\left(\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right)\right)^{-1} \mathrm{Re}_\mu(\gamma(t))} \right] \\
&\leq \lim_{t\to\infty} \left[\frac{(1+\mu(t)\mathrm{Re}_\mu \gamma(t)) h_k(t,t_0) e_{\mathrm{Re}_\mu \gamma(t)}(t,t_0)}{\inf_{t\in[H,\infty)_{\mathbb{T}}} \left[-\left(\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right)\right)^{-1} \mathrm{Re}_\mu(\gamma(t)) \right]} \right].
\end{aligned} \tag{4.12}$$

Now, one may use (3.4) and (4.4) to get the inequality

$$1+\mu(t)\mathrm{Re}_\mu \gamma(t) = \left| 1 + \mu(t) \lim_{s\to t} \left(\frac{\lambda^{\frac{1}{T}\Lambda(t,s)} - 1}{\sigma(t)-s} \right) \right| \leq \max\{1, |\lambda|\}$$

which along with (4.12) implies

$$\lim_{t\to\infty} h_{k+1}(t,t_0) e_{\mathrm{Re}_\mu \gamma(t)}(t,t_0) = 0$$

as desired. □

Theorem 18. *Let* $\{\gamma_i(t)\}_{i=1}^n$ *be the set of conventional eigenvalues of the matrix* $R(t)$ *given in (4.2) and* $\{w_i(t)\}_{i=1}^n$ *be the set of corresponding linearly independent dynamic eigenvectors as defined by Lemma 7. Then,* $\{\gamma_i(t), w_i(t)\}_{i=1}^n$ *is a set of dynamic eigenpairs of* $R(t)$ *with the property that for each* $1 \le i \le n$ *there are positive constants* $D_i > 0$ *such that*

$$\|w_i(t)\| \le D_i \sum_{k=0}^{m_i-1} h_k(t, t_0), \tag{4.13}$$

holds where $h_k(t, t_0)$, $k = 0, 1, \ldots, m_i - 1$, *are the monomials defined as in (4.8) and* m_i *is the dimension of the Jordan block which contains the* i*th eigenvalue, for all* $1 \le i \le n$.

Proof. By Lemma 7, it is obvious that, $\{\gamma_i(t), w_i(t)\}_{i=1}^n$ is the set of eigenpairs of $R(t)$. First, there exists an appropriate $n \times n$ constant, nonsingular matrix S which transforms $\Phi_A(\delta_+^T(t_0), t_0)$ to its Jordan canonical form given by

$$\begin{aligned} J &:= S^{-1}\Phi_A\left(\delta_+^T(t_0), t_0\right)S \\ &= \begin{bmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{m_d}(\lambda_d) \end{bmatrix}_{n\times n}, \end{aligned} \tag{4.14}$$

where $d \le n$, $\sum_{i=1}^d m_i = n$, λ_i are the eigenvalues of $\Phi_A\left(\delta_+^T(t_0), t_0\right)$. By utilizing above determined matrix S, we define the following:

$$\begin{aligned} K(t) &:= S^{-1}R(t)S \\ &= S^{-1}\left(\lim_{s\to t}\frac{\Phi_A\left(\delta_+^T(t_0), t_0\right)^{\frac{1}{T}\Lambda(t,s)} - I}{\sigma(t) - s}\right)S \\ &= \lim_{s\to t}\frac{S^{-1}\Phi_A\left(\delta_+^T(t_0), t_0\right)^{\frac{1}{T}\Lambda(t,s)}S - I}{\sigma(t) - s}. \end{aligned}$$

