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Flat minimal quantizations of Stäckel systems and quantum separability

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HIGHLIGHTS

- Using Stäckel transform, separable Hamiltonians are expressed by flat coordinates.
- The concept of admissible flat minimal quantizations is developed.
- The class of Stäckel systems, separable after minimal flat quantization is established.
- Separability of related stationary Schrödinger equations is presented in explicit form.

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ABSTRACT

In this paper, we consider the problem of quantization of classical Stäckel systems and the problem of separability of related quantum Hamiltonians. First, using the concept of Stäckel transform, natural Hamiltonian systems from a given Riemann space are expressed by some flat coordinates of related Euclidean configuration space. Then, the so-called flat minimal quantization procedure is applied in order to construct an appropriate Hermitian operator in the respective Hilbert space. Finally, we distinguish a class of Stäckel systems which remains separable after any of admissible flat minimal quantizations.

1. Introduction

There exists a connection between classical Hamiltonian systems and quantum systems, through an appropriate quantization procedure [1–4]. It is of great interest to investigate this connection

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as it could help to transfer results from classical theory to quantum theory. One of the particularly interesting problems, is a relation between integrability and in particular separability of classical and quantum systems. Some partial results on that subject can be found in literature [5–8]. In this paper we are going to investigate systematically a separability of quantum systems received from classical Stäckel systems, i.e. these systems for which all constants of motion are quadratic in momenta, by means of an appropriate quantizations. It should be noted that there is a variety of quantization procedures leading to different quantum systems [9]. In this paper we are going to focus on so-called minimal quantizations.

In our approach a minimal quantization depends on a metric tensor from a configuration space. With every classical Stäckel system is associated a natural metric tensor, which can be used to quantize such a system. In [10] it was shown that so called Benenti class of Stäckel systems after such minimal quantization leads to quantum separable systems (the respective system of stationary Schrödinger equations is separable [11,12]). In this paper we are going to consider the whole family of admissible minimal quantizations of Stäckel systems and investigate the problem of their quantum separability.

It is known that for each pair of classical Stäckel systems there exists a Stäckel transform relating them [13,14]. Using this fact we can relate any Stäckel system with a chosen flat system and introduce quantization by means of a natural flat metric induced by that system.

In Section 2 we refer basic notions about Stäckel systems and Stäckel transform. Section 3 contains a description of minimal quantization procedure. In Section 4 we investigate a family of flat minimal quantizations of Benenti class of Stäckel systems. In particular, we prove that for any Benenti system, there exists an n-parameter family of minimal flat quantizations, which preserves quantum separability. In Section 5 we investigate flat minimal quantizations of arbitrary classical Stäckel system. We receive the result that all admissible flat minimal quantizations of any non-Benenti class destroy a quantum separability. Section 6 presents a procedure of deformation of Stäckel systems so as to preserve the separability of deformed operators which however destroy their Hermicity. Finally, in Section 7, we illustrate the theory by few examples.

2. Stäckel systems in flat coordinates

Let us recall basic notions from the theory of separable Hamiltonian systems. Consider a Liouville-integrable system on a $2n$-dimensional phase space $(M, \mathscr{P})$, where $\mathscr{P}$ is a non-degenerated Poisson tensor. Then, there exist n functions H_i in involution with respect to a Poisson bracket:

$$\{H_i, H_j\}_{\mathscr{P}} := \mathscr{P}(\mathrm{d}H_i, \mathrm{d}H_j) = 0, \quad i, j = 1, 2, \ldots, n. \tag{2.1}$$

The functions H_i generate n Hamiltonian dynamic systems

$$u_{t_i} = \mathscr{P}\mathrm{d}H_i, \quad i = 1, 2, \ldots, n,\ u \in M. \tag{2.2}$$

One of the methods of solving the system of equations (2.2) is a Hamilton–Jacobi method. In this method one linearizes equations (2.2) by performing an appropriate canonical transformation of co-ordinates $(q, p) \mapsto (b, a)$, $a_i = H_i$. The generating function $W(q, a)$ of such canonical transformation is then calculated by solving the Hamilton–Jacobi equations

$$H_i\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = a_i, \quad i = 1, 2, \ldots, n. \tag{2.3}$$

A system of equations (2.3) can be solved by separation of variables, i.e. we have to find a canonical transformation $(q, p) \mapsto (\lambda, \mu)$ to a new coordinate system (λ, μ), called separation coordinates, in which (2.3) separates to a system of n decoupled ordinary differential equations, which in turn can be solved by quadratures. In other words, in separation coordinates (λ, μ) there exist the following relations

$$\varphi_i(\lambda_i, \mu_i; a_1, \ldots, a_n) = 0, \quad i = 1, 2, \ldots, n$$

$$a_i \in \mathbb{R}, \qquad \det\left[\frac{\partial \varphi_i}{\partial a_j}\right] \neq 0, \tag{2.4}$$

such that each of these relations involves only a single pair of canonical coordinates. The relations (2.4) are called separation relations [15,16]. In this paper we consider Liouville-integrable systems having separation relations in the following form

$$H_1\lambda_i^{\gamma_1}+H_2\lambda_i^{\gamma_2}+\cdots+H_n\lambda_i^{\gamma_n}=\frac{1}{2}f(\lambda_i)\mu_i^2+\sigma(\lambda_i),\quad i=1,2,\ldots,n, \tag{2.5}$$

where $\gamma_i\in\mathbb{Z}$ and are such that no two γ_i coincide, and f, σ are arbitrary smooth functions. Systems described by separation relations (2.5) are called classical Stäckel systems.

Consider a Stäckel system described by a class of irreducible separation relations given by n copies of the following separation curve (substitution $\lambda=\lambda_i$, $\mu=\mu_i$ for $i=1,2,\ldots,n$ yields n separation relations (2.5))

$$H_1\lambda^{\gamma_1}+H_2\lambda^{\gamma_2}+\cdots+H_n=\frac{1}{2}f(\lambda)\mu^2+\sigma(\lambda), \tag{2.6}$$

where $\gamma_1>\gamma_2>\cdots>\gamma_n=0$, $\gamma_i\in\mathbb{Z}_+$ and f, σ are rational functions. Irreducible means, that the set $\{\gamma_1,\ldots,\gamma_{n-1}\}$ of integers do not have a common divisor α. Otherwise, separation curve (2.6) can be reduced to the one with $\gamma_i\to\frac{\gamma_i}{\alpha}\in\mathbb{Z}_+$ by a transformation $\lambda\mapsto\lambda^{\frac{1}{\alpha}}$. The n copies of (2.6) constitute a system of n equations linear in the unknowns H_i with the solution of the form

