Applied Mathematical Modelling 48 (2017) 806–820

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Dynamic economic lot size model with perishable inventory and capacity constraints

F. Zeynep Sargut*, Gül Işık

Department of Industrial Engineering, Izmir University of Economics, Sakarya cad. No:156, Balçova, Izmir, 35330, Turkey

ARTICLE INFO

Article history:
Received 20 February 2016
Revised 1 February 2017
Accepted 14 February 2017
Available online 21 February 2017

Keywords:
Perishable items
Lot sizing
Capacitated production
Complexity analysis

ABSTRACT

In this study, we consider a dynamic economic lot sizing problem for a single perishable item under production capacities. We aim to identify the production, inventory and backlogging decisions over the planning horizon, where (i) the parameters of the problem are deterministic but changing over time, and (ii) producer has a constant production capacity that limits the production amount at each period and is allowed to backorder the unmet demand later on. All cost functions are assumed to be concave. A similar problem without production capacities was studied in the literature and a polynomial time algorithm was suggested (Hsu, 2003 [1]). We assume age-dependent holding cost functions and the deterioration rates, which are more realistic for perishable items. Backordering cost functions are period-pair dependent. We prove the NP-hardness of the problem even with zero inventory holding and backlogging costs under our assumptions. We show the structural properties of the optimal solution and suggest a heuristic that finds a good production and distribution plan when the production periods are given. We discuss the performance of the heuristic. We also give a Dynamic Programing-based heuristic for the solution of the overall problem.

1. Introduction

Goyal and Giri [2] summarize the ways to model deterioration in the literature. A product can have a fixed lifetime, random lifetime, and proportional inventory decrease in terms of its utility or physical quantity (deterioration with age). We present an inventory model for a producer of a perishable item deteriorating with age over time, such as fresh produce, dairy goods, and baked goods. To clarify the objective of our paper, let us consider a large sized baking company that ships to bakery stores in different locations. Stores have their demand forecasts for a specific product and the baking company plans its production based on aggregate units demanded from all stores. Each store represents a different customer class and has a different priority. The amount to be shipped at each period is distributed among the stores based on their priorities. Stores can wait for unmet demand for some time. Moreover, they have different seasonality in demand of this product because of their locations. At different periods, a different store's demand can be dominant. Therefore, the backlogging cost depends on its period. Moreover, it depends on how long unsatisfied stores are going to wait. In another setting, we can consider a fresh produce producer selling his products to companies using them as raw materials for producing consumer products with higher added value.

* Corresponding author.
E-mail address: zeynepsargut@gmail.com (F.Z. Sargut).

http://dx.doi.org/10.1016/j.apm.2017.02.024

Bakker et al. [3] give an up-to-date advances made in the field of inventory control of perishable items since 2001. Taleizadeh and Nematollahi [4] consider an Economic Order Quantity (EOQ) model for a perishable item over a finite planning horizon with backlogging and delayed payment. Sanni and Chukwu [5] develop an EOQ model with deterioration having three-parameter Weibull distribution, ramp-type demand, and backlogging. In [6], deteriorating inventory is incorporated into material requirement planning systems with the assumption of a fixed deterioration rate for each period. In a recent book on perishable inventories [7], it is stated that the vast majority of the literature on perishable inventories are related to stochastic demand.

We consider a finite horizon problem in a deterministic environment, where the parameters of the problem are deterministic but changing over time, namely Economic Lot Sizing problem (ELS). ELS has been an active area of research since its introduction by [8]. The objective is to minimize the total cost of production and inventory during a finite planning horizon. Holding cost in a period depends on the current period and the quantity on hand. Wagner and Whitin [8] show that in the case of fixed-plus-linear production and linear holding cost functions and unlimited production, there is an optimal solution with a special structure: production may occur only in periods where the initial inventory is zero, called the Zero Inventory Property (ZIP). Because of this property, Dynamic Programing (DP) formulation is possible. Later on, the versions of ELS with backlogging [9], stationary production capacities [10] and more general cost functions appear in the literature. Zangwill [11] shows that ZIP holds when all cost functions are concave.

Chan et al. [12] consider ELS with modified all-unit discount ordering and linear holding cost functions. They prove the NP-hardness of the problem and suggest an approximation algorithm that assumes ZIP, which may not hold in the optimal solution.

Florian et al. [13] and Bitran and Yanasse [10] show that with non-stationary production capacities, ELS is NP-hard even when the production cost functions have fixed-charge structure and holding costs are linear. Florian and Klein [14] give an $O(n^4)$ time Dynamic Programing algorithm to find an optimal solution (n is the number of periods) with concave costs and stationary production capacities. This algorithm is based on the special structure of the independent subparts of the optimal solution. We can classify the cost functions based on which items are subject to that cost function as (i) Period dependent, (ii) Period-pair dependent (PPD), (iii) Age-dependent (AD) cost functions.