This along with [16, Theorem A.6] yields

$$K(t) = \lim_{s\to t}\frac{J^{\frac{1}{T}\Lambda(t,s)} - I}{\sigma(t) - s}.$$

Note that, $K(t)$ has the block diagonal form

$$K(t) = \mathrm{diag}[K_1(t), \ldots, K_d(t)]$$

in which each $K_i(t)$ given by

$$K_i(t) := \lim_{s\to t} K_i(t) := \lim_{s\to t}\begin{bmatrix} \frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)}-1}{\sigma(t)-s} & \frac{\frac{1}{T}\Lambda(t,s)\lambda_i^{\frac{1}{T}\Lambda(t,s)-1}}{(\sigma(t)-s)2!} & \cdots & \frac{\left(\prod_{k=0}^{n-2}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)\lambda_i^{\frac{1}{T}\Lambda(t,s)-n+1}}{(n-1)!(\sigma(t)-s)} \\ & \frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)}-1}{\sigma(t)-s} & \cdots & \frac{\left(\prod_{k=0}^{n-3}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)\lambda_i^{\frac{1}{T}\Lambda(t,s)-n+2}}{(n-2)!(\sigma(t)-s)} \\ & & \ddots & \vdots \\ & & & \frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)}-1}{\sigma(t)-s} \end{bmatrix}_{m_i\times m_i}.$$

It should be mentioned that, since $R(t)$ and $K(t)$ are similar, they have the same conventional eigenvalues

$$\gamma_i(t) = \lim_{s\to t}\left(\frac{\lambda_i^{\frac{1}{T}[\Lambda(t,s)]} - 1}{\sigma(t) - s}\right), \quad i = 1, 2, \ldots, n,$$

with corresponding multiplicities. Moreover, if we set the dynamic eigenvalues of $K(t)$ to be same as conventional eigenvalues $\gamma_i(t)$, then the corresponding dynamic eigenvectors $\{u_i(t)\}_{i=1}^n$ of $K(t)$ can be given by $u_i(t) = S^{-1}w_i(t)$.

We can prove this claim by showing that $\{\gamma_i(t), u_i(t)\}_{i=1}^n$ is a set of dynamic eigenpairs of $K(t)$. By Definition 17, we can write that

$$\begin{aligned} u_i^\Delta(t) &= S^{-1}w_i^\Delta(t) \\ &= S^{-1}R(t)w_i(t) - S^{-1}\gamma_i(t)w_i^\sigma(t) \\ &= K(t)S^{-1}w_i(t) - \gamma_i(t)S^{-1}w_i^\sigma(t) \\ &= K(t)u_i(t) - \gamma_i(t)u_i^\sigma(t), \end{aligned} \tag{4.15}$$

for all $1 \le i \le n$ and this proves our claim. Now, we have to show that $u_i(t)$ satisfies (4.13). Since $\{\gamma_i(t), u_i(t)\}_{i=1}^n$ is the set of dynamic eigenpairs of $K(t)$, it satisfies (4.15) for all $1 \le i \le n$. By choosing the ith block of $K(t)$ with dimension $m_i \times m_i$, we can construct the following linear dynamic system:

$$v^\Delta(t) = \tilde{K}_i(t)v(t) = \lim_{s\to t}\begin{bmatrix} 0 & \frac{\frac{1}{T}\Lambda(t,s)}{(\sigma(t)-s)\lambda_i} & \frac{\left(\frac{1}{T}\Lambda(t,s)\right)\left(\frac{1}{T}\Lambda(t,s)-1\right)}{(\sigma(t)-s)\lambda_i 2!} & \cdots & \frac{\left(\prod_{k=0}^{n-2}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)}{(n-1)!(\sigma(t)-s)\lambda_i^{n-1}} \\ & 0 & \frac{\frac{1}{T}\Lambda(t,s)}{(\sigma(t)-s)\lambda_i} & & \frac{\left(\prod_{k=0}^{n-3}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)}{(n-2)!(\sigma(t)-s)\lambda_i^{n-2}} \\ & & 0 & \ddots & \vdots \\ & & & \ddots & \frac{\frac{1}{T}\Lambda(t,s)}{(\sigma(t)-s)\lambda_i} \\ & & & & 0 \end{bmatrix} v(t), \tag{4.16}$$