$$H_r=\frac{1}{2}(A_r)^{ii}\mu_i^2+V_r(\lambda)=\frac{1}{2}(K_rG)^{ii}\mu_i^2+V_r(\lambda),\quad r=1,\ldots,n, \tag{2.7}$$

where K_r are Killing tensors of the metric tensor $G=A_1$ and $K_1=I$ (K_r and G are diagonal in separation coordinates (λ,μ)). Introducing a Stäckel matrix

$$S_\gamma=\begin{pmatrix}\lambda_1^{\gamma_1} & \lambda_1^{\gamma_2} & \cdots & 1\\ \vdots & \vdots & & \vdots\\ \lambda_n^{\gamma_1} & \lambda_n^{\gamma_2} & \cdots & 1\end{pmatrix} \tag{2.8}$$

separation relations following from (2.6) can be written in a compact form

$$S_\gamma\mathbf{H}=\mathbf{U}, \tag{2.9}$$

where $\mathbf{H}=(H_1,\ldots,H_n)^T$ and $\mathbf{U}=(\frac{1}{2}f(\lambda_1)\mu_1^2+\sigma(\lambda_1),\ldots,\frac{1}{2}f(\lambda_n)\mu_n^2+\sigma(\lambda_n))^T$ is a Stäckel vector. It also means that tensor A_r and potential V_r in (2.7) can be expressed as

$$A_r=\mathrm{diag}((S_\gamma^{-1})_r^1f(\lambda_1),\ldots,(S_\gamma^{-1})_r^nf(\lambda_n)),\qquad V_r=(S_\gamma^{-1})_r^i\sigma(\lambda_i)\quad r=1,\ldots,n, \tag{2.10}$$

and hence

$$H_r=\frac{1}{2}(S_\gamma^{-1})_r^if(\lambda_i)\mu_i^2+(S_\gamma^{-1})_r^i\sigma(\lambda_i). \tag{2.11}$$

The Stäckel matrix S_γ, or equivalently the set $\gamma=\{\gamma_1,\gamma_2,\ldots,1\}$, determines a given class of Stäckel systems [16] and we will call it a γ-class of classical Stäckel systems. For a fixed S_γ the metric tensor G is determined by $f(\lambda)$ and the separable potentials $V_r(\lambda)$ are determined by $\sigma(\lambda)$. In general metric G is non-flat.

There is one distinguished class of (2.6) when $\gamma_k=n-k$, i.e.

$$H_1\lambda^{n-1}+H_2\lambda^{n-2}+\cdots+H_n=\frac{1}{2}f(\lambda)\mu^2+\sigma(\lambda), \tag{2.12}$$

called Benenti class.

Notice, that all Stäckel systems (2.6) of two degrees of freedom ($n=2$) are of Benenti type, as the only separation curve (2.12) is irreducible in that case.

For Benenti class, in separation coordinates (λ,μ), the Stäckel matrix

$$S=\begin{pmatrix}\lambda_1^{n-1} & \cdots & 1\\ \vdots & \ddots & \vdots\\ \lambda_n^{n-1} & \cdots & 1\end{pmatrix} \tag{2.13}$$

is a Vandermonde matrix and metric tensors are

$$G^{ii} = \frac{f(\lambda_i)}{\Delta_i}, \quad \Delta_i = \prod_{k\neq i}(\lambda_i - \lambda_k),\ i = 1, \dots, n. \tag{2.14}$$

All metric tensors (2.14) have a common set of Killing tensors (also diagonal)

$$(K_r)^i_i = -\frac{\partial \rho_r}{\partial \lambda_i}, \quad r = 1, \dots, n, \tag{2.15}$$

where $\rho_r(\lambda)$ are signed symmetric polynomials (Viéte polynomials)

$$\rho_1 = -(\lambda_1 + \cdots + \lambda_n), \dots, \rho_n = (-1)^n \lambda_1\lambda_2\cdots\lambda_n. \tag{2.16}$$

The matrix

$$F = S^{-1}\Lambda S, \quad \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \tag{2.17}$$

is a recursion matrix [14] for basic potentials $\sigma(\lambda) = \lambda^k$

$$\mathbf{V}^{(k)} = F^k \mathbf{V}^{(0)}, \quad k \in \mathbb{Z}, \tag{2.18}$$

where $\mathbf{V}^{(k)} = (V_1^{(k)}, \dots, V_r^{(k)})^T$, $V^{(0)} = (0, \dots, 0, 1)^T$ are separable potentials determined respectively by $\sigma(\lambda) = \lambda^k$ and $\sigma(\lambda) = 1$ from separation curve (2.12). In explicit form

$$F = \begin{pmatrix} -\rho_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{n-1} & 0 & \cdots & 1 \\ -\rho_n & 0 & \cdots & 0 \end{pmatrix}. \tag{2.19}$$

Benenti class of Stäckel systems contains a sub-class of systems with flat metrics G (of mixed signature in general), when

$$f(\lambda) = \prod_{k=1}^{m}(\lambda - \beta_k) =: f_{\mathrm{flat}}(\lambda), \quad m = 0, 1, \dots, n. \tag{2.20}$$

In such case a phase space M is the cotangent bundle to the pseudo-Euclidean space $E^{r,s}$: $M = T^*E^{r,s}$.

The important fact about Stäckel systems (2.6) is the existence of a so called Stäckel transform [13,14] relating all of them. In [14] it was proved that from a set of Benenti systems with fixed metric tensor $\bar{G}$ (by fixing $\bar{f}(\lambda)$), one can construct the rest of Stäckel systems (2.6), both from Benenti class as well as from other classes. The transformation is known as a Stäckel transform:

$$\begin{gathered} \bar{H}_1\lambda^{n-1} + \bar{H}_2\lambda^{n-2} + \cdots + \bar{H}_n = \frac{1}{2}\bar{f}(\lambda)\mu^2 + \bar{\sigma}(\lambda) \\ \Big\downarrow \text{Stäckel transform} \\ H_1\lambda^{\gamma_1} + H_2\lambda^{\gamma_2} + \cdots + H_n = \frac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda). \end{gathered} \tag{2.21}$$

Explicitly it is given in a matrix form

$$\mathbf{H} = W_\gamma R(F)\bar{\mathbf{H}}, \tag{2.22}$$

where $\mathbf{H} = (H_1, \dots, H_n)^T$, $\bar{\mathbf{H}} = (\bar{H}_1, \dots, \bar{H}_n)^T$, $W_\gamma = S_\gamma^{-1}S$, where S_γ, S are respective Stäckel matrices (2.8), (2.13) and $R(F) = f(F)\bar{f}^{-1}(F)$. What is important, the inverse of the matrix W_γ is expressible by basic potentials $V_i^{(\gamma_j)}$ (2.18)

$$(S^{-1}S_\gamma)_{ij} = (W_\gamma^{-1})_{ij} = V_i^{(\gamma_j)}. \tag{2.23}$$

Now, let us choose $\bar{f}(\lambda) = \bar{f}_{\text{flat}}(\lambda)$ and write $\{\bar{H}_r\}$ in respective flat coordinates (x, y) (not necessary orthogonal). It means that all Stäckel Hamiltonians $\{H_r\}$ of (2.6) can be expressed by a flat coordinates as well, so can be considered as some quadratic in momenta functions on a phase space $M = \mathbb{R}^{2n}$.