where $\tilde{K}_i(t)(t) := K_i(t) \ominus \gamma_i(t)I$. There are m_i linearly independent solutions of (4.16). Let us denote these solutions by $v_{i,j}(t)$, where i corresponds to the ith block matrix $K_i(t)$ and $j = 1, \ldots, m_i$. For $1 \le i \le d$, we define $l_i = \sum_{s=0}^{i-1} m_s$, with $m_0 = 0$. Then, the form of an arbitrary $n \times 1$ column vector u_{l_i+j} for $i \le j \le m$ can be given as

$$u_{l_i+j} = [\underbrace{0,\ldots,0}_{m_1+\ldots+m_{i-1}}, \underbrace{v_{i,j}^T(t)}_{m_i}, \underbrace{0,\ldots,0}_{m_{i+1},\ldots,m_d}]_{1\times n}. \tag{4.17}$$

When we consider the all vector solutions of (4.15), the solution of the $n \times n$ matrix dynamic equation

$$U^\Delta(t) = K(t)U(t) - U^\sigma(t)\Gamma(t),$$

where $\Gamma(t) := \mathrm{diag}[\gamma_1(t), \ldots, \gamma_n(t)]$, can be written as

$$U(t) := \left[u_1, \ldots, u_{m_1}, \ldots, u_{\left(\sum_{k=1}^{i-1} m_k\right)}, \ldots, u_{\left(\sum_{k=1}^{i} m_k\right)}, \ldots, u_{\left(\sum_{k=1}^{d} m_k\right)-1}, u_n\right]$$

$$= \begin{bmatrix} \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,m_1} \\ & v_{1,1} & \ddots & v_{1,m_1-1} \\ & & \ddots & \vdots \\ & & & v_{1,1} \end{bmatrix}_{m_1\times m_1} & & \\ & \ddots & \\ & & \begin{bmatrix} v_{d,1} & v_{d,2} & \cdots & v_{d,m_d} \\ & v_{d,1} & \ddots & v_{d,m_d-1} \\ & & \ddots & \vdots \\ & & & v_{d,1} \end{bmatrix}_{m_d\times m_d} \end{bmatrix}_{n\times n}.$$

The m_i linearly independent solutions of (4.16) have the form

$$\begin{aligned} v_{i,1}(t) &:= [v_{i,m_i}(t), 0, \ldots, 0]^T_{m_i\times 1}, \\ v_{i,2}(t) &:= [v_{i,m_i-1}(t), v_{i,m_i}(t), 0, \ldots, 0]^T_{m_i\times 1}, \\ &\vdots \\ v_{i,m_i}(t) &:= [v_{i,1}(t), v_{i,2}(t), \ldots, v_{i,m_i-1}(t), v_{i,m_i}(t)]^T_{m_i\times 1}. \end{aligned}$$

Then, we have the dynamic equations

$$\begin{aligned} v_{i,m_i}^\Delta(t) &= 0, \\ v_{i,m_i-1}^\Delta(t) &= \lim_{s\to t}\frac{[\Lambda(t,s)]}{T(\sigma(t)-s)\lambda_i} v_{i,m_i}(t), \\ v_{i,m_i-2}^\Delta(t) &= \lim_{s\to t}\frac{\left(\prod_{k=0}^{1}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)}{2(\sigma(t)-s)\lambda_i^2} v_{i,m_i}(t) + \lim_{s\to t}\frac{\Lambda(t,s)}{T(\sigma(t)-s)\lambda_i} v_{i,m_i-1}(t), \\ &\vdots \\ v_{i,1}^\Delta(t) &= \lim_{s\to t}\frac{\left(\prod_{k=0}^{m_i-2}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)}{(m_i-1)!(\sigma(t)-s)\lambda_i^{m_i-1}} v_{i,m_i}(t) \end{aligned}$$

$$+ \lim_{s\to t} \frac{\left(\prod_{k=0}^{m_i-3}[\frac{1}{T}\Lambda(t,s)-k]\right)}{(m_i-2)!(\sigma(t)-s)\lambda^{m_i-2}} v_{i,m_i-1}(t)$$
$$+ \ldots + \lim_{s\to t} \frac{\left(\prod_{k=0}^{1}[\frac{1}{T}\Lambda(t,s)-k]\right)}{2(\sigma(t)-s)\lambda_i^2} v_{i,3}(t) + \lim_{s\to t} \frac{\Lambda(t,s)}{T(\sigma(t)-s)\lambda_i} v_{i,2}(t).$$

Moreover, we have the following solutions:

$$v_{i,m_i}(t)=1, \quad v_{i,m_i-1}(t)=\int_{t_0}^{t} \lim_{s\to\tau} \frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i} v_{i,m_i}(\tau)\Delta\tau,$$

$$v_{i,m_i-2}(t) = \int_{t_0}^{t} \lim_{s\to\tau} \frac{\left(\prod_{k=0}^{1}[\frac{1}{T}\Lambda(\tau,s)-k]\right)}{2(\sigma(\tau)-s)\lambda_i^2} v_{i,m_i}(\tau)\Delta\tau + \int_{t_0}^{t} \lim_{s\to\tau} \frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i} v_{i,m_i-1}(\tau)\Delta\tau,$$
$$\vdots$$
$$v_{i,1}(t) = \int_{t_0}^{t} \lim_{s\to\tau} \frac{\left(\prod_{k=0}^{m_i-2}\frac{1}{T}\Lambda(\tau,s)-k]\right)}{(m_i-1)!(\sigma(\tau)-s)\lambda_i^{m_i-1}} v_{i,m_i}(\tau)\Delta\tau$$
$$+ \int_{t_0}^{t} \lim_{s\to\tau} \frac{\left(\prod_{k=0}^{m_i-3}\frac{1}{T}\Lambda(\tau,s)-k]\right)}{(m_i-2)!(\sigma(\tau)-s)\lambda_i^{m_i-2}} v_{i,m_i-1}(\tau)\Delta\tau + \ldots + \int_{t_0}^{t} \lim_{s\to\tau} \frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i} v_{i,2}(\tau)\Delta\tau.$$

Then we can show that each $v_{i,j}$ is bounded. There exist constants $B_{i,j}$, $i=1,\ldots,d$ and $j=1,\ldots,m_i$, such that

$$\left|v_{i,m_i}(t)\right| = 1 \le B_{i,m_i} h_0(t,t_0) = B_{i,m_i},$$
$$\left|v_{i,m_i-1}(t)\right| \le \int_{t_0}^{t} \lim_{s\to\tau}\left(\frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i}\right) v_{i,m_i}(\tau)\Delta\tau \le \frac{1}{T\lambda_i}\int_{t_0}^{t} \lim_{s\to\tau}\left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s}\right) h_0(\tau,t_0)\Delta\tau$$
$$\le \frac{h_1(t,t_0)}{T\lambda_i} \le B_{i,m_i-1} h_1(t,t_0),$$
$$\left|v_{i,m_i-2}(t)\right| \le \int_{t_0}^{t} \left|\lim_{s\to\tau} \frac{\left(\prod_{k=0}^{1}[\frac{1}{T}\Lambda(\tau,s)-k]\right)}{2(\sigma(\tau)-s)\lambda_i^2}\right| v_{i,m_i}(\tau)\Delta\tau + \int_{t_0}^{t} \lim_{s\to\tau}\left(\frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i}\right) v_{i,m_i-1}(\tau)\Delta\tau.$$

Since

$$0 \le \Theta(\sigma(\tau)) - \Theta(s) \le T \text{ as } s\to\tau,$$

we get

$$\left|\frac{1}{T}\Lambda(\tau,s)-k\right| \le k \text{ as } s\to\tau \text{ for } k=1,2,\ldots.$$

Then

$$|v_{m_i-2}(t)| \le \frac{1}{2T\lambda_i^2}\int_{t_0}^{t} \lim_{s\to\tau}\left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s}\right) h_0(\tau,t_0)\Delta\tau + \frac{1}{T^2\lambda_i^2}\int_{t_0}^{t} \lim_{s\to\tau}\left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s}\right) h_1(\tau,t_0)\Delta\tau$$
$$= \frac{h_1(t,t_0)}{2T\lambda_i^2} + \frac{h_2(t,t_0)}{T^2\lambda_i^2}$$
$$\le B_{i,m_i-2}\sum_{j=1}^{2} h_j(t,t_0)$$
$$\vdots$$
$$|v_1| \le B_{i,1}\sum_{j=1}^{m_i-1} h_j(t,t_0).$$