Consider Stäckel Hamiltonians (2.22) written in a flat coordinates (x, y) of the metric tensor $\bar{G}$ (2.20)

$$H_r = \frac{1}{2}A_r^{ij}y_iy_j + V_r(x), \quad r = 1, \ldots, n. \tag{2.24}$$

There are two natural settings for Hamiltonians (2.24) as functions on a phase space $M = T^*\mathcal{Q}$ (a cotangent bundle to a configuration space $\mathcal{Q}$). We can consider $\mathcal{Q}$ as two different pseudo-Riemannian spaces. Either $\mathcal{Q} = (\mathbb{R}^n, \bar{g}) = E^{r,s}$ or $\mathcal{Q} = (\mathbb{R}^n, g)$, where $\bar{g} = \bar{G}^{-1}$, $g = G^{-1}$, and $G = A_1$. In the first case we simple have $\mathcal{Q} = E^{r,s}$, while in the second case the curvature tensor is nonzero and there are regions of $\mathbb{R}^n$ where g is degenerated so $\mathcal{Q}$ is not pseudo-Euclidean any more. Moreover, the second case is natural for classical separability theory, as then

$$H_r = \frac{1}{2}A_r^{ij}y_iy_j + V_r(x) = \frac{1}{2}(K_rG)^{ij}y_iy_j + V_r(x), \tag{2.25}$$

$K_1 = I$ and K_r are Killing tensors of the metric G, non-flat in general. Obviously, in the first case, Hamiltonians (2.24) can be written as

$$H_r = \frac{1}{2}A_r^{ij}y_iy_j + V_r(x) = \frac{1}{2}(T_r\bar{G})^{ij}y_iy_j + V_r(x). \tag{2.26}$$

Although tensors T_r are not Killing tensors for the flat metric $\bar{G}$, but the representation (2.26) will be useful for admissible quantizations of H_r.

3. Minimal quantizations of Stäckel systems

Let $(\mathcal{Q}, g)$ be a pseudo-Riemannian configuration space and

$$H = \frac{1}{2}A^{ij}p_ip_j + V(q) \tag{3.1}$$

be a function on $T^*\mathcal{Q}$, written in some local canonical chart (q, p) and associated with a symmetric contravariant two-tensor A on $\mathcal{Q}$. A minimal quantization procedure [17,9,11,12] associates with (3.1) a self-adjoint linear operator

$$\hat{H} = -\frac{1}{2}\hbar^2\,\nabla_iA^{ij}\nabla_j + V(q) \tag{3.2}$$

acting in a Hilbert space $L^2(\mathcal{Q}, \omega_g)$ of square integrable functions defined on the configuration space $\mathcal{Q}$ with respect to the metric volume form ω_g. By ∇_i we denote the covariant derivative with respect to the Levi-Civita connection.

Hence, for Stäckel Hamiltonians (2.24) we can apply either flat or non-flat minimal quantization related with representations (2.25) and (2.26), respectively. In [10] we analyzed the non-flat case. In the following paper we consider all admissible flat minimal quantizations and compare them with the non-flat one.

For a non-flat case (2.25) the related set of quantum operators is

$$\hat{H}_r = -\frac{1}{2}\hbar^2\,\nabla_iA^{ij}\nabla_j + V_r(x), \quad r = 1, \ldots, n \tag{3.3}$$

where ∇_i is the covariant derivative with respect to the connection generated by metric g and for the flat representation (2.26) respectively

$$\hat{\hat{H}}_r = -\frac{1}{2}\hbar^2\,\bar{\nabla}_iA^{ij}\bar{\nabla}_j + V_r(x), \quad r = 1, \ldots, n, \tag{3.4}$$

where $\bar{\nabla}_i$ is the covariant derivative with respect to the connection generated by a flat metric $\bar{g}$. In order to investigate a separability of (3.3) and (3.4), let us rewrite the operators in separation coordinates (λ, μ) [12]

$$\hat{H}_r = -\frac{1}{2}\hbar^2 G^{ii}\left(K_r^{(i)}\partial_i^2 + (\partial_i K_r^{(i)})\partial_i - K_r^{(i)}\Gamma_i\partial_i\right) + V_r(\lambda), \tag{3.5a}$$

$$\hat{\hat{H}}_r = -\frac{1}{2}\hbar^2 \bar{G}^{ii}\left(T_r^{(i)}\partial_i^2 + (\partial_i T_r^{(i)})\partial_i - T_r^{(i)}\bar{\Gamma}_i\partial_i\right) + V_r(\lambda), \tag{3.5b}$$

where Γ_i ($\bar{\Gamma}_i$) is the contracted Christoffel symbol defined by $\Gamma_i = g_{il}G^{jk}\Gamma^l_{jk}$ and in orthogonal coordinates

$$\Gamma_i = \frac{1}{2}\partial_i \ln|G| - \partial_i \ln G^{ii}, \tag{3.6}$$

$K_r^{(i)} \equiv (K_r)^i_i$, $T_r^{(i)} \equiv (T_r)^i_i$, and $\partial_i = \frac{\partial}{\partial\lambda_i}$. As all K_r are Killing tensors for the metric G so $\partial_i K_r^{(i)} = 0$ [12]. Thus, (3.5) can be written in the form

$$\hat{H}_r = -\frac{1}{2}\hbar^2 A_r^{ii}\left(\partial_i^2 - \Gamma_i\partial_i\right) + V_r(\lambda), \tag{3.7a}$$

$$\hat{\hat{H}}_r = -\frac{1}{2}\hbar^2 A_r^{ii}\left(\partial_i^2 + (\partial_i \ln T_r^{(i)})\partial_i - \bar{\Gamma}_i\partial_i\right) + V_r(\lambda). \tag{3.7b}$$

A necessary and sufficient condition for separability of operators (3.7a) is a Robertson condition [11]

$$\Gamma_i = \Gamma_i(\lambda_i) \Leftrightarrow \partial_j\Gamma_i = 0, \quad j \neq i,$$

while a necessary and sufficient condition for separability of operators (3.7b) takes the form

$$\partial_i \ln(T_r^{(i)}) - \bar{\Gamma}_i = \Xi_i(\lambda_i) \Leftrightarrow \partial_j \Xi_i = 0, \quad j \neq i.$$