If we set $\beta_i := \max_{j=1,\ldots,m_i}\{B_{i,j}\}$ for each $1\le i\le d$, we obtain

$$\|u_{l_i+j}(t)\| \le \beta_i \sum_{k=0}^{m_i-1} h_k(t,t_0)$$

for $1\le i\le d$ and $j=1,2,\ldots,m_i$. Since $w_i = Su_i$ we have

$$\|w_i(t)\| = \|Su_i(t)\| \leq \|S\|\beta_i \sum_{k=0}^{m_i-1} h_k(t,t_0)$$
$$= D_i \sum_{k=0}^{m_i-1} h_k(t,t_0),$$

where $D_i := \|S\|\beta_i$, for all $1 \leq i \leq n$. The proof is complete. □

Definition 19 ([16, Definition 7.8]). Let $\mathbb{C}_\mu := \{z \in \mathbb{C} : z \neq -\frac{1}{\mu(t)}\}$. Given an element $t \in \mathbb{T}^k$ with $\mu(t) > 0$, the Hilger circle is defined by

$$\mathcal{H}_t := \{z \in \mathbb{C}_\mu : \mathrm{Re}_\mu(z) < 0\}.$$

If $\mu(t) = 0$, Hilger circle becomes

$$\mathcal{H}_t := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}.$$

Now, we can state the main stability theorem. This theorem shows strong relationship between the stability results of the T-periodic time varying linear dynamic system (4.1) and the eigenvalues of the corresponding time varying linear dynamic system (4.3).

Theorem 19 (Floquet stability theorem). *Let $\mathbb{T}$ be a periodic time scale in shifts that is unbounded above. We get the following stability results of the solutions of the system (4.1) based on the eigenvalues $\{\gamma_i(t)\}_{i=1}^n$ of system (4.3):*

1. *If there is a positive constant H such that*

$$\inf_{t\in[H,\infty)_\mathbb{T}} \left[-\left(\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) \right)^{-1} \mathrm{Re}_\mu \gamma_i(t) \right] > 0 \tag{4.18}$$

for all $i = 1, \dots, n$, then the system (4.1) is asymptotically stable. Moreover, if there are positive constants H and ε such that (4.18) and

$$-\mathrm{Re}_\mu \gamma_i(t) \geq \varepsilon \tag{4.19}$$

for all $t \in [H,\infty)_\mathbb{T}$ and all $i = 1, \dots, n$, then the system (4.1) is exponentially stable.

2. *If there is a positive constant H such that*

$$\inf_{t\in[H,\infty)_\mathbb{T}} \left[-\left(\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) \right)^{-1} \mathrm{Re}_\mu \gamma_i(t) \right] \geq 0 \tag{4.20}$$

for all $i = 1, \dots, n$, and if, for each characteristic exponent with

$$\mathrm{Re}_\mu(\gamma_i(t)) = 0 \text{ for all } t \in [H,\infty)_\mathbb{T},$$

the algebraic multiplicity equals the geometric multiplicity, then the system (4.1) is stable; otherwise the system (4.1) is unstable.

3. *If there exists a number $H \in \mathbb{R}$ such that*

$$\mathrm{Re}_\mu(\gamma_i(t)) > 0$$

for all $t \in [H,\infty)_\mathbb{T}$ and some $i = 1, \dots, n$, then the system (4.1) is unstable.

Proof. Let $e_R(t,t_0)$ be the transition matrix of the system (4.3) and $R(t)$ be defined as in (4.2). Given the conventional eigenvalues $\{\gamma_i(t)\}_{i=1}^n$ of $R(t)$, we can define the set of dynamic eigenpairs $\{\gamma_i(t), w_i(t)\}_{i=1}^n$ and from Theorem 18, the dynamic eigenvector $w_i(t)$ satisfies (4.13). Moreover, let us define $W(t)$ as the following:

$$W(t) = e_R(t,\tau)e_{\ominus\Xi}(t,\tau) \tag{4.21}$$

and we have

$$e_R(t,\tau) = W(t)e_\Xi(t,\tau), \tag{4.22}$$

where $\tau \in \mathbb{T}$ and $\Xi(t)$ is given as in Lemma 7. Employing (4.22), we can write that

$$e_R(\tau,t_0) = e_\Xi(\tau,t_0)W^{-1}(t_0). \tag{4.23}$$

By combining (4.22) and (4.23), the transition matrix of the system (4.3) can be represented by

$$e_R(t,t_0) = W(t)e_\Xi(t,t_0)W^{-1}(t_0), \tag{4.24}$$

where $W(t) := [w_1(t), w_2(t), \dots, w_n(t)]$. Furthermore, we can denote the matrix $W^{-1}(t_0)$ as follows:

$$W^{-1}(t_0) = \begin{bmatrix} v_1^T(t_0) \\ v_2^T(t_0) \\ \vdots \\ v_n^T(t_0) \end{bmatrix}.$$

Since $\Xi(t)$ is a diagonal matrix, we can write (4.24) as

$$e_R(t,t_0) = \sum_{i=1}^{n} e_{\gamma_i}(t,t_0) W(t) F_i W^{-1}(t_0), \tag{4.25}$$

where $F_i := \delta_{i,j}$ is $n \times n$ matrix. Using $v_i^T(t) w_j(t) = \delta_{i,j}$ for all $t \in \mathbb{T}$, we rewrite F_i as follows:

$$F_i = W^{-1}(t)[0, \dots, 0, w_i(t), 0, \dots, 0]. \tag{4.26}$$

By means of (4.25) and (4.26) we have

$$e_R(t,t_0) = \sum_{i=1}^{n} e_{\gamma_i}(t,t_0) w_i(t) v_i^T(t_0) = \sum_{i=1}^{n} \chi_i(t) v_i^T(t_0),$$

where $\chi_i(t)$ is mode vector of system (4.3).

Case 1. By (4.6), for each $1 \le i \le n$, we can write that

$$\begin{aligned} \|\chi_i(t)\| &\le D_i \sum_{k=0}^{d_i-1} h_k(t,t_0) |e_{\gamma_i}(t,t_0)| \\ &\le D_i \sum_{k=0}^{d_i-1} h_k(t,t_0) e_{\mathrm{Re}_\mu(\gamma_i)}(t,t_0) \end{aligned}$$

where D_i is as in Theorem 18, d_i represents the dimension of the Jordan block which contains ith eigenvalue of $R(t)$. Using Lemma 8 we get

$$\lim_{t\to\infty} h_k(t,t_0) e_{\mathrm{Re}_\mu(\gamma_i)}(t,t_0) = 0$$

for each $1 \le i \le n$ and all $k = 1, 2, \dots, d_i - 1$. This along with Theorem 17 implies that (4.3) is asymptotically stable. By Theorem 6, since the solutions of (4.1) and (4.3) are related by Lyapunov transformation, we can state that solution of (4.1) is asymptotically stable. For the second part, we first write

$$\begin{aligned} \|\chi_i(t)\| &\le D_i \sum_{k=0}^{d_i-1} h_k(t,t_0) |e_{\gamma_i}(t,t_0)| \\ &\le D_i \sum_{k=0}^{d_i-1} h_k(t,t_0) e_{\mathrm{Re}_\mu(\gamma_i)\oplus\varepsilon}(t,t_0) e_{\ominus\varepsilon}(t,t_0). \end{aligned} \tag{4.27}$$

If (4.19) holds, then $\mathrm{Re}_\mu(\gamma_i \oplus \varepsilon)$ satisfies (4.9). Hence, by Lemma 8 the term $h_k(t,t_0) e_{\mathrm{Re}_\mu(\gamma_i)\oplus\varepsilon}(t,t_0)$ converges to zero as $t \to \infty$. That is, there is an upper bound C_ε for the sum $\sum_{k=0}^{d_i-1} h_k(t,t_0) e_{\mathrm{Re}_\mu(\gamma_i)\oplus\varepsilon}(t,t_0)$. This along with (4.27) yields

$$\|\chi_i(t)\| \le D_i C_\varepsilon e_{\ominus\varepsilon}(t,t_0).$$

Thus, Theorem 17 implies that (4.3) is exponentially stable. Using the above given argument (4.1) is exponentially stable.