Indeed, if operators (3.7) are of the form

$$\begin{aligned}\hat{B}_r &= -\frac{1}{2}\hbar^2 A_r^{ii}\left(\partial_i^2 + \Xi_i\partial_i\right) + V_r(\lambda),\\ &= -\frac{1}{2}\hbar^2 \left(S^{-1}\right)^i_r f(\lambda_i)\left(\partial_i^2 + \Xi_i\partial_i\right) + \left(S^{-1}\right)^i_r \sigma(\lambda_i), \quad r = 1, \dots, n,\end{aligned} \tag{3.8}$$

where $\hat{B}_r = \hat{H}_r(\hat{\hat{H}}_r)$ and $\Xi_i = \Xi_i(\lambda_i)$, then application of Stäckel matrix S to the system of eigenvalue problems for (3.8)

$$S\begin{pmatrix}\hat{B}_1\Psi\\ \vdots\\ \hat{B}_n\Psi\end{pmatrix} = S\begin{pmatrix}E_1\Psi\\ \vdots\\ E_n\Psi\end{pmatrix} \tag{3.9}$$

separates (3.9) onto n one-dimensional eigenvalue problems

$$(E_1\lambda_i^{\gamma_1} + E_2\lambda_i^{\gamma_2} + \dots + E_n)\psi_i(\lambda_i) = -\frac{1}{2}\hbar^2 f(\lambda_i)\left[\frac{d^2\psi_i(\lambda_i)}{d\lambda_i^2} + \Xi_i(\lambda_i)\frac{d\psi_i(\lambda_i)}{d\lambda_i}\right] + \sigma(\lambda_i)\psi_i(\lambda_i),$$

where $\Psi(\lambda_1, \dots, \lambda_n) = \prod_{i=1}^n \psi_i(\lambda_i)$. In the case when $\Xi_i(\lambda_i) = \Xi(\lambda_i)$, $i = 1, \dots, n$, we have n copies of one-dimensional eigenvalue problem

$$(E_1\lambda^{\gamma_1} + E_2\lambda^{\gamma_2} + \dots + E_n)\psi(\lambda) = -\frac{1}{2}\hbar^2 f(\lambda)\left[\frac{d^2\psi(\lambda)}{d\lambda^2} + \Xi(\lambda)\frac{d\psi(\lambda)}{d\lambda}\right] + \sigma(\lambda)\psi(\lambda),$$

where $\Psi(\lambda_1, \dots, \lambda_n) = \prod_{i=1}^n \psi(\lambda_i)$.

4. Minimal flat quantization of Benenti class

First, let us analyze the case of two quantizations inside the Benenti class, where in (2.22) $W_\gamma = I$. Assume that $\{\bar{H}_r\}$ is a Benenti system with a flat metric generated by $\bar{f}_{\text{flat}}(\lambda)$. Then, any other Benenti system $\{H_r\}$ is given by

$$\mathbf{H} = R(F)\bar{\mathbf{H}}, \qquad R(F) = f(F)\bar{f}_{\text{flat}}^{-1}(F) \tag{4.1}$$

and separation curves for $\{\bar{H}_r\}$ and $\{H_r\}$ are

$$\begin{gathered}\bar{H}_1\lambda^{n-1} + \bar{H}_2\lambda^{n-2} + \cdots + \bar{H}_n = \frac{1}{2}\bar{f}_{\text{flat}}(\lambda)\mu^2 + \bar{\sigma}(\lambda)\\ \Big\downarrow R(F)\\ H_1\lambda^{n-1} + H_2\lambda^{n-2} + \cdots + H_n = \frac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda),\end{gathered} \tag{4.2}$$

$\sigma(\lambda) = R(\lambda)\bar{\sigma}(\lambda) = f(\lambda)\bar{\sigma}(\lambda)/\bar{f}_{\text{flat}}(\lambda)$. The relation (4.2) follows from the following relations which hold in separation coordinates:

$$A_r = \sum_k R(F)_{rk}\bar{A}_k = R(\Lambda)\bar{A}_r, \tag{4.3}$$

$$\mathbf{V} = R(F)\bar{\mathbf{V}} = S^{-1}R(\Lambda)\bar{\boldsymbol{\sigma}}(\lambda), \quad \bar{\boldsymbol{\sigma}}(\lambda) = (\bar{\sigma}(\lambda_1), \ldots, \bar{\sigma}(\lambda_n))^T. \tag{4.4}$$

Indeed, for rational $f(\lambda)$ (4.3) follows from the fact that it is fulfilled for $R(F) = F - \beta I$ and $R(F) = (F - \beta I)^{-1}$. To prove (4.4) observe that $\bar{\mathbf{V}} = S^{-1}\bar{\boldsymbol{\sigma}}(\lambda)$ and $R(F) = S^{-1}R(\Lambda)S$. Hence $\mathbf{V} = R(F)S^{-1}\bar{\boldsymbol{\sigma}}(\lambda) = S^{-1}R(\Lambda)\bar{\boldsymbol{\sigma}}(\lambda)$.

Now, let us go back to operators (3.7). As for metric (2.14) from Benenti class $\Gamma_i = -\frac{1}{2}\frac{\partial_i f(\lambda_i)}{f(\lambda_i)}$ and as follows from (4.3) $T_r^{(i)} = R(\lambda_i)\bar{K}_r^{(i)}$ then, using the relation (2.10), we have

$$\hat{H}_r = -\frac{1}{2}\hbar^2(S^{-1})_r^i\left(f(\lambda_i)\partial_i^2 + \frac{1}{2}\frac{\mathrm{d}f(\lambda_i)}{\mathrm{d}\lambda_i}\partial_i\right) + (S^{-1})_r^i\sigma(\lambda_i), \tag{4.5a}$$

$$\hat{\bar{H}}_r = -\frac{1}{2}\hbar^2(S^{-1})_r^i\left[f(\lambda_i)\partial_i^2 + \left(\frac{\mathrm{d}f(\lambda_i)}{\mathrm{d}\lambda_i} - \frac{1}{2}\frac{f(\lambda_i)}{\bar{f}_{\text{flat}}(\lambda_i)}\frac{\mathrm{d}\bar{f}_{\text{flat}}(\lambda_i)}{\mathrm{d}\lambda_i}\right)\partial_i\right] + (S^{-1})_r^i\sigma(\lambda_i), \tag{4.5b}$$

so Eqs. (4.5) take the form (3.8) with $\Xi(\lambda_i) = \frac{1}{2}\frac{\mathrm{d}f(\lambda_i)}{\mathrm{d}\lambda_i}$ in the case of Eq. (4.5a) and $\Xi(\lambda_i) = \frac{\mathrm{d}f(\lambda_i)}{\mathrm{d}\lambda_i} - \frac{1}{2}\frac{f(\lambda_i)}{\bar{f}_{\text{flat}}(\lambda_i)}\frac{\mathrm{d}\bar{f}_{\text{flat}}(\lambda_i)}{\mathrm{d}\lambda_i}$ in the case of Eq. (4.5b). As a consequence all operators $\{\hat{H}_r\}$ as well as $\{\hat{\bar{H}}_r\}$ have common eigenfunctions:

$$\hat{H}_r\Psi = E_r\Psi, \qquad \hat{\bar{H}}_r\bar{\Psi} = \bar{E}_r\bar{\Psi}, \quad r = 1, \ldots, n, \tag{4.6}$$

where $\Psi(\lambda_1, \ldots, \lambda_n) = \prod_{k=1}^n \psi(\lambda_k)$, $\bar{\Psi}(\lambda_1, \ldots, \lambda_n) = \prod_{k=1}^n \bar{\psi}(\lambda_k)$, and $\psi(\lambda_k)$ and $\bar{\psi}(\lambda_k)$ are n copies of one-dimensional eigenvalue problems

$$\begin{aligned}(E_1\lambda^{n-1} + E_2\lambda^{n-2} + \cdots + E_n)\psi(\lambda) \;=\; &-\frac{1}{2}\hbar^2\left(f(\lambda)\frac{\mathrm{d}^2\psi(\lambda)}{\mathrm{d}\lambda^2} + \frac{1}{2}\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}\psi(\lambda)}{\mathrm{d}\lambda}\right)\\ &+\sigma(\lambda)\psi(\lambda),\end{aligned} \tag{4.7a}$$

$$\begin{aligned}&(\bar{E}_1\lambda^{n-1} + \bar{E}_2\lambda^{n-2} + \cdots + \bar{E}_n)\bar{\psi}(\lambda)\\ &\quad = -\frac{1}{2}\hbar^2\left[f(\lambda)\frac{\mathrm{d}^2\bar{\psi}(\lambda)}{\mathrm{d}\lambda^2} + \left(\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} - \frac{1}{2}\frac{f(\lambda)}{\bar{f}_{\text{flat}}(\lambda)}\frac{\mathrm{d}\bar{f}_{\text{flat}}(\lambda)}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}\bar{\psi}(\lambda)}{\mathrm{d}\lambda}\right] + \sigma(\lambda)\bar{\psi}(\lambda).\end{aligned} \tag{4.7b}$$

Eqs. (4.7a) and (4.7b) represent the non-flat and flat minimal quantizations of separation curve (4.2). Moreover,

$$[\hat{H}_r, \hat{H}_s] = 0, \qquad [\hat{\bar{H}}_r, \hat{\bar{H}}_s] = 0. \tag{4.8}$$

The first set of commutation relations was proved in [10] and follows from the fulfillment of the pre-Robertson condition [12]

$$\partial_i^2 \Gamma_j - \Gamma_i \partial_i \Gamma_j = 0, \quad i \neq j$$

for $\Gamma_i = -\frac{1}{2}\partial_i \ln f(\lambda_i)$. The second set of commutation relations follows from the analog of the pre-Robertson condition

$$\partial_i^2 \Xi_i - \Xi_j \partial_i \Xi_j = 0, \quad i \neq j, \tag{4.9}$$

where

$$\Xi_i = \bar{\Gamma}_i - \partial_i \ln R(\lambda_i) = -\frac{1}{2}\partial_i \ln f_{\text{flat}}(\lambda_i) - \partial_i \ln R(\lambda_i).$$

The condition (4.9) can be obtain repeating the procedure from [12] (Section 5) under substitution $K_r^{(i)} \to R(\Lambda)K_r^{(i)}$.

Summarizing that part, we proved that for any classical Benenti system, there exists an n-parameter family (2.20) of minimal flat quantizations, which preserves quantum separability.

5. Minimal flat quantization for arbitrary γ-class

Let us consider the case $R = 1$. Then,

$$\mathbf{H} = W_\gamma \bar{\mathbf{H}}, \tag{5.1}$$

where separation curves for $\bar{H}_r$ and H_r are

$$\begin{array}{c} \bar{H}_1\lambda^{n-1} + \bar{H}_2\lambda^{n-2} + \cdots + \bar{H}_n = \frac{1}{2}\bar{f}_{\text{flat}}(\lambda)\mu^2 + \bar{\sigma}(\lambda) \\ \Big\downarrow W_\gamma \\ H_1\lambda^{\gamma_1} + H_2\lambda^{\gamma_2} + \cdots + H_n = \frac{1}{2}\bar{f}_{\text{flat}}(\lambda)\mu^2 + \bar{\sigma}(\lambda), \end{array} \tag{5.2}$$

and Hamiltonian operators for non-flat and flat minimal quantizations of H_i are of the form (3.7). Γ_i in (3.7a) is a reduced Christoffel symbol for a metric tensor G, and it was proved in [10] that for arbitrary γ-class $\partial_j \Gamma_i \neq 0, j \neq i$ and we loose a separability. In operator $\hat{\bar{H}}_r$ from (3.7b) $\bar{\Gamma}_i = -\frac{1}{2}\frac{\partial_i \bar{f}_{\text{flat}}(\lambda_i)}{\bar{f}_{\text{flat}}(\lambda_i)}$, hence does not depend on $\lambda_j \neq \lambda_i$, so we have to analyze only the term $\partial_i T_r^{(i)}/T_r^{(i)}$. A very useful form of $T_r^{(i)}$ was derived in [18]. Consider polynomial $P = \sum_{r=1}^n H_r\lambda^{\gamma_r}$ from separation curve (5.2). Its order is γ_1 which we denote by $\gamma_1 = n+k-1$. Notice that for $k = 0$ we are in the Benenti class. There is k missing monomials λ^{n+k-n_i} in polynomial P, enumerated by $(n_1, \ldots, n_k)$. For example, if $P = H_1\lambda^4 + H_2\lambda + H_3$, then $n = 3$, $k = 2$, $n_1 = 2$, $n_2 = 3$. In [18] was proved that

$$T_r^{(i)} = \frac{1}{\varphi}\chi_r^{(i)}, \tag{5.3}$$

where $\chi_r^{(i)}$ is λ_i independent and

$$\varphi = \det\begin{pmatrix} \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \vdots & \ddots & \vdots \\ \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{pmatrix} \tag{5.4}$$

where $\rho_0 = 1$, $\rho_m = 0$ for $m > n$ and $m < 0$, and remaining ρ_m are given by (2.16). Hence, (3.7b) takes the form

$$\hat{\bar{H}}_r = -\frac{1}{2}\hbar^2 A_r^{ii}\left(\partial_i^2 - \left(\frac{\partial_i\varphi}{\varphi} - \frac{1}{2}\frac{\partial_i\bar{f}_{\mathrm{flat}}}{\bar{f}_{\mathrm{flat}}}\right)\partial_i\right) + V_r(\lambda). \tag{5.5}$$

It can be proved that for any φ, $\partial_j\left(\frac{\partial_i\varphi}{\varphi}\right) \neq 0$ for $j \neq i$. As a result, all admissible flat minimal quantizations of a non-Benenti γ class destroy a quantum separability.