Case 2. Assume that $\mathrm{Re}_\mu[\gamma_k(t)] = 0$ for some $1 \le k \le n$ with equal algebraic and geometric multiplicities corresponding to $\gamma_k(t)$. Then the Jordan block of $\gamma_k(t)$ is 1×1 and this implies

$$\chi_k(t) = \beta_k e_{\gamma_k}(t,t_0).$$

Thus,

$$\begin{aligned} \lim_{t\to\infty} \|\chi_k(t)\| &\le \lim_{t\to\infty} \beta_k |e_{\gamma_k}(t,t_0)| \\ &\le \lim_{t\to\infty} \beta_k e_{\mathrm{Re}_\mu(\gamma_k)}(t,t_0) \\ &= 0. \end{aligned}$$

By Theorem 17, the system (4.3) is stable. By Theorem 6, the solutions of (4.1) and (4.3) are related by Lyapunov transformation. This implies that the system (4.1) is stable.

Case 3. Suppose that $\text{Re}_\mu(\gamma_i(t)) > 0$ for some $i = 1, \ldots, n$. Then, we have

$$\lim_{t\to\infty} \|e_R(t, t_0)\| = \infty,$$

and by the relationship between solutions of (4.1) and (4.3), we can write that

$$\lim_{t\to\infty} \|\Phi_A(t, t_0)\| = \infty.$$

Therefore, (4.1) is unstable. □

Remark 5. In the case when the time scale is additive periodic, Theorem 19 gives its additive counterpart [16, Theorem 7.9]. For an additive time scale the graininess function $\mu(t)$ is bounded above by the period of the time scale. However, this is not true in general for the times scales that are periodic in shifts. The highlight of Theorem 19 is to rule out strong restriction that obliges the time scale to be additive periodic. Hence, unlike [16, Theorem 7.9] our stability theorem (i.e. Theorem 19) is valid for q-difference systems.

We can state the following corollary as a consequence of Theorem 19.

Corollary 5. *Consider the T-periodic linear dynamic system (3.1);*

1. *If all the Floquet multipliers have modulus less than* 1*, then the system (3.1) is exponentially stable;*
2. *If all of the Floquet multipliers have modulus less than or equal to* 1*, and if, for each Floquet multiplier with modulus less than* 1*, the algebraic multiplicity equals to geometric multiplicity, then the system (3.1) is stable, otherwise the system (3.1) is unstable, growing at rates of generalized polynomials of t;*
3. *If at least one of the Floquet multipliers have modulus greater than* 1*, then the system (3.1) is unstable.*

Now, we can revisit our examples to make stability analysis:

Example 9. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$ and consider the following system

$$\begin{aligned} x^{\Delta}(t) &= A(t)x(t) \\ &= \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix} x(t). \end{aligned} \tag{4.28}$$

As we did in Example 6 we obtain $R(t)$ as follows:

$$R(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix}.$$

Then $R(t)$ has eigenvalues $\gamma_{1,2}(t) = 1/t$ and

$$\begin{aligned} \text{Re}_\mu(\gamma_{1,2}(t)) &= \frac{|\mu(t)\gamma_{1,2}(t) + 1| - 1}{\mu(t)} \\ &= \frac{|(qt - t)\frac{1}{t} + 1| - 1}{qt - t} \\ &= \frac{q-1}{qt-t} \\ &= \frac{1}{t} > 0. \end{aligned}$$

Thus, we can conclude by the preceding theorem that the system (4.28) is unstable.

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