6. Separable deformations of Stäckel Hamiltonians

In order to make all Hamiltonians $\hat{\bar{H}}_r$ separable, we need to get rid of the terms

$$\frac{1}{2}\hbar^2\left(A_r^{ii}\frac{\partial_i\varphi}{\varphi}\right)\partial_i \tag{6.1}$$

from (5.5). Terms (6.1) are generated by appropriate linear in momenta terms in Hamiltonians H_r. Define a vector field u_r with components

$$u_r^i = A_r^{ii}\frac{\partial_i\varphi}{\varphi} \tag{6.2}$$

in separation coordinates. Then, consider a deformed Hamiltonians in flat coordinates

$$H_r(\hbar) = \frac{1}{2}A_r^{ij}y_iy_j - \frac{1}{2}i\hbar u_r^i(x)y_i + V_r(x) + \frac{1}{4}\hbar^2 w_r(x), \tag{6.3}$$

where $w_r = \sum_i \frac{\partial u_r^i}{\partial x_i}$. Appropriate quantum operator in flat minimal quantization takes a form

$$\hat{\bar{H}}_r = -\frac{1}{2}\hbar^2\bar{\nabla}_i A_r^{ij}\bar{\nabla}_j - \frac{1}{4}\hbar^2(\bar{\nabla}_i u_r^i + u_r^i\bar{\nabla}_i) + \frac{1}{4}\hbar^2 w_r(x) + V_r(x) \tag{6.4}$$

and in separation coordinates

$$\hat{\bar{H}}_r = -\frac{1}{2}\hbar^2 A^{ii}\left[\partial_i^2 + \frac{1}{2}(\partial_i \ln\bar{f}_{\mathrm{flat}}(\lambda_i))\partial_i\right] + V_r(\lambda). \tag{6.5}$$

Hence all $\hat{\bar{H}}_r$ separate to a single one-dimensional eigenvalue problem:

$$(E_1\lambda^{\gamma_1} + E_2\lambda^{\gamma_2} + \cdots + E_n)\bar{\psi}(\lambda) = -\frac{1}{2}\hbar^2\left(\bar{f}_{\mathrm{flat}}(\lambda)\frac{d^2\bar{\psi}(\lambda)}{d\lambda^2} + \frac{1}{2}\frac{d\bar{f}_{\mathrm{flat}}(\lambda)}{d\lambda}\frac{d\bar{\psi}(\lambda)}{d\lambda}\right) + \bar{\sigma}(\lambda)\bar{\psi}(\lambda). \tag{6.6}$$

Nevertheless $\hat{\bar{H}}_r$ are not Hermitian anymore, since the extra terms $-\frac{1}{4}\hbar^2(\bar{\nabla}_i u_r^i + u_r^i\bar{\nabla}_i)$ are anti-Hermitian operators in a Hilbert space $L^2(\mathcal{Q}, \omega_g)$.

7. Examples

As first example let us consider a pseudo-Euclidean space $E^{2,1}$ with signature $(++-)$ and flat non-orthogonal coordinates (x_1, x_1, x_3) such that

$$\bar{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7.1}$$

Then, consider the following Stäckel geodesic system on $T^*E^{2,1}$

$$\bar{h}_1 = \bar{G}^{ij}y_iy_j = y_1y_3 + \frac{1}{2}y_2^2,$$
$$\bar{h}_2 = (\bar{K}_2\bar{G})^{ij}y_iy_j = \frac{1}{8}x_1^2y_1^2 - \frac{1}{4}x_1x_3y_2^2 + \frac{1}{8}x_3^2y_3^2 + \left(\frac{1}{4}x_1x_2 + 1\right)y_1y_2 - \frac{1}{4}\left(x_1x_3 + x_2^2\right)y_1y_3 - \frac{1}{4}x_2x_3y_2y_3,$$
$$\bar{h}_3 = (\bar{K}_3\bar{G})^{ij}y_iy_j = \left(\frac{1}{4}x_1x_2 + \frac{1}{2}\right)y_1^2 - \frac{1}{4}x_1x_3y_1y_2 - \frac{1}{4}x_2x_3y_1y_3 + \frac{1}{4}x_3^2y_2y_3.$$

One can check that $\{\bar{h}_i, \bar{h}_j\} = 0$. The transformation to separation coordinates (λ, μ) is generated by [19]

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{2}x_1x_3 + \frac{1}{4}x_2^2,$$
$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = -\frac{1}{2}x_2x_3, \tag{7.2}$$
$$\lambda_1\lambda_2\lambda_3 = \frac{1}{4}x_3^2$$

and the related separation curve is

$$\bar{h}_1\lambda^2 + \bar{h}_2\lambda + \bar{h}_3 = \frac{1}{2}\lambda^3\mu^2$$

operator F (2.19) in x-coordinates is

$$F = \begin{pmatrix} \frac{1}{2}x_1x_3 + \frac{1}{4}x_2^2 & 1 & 0 \\ \frac{1}{2}x_2x_3 & 0 & 1 \\ \frac{1}{4}x_3^2 & 0 & 0 \end{pmatrix},$$

so separable potentials $\bar{V}_r^{(k)}$ are given by (2.18). For example, the first nontrivial potential is

$$\bar{\mathbf{V}}^{(3)} = F^3\bar{\mathbf{V}}^{(0)} = \begin{pmatrix} \frac{1}{2}x_1x_3 + \frac{1}{4}x_2^2 \\ \frac{1}{2}x_2x_3 \\ \frac{1}{4}x_3^2 \end{pmatrix}$$

and separation curve for Hamiltonians $\bar{H}_i = \bar{h}_i + \bar{V}_i^{(k)}$, $i = 1, 2, 3$, takes the form

$$\bar{H}_1\lambda^2 + \bar{H}_2\lambda + \bar{H}_3 = \frac{1}{2}\lambda^3\mu^2 + \lambda^k.$$

Now, let us consider the following Stäckel transform

$$\bar{H}_1\lambda^2 + \bar{H}_2\lambda + \bar{H}_3 = \frac{1}{2}\lambda^3\mu^2 + \lambda^{r-s+3}$$
$$\Big\downarrow R(F) = F^{s-3}$$
$$H_1\lambda^2 + H_2\lambda + H_3 = \frac{1}{2}\lambda^s\mu^2 + \lambda^r \tag{7.3}$$

so, $\mathbf{H} = F^{s-3}\bar{\mathbf{H}}$ and in particular, for $s = 4$ and $r = 4$, we have for $H_i = h_i + V_i^{(4)}$

$$h_1 = \frac{1}{8}x_1^2y_1^2 + \frac{1}{8}x_2^2y_2^2 + \frac{1}{8}x_3^2y_3^2 + \left(\frac{1}{4}x_1x_2 + 1\right)y_1y_2 + \frac{1}{4}x_1x_3y_1y_3 + \frac{1}{4}x_2x_3y_2y_3,$$
$$h_2 = \left(\frac{1}{4}x_1x_2 + \frac{1}{2}\right)y_1^2 + \frac{1}{4}x_2x_3y_2^2 - \frac{1}{4}x_1x_3y_1y_2 + \frac{1}{4}x_2x_3y_1y_3 + \frac{1}{4}x_3^2y_2y_3,$$
$$h_3 = \frac{1}{4}x_3^2y_1y_3 + \frac{1}{8}x_3^2y_2^2$$

and

$$V_1^{(4)} = \frac{1}{4}x_1^2x_3^2\frac{1}{4}x_1x_2^2x_3 + \frac{1}{16}x_2^4 + \frac{1}{2}x_2x_3,$$
$$V_2^{(4)} = \frac{1}{4}x_1x_2x_3^2 + \frac{1}{8}x_2^3x_3 + \frac{1}{4}x_2x_3,$$
$$V_3^{(4)} = \frac{1}{16}x_3^2\left(2x_1x_3 + x_2^2\right). \tag{7.4}$$

Of course, again canonical transformation generated by (7.2) is a transformation to separation coordinates, with separation curve (7.3) and $s = r = 4$.

As was considered in previous sections, we have two natural minimal quantizations. One, the flat minimal quantization expressed by Levi–Civita connection of metric $\bar{g}$ (7.1) and second, expressed by Levi–Civita connection of metric tensor $g = G^{-1}$, where

$$G = \begin{pmatrix} \frac{1}{4}x_1^2 & \frac{1}{4}x_1x_2 + 1 & \frac{1}{4}x_1x_3 \\ \frac{1}{4}x_1x_2 + 1 & \frac{1}{4}x_2^2 & \frac{1}{4}x_2x_3 \\ \frac{1}{4}x_1x_3 & \frac{1}{4}x_1x_3 & \frac{1}{4}x_3^2 \end{pmatrix}$$

is generated by $h_1 = \frac{1}{2}G^{ij}y_iy_j$. Notice that in the second case the configuration space $(\mathbb{R}^3, g)$ is not pseudo-Euclidean any more as g has constant Ricci scalar $R_S = \frac{3}{2}$ and is not non-degenerated on the whole $\mathbb{R}^3$. Thus, the second admissible minimal quantization is non-flat.

In flat quantization, related to metric tensor $\bar{g}$ Christoffel symbols vanish and quantum operators $\hat{\bar{H}}_r$ related to classical Hamiltonian functions H_r are

$$\begin{aligned}\hat{\bar{H}}_1 \;=&\; -\hbar^2\left[\frac{1}{8}\left(x_1^2\partial_1^2 + x_2^2\partial_2^2 + x_3^2\partial_3^2\right) + \left(\frac{1}{4}x_1x_2 + 1\right)\partial_1\partial_2 + \frac{1}{4}x_1x_3\partial_1\partial_3 + \frac{1}{4}x_2x_3\partial_2\partial_3\right.\\ &+\left.\frac{1}{2}\left(x_1\partial_1 + x_2\partial_2 + x_3\partial_3\right)\right] + V_1^{(r)},\\ \hat{\bar{H}}_2 \;=&\; -\hbar^2\left[\left(\frac{1}{4}x_1x_2 + \frac{1}{2}\right)\partial_1^2 + \frac{1}{4}x_2x_3\partial_2^2 - \frac{1}{4}x_1x_3\partial_1\partial_2 + \frac{1}{4}x_2x_3\partial_1\partial_3\right.\\ &+\left.\frac{1}{4}x_3^2\partial_2\partial_3 + \frac{3}{8}x_2\partial_1 + \frac{3}{8}x_3\partial_2\right] + V_2^{(r)},\\ \hat{\bar{H}}_3 =&\, -\hbar^2\left[\frac{1}{4}x_3^2\partial_1\partial_3 + \frac{1}{8}x_3^2\partial_2^2 + \frac{1}{4}x_3\partial_1\right] + V_3^{(r)}.\end{aligned} \tag{7.5}$$

Obviously these operators are Hermitian in $L^2(\mathcal{Q}, \omega_g)$. Substituting $r = 4$ (7.4) one can check directly the commutativity of operators $\hat{\bar{H}}_r$ (7.5).

In (λ, μ) coordinates eigenvalue problems (4.6) reduce to three copies of one-dimensional eigenvalue problem

$$(\bar{E}_1\lambda^2 + \bar{E}_2\lambda + \bar{E}_3)\bar{\psi}(\lambda) = -\frac{1}{2}\hbar^2\left[\lambda^s\frac{\mathrm{d}^2\bar{\psi}}{\mathrm{d}\lambda^2} + \left(s - \frac{3}{2}\right)\lambda^{s-1}\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}\lambda}\right] + \lambda^r\bar{\psi}$$

for operators $\hat{\bar{H}}_r$ of minimal flat quantization and

$$(E_1\lambda^2 + E_2\lambda + E_3)\psi(\lambda) = -\frac{1}{2}\hbar^2\left[\lambda^s \frac{d^2\psi}{d\lambda^2} + \frac{1}{2}\lambda^{s-1}\frac{d\psi}{d\lambda}\right] + \lambda^r\psi,$$

for operators $\hat{H}_r$ of minimal non-flat quantization with $f(\lambda) = \lambda^s$.

As our second example let us consider again a pseudo-Euclidean space $E^{2,1}$ with signature $(++-)$ and flat, non-orthogonal coordinates (x_1, x_1, x_3) such that

$$\bar{g} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.6}$$

Then, consider the following Stäckel geodesic system on $T^*E^{2,1}$

$$\begin{aligned}
\bar{h}_1 &= \bar{G}^{ij}y_iy_j = y_1y_2 + \frac{1}{2}y_3^2, \\
\bar{h}_2 &= (\bar{K}_2\bar{G})^{ij}y_iy_j = \frac{1}{2}y_1^2 - \frac{1}{2}x_2y_2^2 + \frac{1}{2}x_1y_3^2 + \frac{1}{2}x_1y_1y_2 - \frac{1}{2}x_3y_2y_3, \\
\bar{h}_3 &= (\bar{K}_3\bar{G})^{ij}y_iy_j = \frac{1}{8}x_3^2y_2^2 + \left(\frac{1}{8}x_1^2 + \frac{1}{2}x_2\right)y_3^2 - \frac{1}{2}x_3y_1y_3 - \frac{1}{4}x_1x_3y_2y_3.
\end{aligned}$$

One can check that $\{\bar{h}_i, \bar{h}_j\} = 0$. The transformation to separation coordinates (λ, μ) is generated by [19]

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 = -x_1, \\
&\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = x_2 + \frac{1}{4}x_1^2, \\
&\lambda_1\lambda_2\lambda_3 = \frac{1}{4}x_3^2.
\end{aligned} \tag{7.7}$$

The related separation curve is

$$\bar{h}_1\lambda^2 + \bar{h}_2\lambda + \bar{h}_3 = \frac{1}{2}\lambda\mu^2,$$

operator F (2.19) in x-coordinates takes the form

$$F = \begin{pmatrix} -x_1 & 1 & 0 \\ -x_2 - \frac{1}{4}x_1^2 & 0 & 1 \\ \frac{1}{4}x_3^2 & 0 & 0 \end{pmatrix}$$

so, separable potentials $\bar{V}_r^{(k)}$ are given by (2.18). For example, the $\bar{V}^{(4)}$ potential and separation curve for Hamiltonians $\bar{H}_i = \bar{h}_i + \bar{V}_i^{(4)}$ are

$$\bar{\mathbf{V}}^{(4)} = F^4\bar{\mathbf{V}}^{(0)} = \begin{pmatrix} \frac{3}{4}x_1^2 - x_2 \\ \frac{1}{4}x_1^3 + x_1x_2 + \frac{1}{4}x_3^2 \\ -\frac{1}{4}x_1x_3^2 \end{pmatrix}$$

$$\bar{H}_1\lambda^2 + \bar{H}_2\lambda + \bar{H}_3 = \frac{1}{2}\lambda\mu^2 + \lambda^4.$$

First, let us consider the following Stäckel transform

$$\bar{H}_1\lambda^2+\bar{H}_2\lambda+\bar{H}_3=\frac{1}{2}\lambda\mu^2+\lambda^4$$
$$\Big\downarrow W_\gamma$$
$$H_1\lambda^3+H_2\lambda+H_3=\frac{1}{2}\lambda\mu^2+\lambda^4, \tag{7.8}$$

where $\gamma=(3,1,0)$ and from (2.23)

$$W_\gamma=\begin{pmatrix} -\frac{1}{x_1} & 0 & 0\\ -\frac{1}{4}\frac{x_1^2+4x_2}{x_1} & 1 & 0\\ \frac{1}{4}\frac{x_3^2}{x_1} & 0 & 1\end{pmatrix}.$$

Then, according to (5.1)

$$H_1=-\frac{1}{x_1}y_1y_2-\frac{1}{2}\frac{1}{x_1}y_3^2-\frac{3}{4}x_1+\frac{x_2}{x_1},$$
$$H_2=\frac{1}{2}y_1^2-\frac{1}{2}x_2y_2^2+\frac{1}{8}\left(3x_1-4\frac{x_2}{x_1}\right)y_3^2+\frac{1}{4}\left(x_1-4\frac{x_2}{x_1}\right)y_1y_2-\frac{1}{2}x_3y_2y_3$$
$$+\frac{1}{16}x_1^3+\frac{1}{2}x_1x_2+\frac{1}{4}x_3^2+\frac{x_2^2}{x_1},$$
$$H_3=\frac{1}{8}x_3^2y_2^2+\frac{1}{8}\left(x_1^2+4x_2+\frac{x_3^2}{x_1}\right)y_3^2+\frac{1}{4}\frac{x_3^2}{x_1}y_1y_2-\frac{1}{2}x_3y_1y_3$$
$$-\frac{1}{4}x_1x_3y_2y_3-\frac{1}{16}x_1x_3^2-\frac{1}{4}\frac{x_2x_3^2}{x_1},$$

where

$$A_1=\begin{pmatrix}0 & -\frac{1}{x_1} & 0\\ -\frac{1}{x_1} & 0 & 0\\ 0 & 0 & -\frac{1}{x_1}\end{pmatrix},\qquad A_2=\begin{pmatrix}1 & \frac{1}{4}x_1-\frac{x_2}{x_1} & 0\\ \frac{1}{4}x_1-\frac{x_2}{x_1} & -x_2 & -\frac{1}{2}x_3\\ 0 & -\frac{1}{2}x_3 & \frac{3}{4}x_1-\frac{x_2}{x_1}\end{pmatrix},$$

$$A_3=\begin{pmatrix}0 & \frac{1}{4}\frac{x_3^2}{x_1} & -\frac{1}{2}x_3\\ \frac{1}{4}\frac{x_3^2}{x_1} & \frac{1}{4}x_3^2 & -\frac{1}{4}x_1x_3\\ -\frac{1}{2}x_3 & -\frac{1}{4}x_1x_3 & \frac{1}{4}x_1^2+x_2+\frac{1}{4}\frac{x_3^2}{x_1}\end{pmatrix}.$$

Of course, again canonical transformation generated by (7.7) is a transformation to separation coordinates, with separation curve (7.8).

We have two natural minimal quantizations. One, the flat minimal quantization expressed by Levi–Civita connection of metric $\bar{g}$ (7.6) and second, expressed by Levi–Civita connection of metric tensor $g=G^{-1}$, where $G=A_1$.

In (λ,μ) coordinates Hamiltonian operators for non-flat and flat minimal quantizations are given respective by (3.7a) and (5.5). As

$$\Gamma_i=-\frac{1}{2}\left(\frac{1}{\lambda_i}+\frac{1}{\lambda_1+\lambda_2+\lambda_3}\right),\qquad \frac{\partial_i\varphi}{\varphi}=\frac{1}{\lambda_1+\lambda_2+\lambda_3},$$

hence both quantizations are non-separable.

The deformation (6.3) of classical Hamiltonians, with respective vector fields

$$u_1 = \left(0, -\frac{1}{x_1^2}, 0\right), \qquad u_2 = \left(\frac{1}{x_1}, \frac{1}{4} - \frac{x_2}{x_1^2}, 0\right), \qquad u_3 = \left(0, \frac{1}{4}\frac{x_3^2}{x_1^2}, -\frac{x_3}{x_1}\right),$$

leads to commuting (non-Hermitian) operators (6.4) and the following one-dimensional eigenvalue problem

$$(E_1\lambda^3 + E_2\lambda + E_3)\bar{\psi}(\lambda) = -\frac{1}{2}\hbar^2\left(\lambda\frac{d^2\bar{\psi}}{d\lambda^2} + \frac{1}{2}\frac{d\bar{\psi}}{d\lambda}\right) + \lambda^4\bar{\psi}(\lambda).$$

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