Hsu and Lowe [15] consider ELS with backlogging, where the backlogging and holding cost functions are period-pair dependent. Period-pair dependent cost means at period t (i) cost is only incurred for items that satisfy the demand of period t, and (ii) this cost is different based on its production period. For the holding cost, it is the total cost of carrying items from its production period to the demand period. They show that under some assumptions on the cost functions, the problem is solvable in polynomial time. However, their model does not consider deteriorating inventory. In age-dependent cost case, the cost function depends on both the current period and the age of the item during the period. It is incurred for all existing items in that period based on their age.

According to Friedman and Hoch [16], ZIP may not hold even in the case of (i) deterioration rates that only depend on the age of the inventory, (ii) linear age-dependent holding cost functions, and (iii) fixed-plus-linear production cost functions. Hsu [17] introduces the dynamic uncapacitated economic lot-sizing problem with perishable inventory under age-dependent holding costs and deterioration rates, where all cost functions are nondecreasing concave. He proves that, under his assumptions, there is an optimal solution composed of subplans of each including some consecutive demand periods and a prior subplan is satisfied by a prior production period. This property of the optimal solution is named as Internal Division Property (IDP). His Dynamic Programing algorithm is based on IDP and runs in $O(n^4)$ time.

Hsu [1] extends Hsu [17] by allowing backlogging in the model and gives an algorithm that runs in $O(n^4)$ time under some assumptions on cost functions and demand. These studies show that without these assumptions, IDP may not hold. Chu et al. [18] introduce a more general version of Chan et al. [12] by considering general economies of scale production cost functions and perishability. The later study considers period-pair-dependent deterioration rates and holding cost functions. Moreover, they assume that the marginal cost of holding additional units of inventory is not higher than that for an older stock. Since the problem is NP-hard, they propose an approximation scheme that gives a solution with an objective value not more than 1.52 times of the optimal objective value. Bai et al. [19] extends Chu et al. [18] by including backlogging. In this study, we extend Hsu [1] by considering finite production capacities.

The rest of the paper is organized as follows. In Section 2, we present our model and show that it is equivalent to a minimum cost network flow problem on a specially constructed network with flow loss. In Section 3, we give some structural properties of the optimal solution and prove that our problem is NP-hard. In Section 4, we introduce our solution approach and in Section 5, we discuss our experimental results. In Section 6, we conclude the paper.

2. Problem formulation

Consider a planning horizon of n periods. We define

- x_t as the production volume in period t, assumed to be available at the beginning of the period t, where $x_t \geq 0$ for $1 \leq t \leq n$.
- z_{it} as the amount of production in period i used to satisfy demand in period t, where $z_{it} \geq 0$ for $1 \leq i, t \leq n$. When $i < t$, it is amount of inventory held, when $i > t$ it is amount of backlogging. Demand satisfaction occurs at the beginning of the production period just after the production.

- y_{it} as the amount of inventory left at the beginning of period t of production at period i after z_{it} is deducted, where $y_{it} \geq 0$ for $1 \leq i \leq t \leq n$.

We also define the parameters of the problem.

- d_t is the demand of the item in period t, where $d_t \geq 0$
- C is the fixed production capacity of a period, where $C \geq 0$
- α_{it} is the deterioration rate during period t for y_{it}, in other words the fraction of y_{it} lost during period t, where $1 \leq i \leq t \leq n$ and $0 \leq \alpha_{it} \leq 1$

We define the cost functions below.

- $c_t(x)$ is the production cost of producing x units in period t
- $H_{it}(y)$ is the cost function for carrying y units during period t, produced in period i and left at the beginning of t, where $i \leq t$
- $B_{it}(z)$ is the cost of backlogging z units from the demand of period t from production in period i, where $i > t$

The production, inventory holding and backlogging functions are nondecreasing concave functions where $c_t(0) = 0$, $H_{it}(0) = 0$ for $1 \leq i \leq t \leq n$ and $B_{it}(0) = 0$ for $1 \leq t < i \leq n$. The inventory holding cost functions are age dependent, in other words all items produced at period i and kept through period t are subject to cost function H_{it}. Moreover, the deterioration rates are age dependent. In period-pair-dependent case, we have a positive quantity for H_{it} if some of the demand of period t is satisfied from period i. On the other hand the backordering cost functions are period-pair dependent, i.e. cost function depends on both periods when the order is placed and filled. As given in [15], period-pair dependent backordering cost is realistic for a product with seasonal demand. In high demand season, customers can be more impatient to wait than in low demand season. Also, backordering cost increases as the waiting time increases. We assume zero inventory and backlogging levels at the beginning and the end of the planning horizon.

Now, we can present our problem (P).

$$(P) \qquad \text{Minimize} \sum_{t=1}^{n} \left[c_t(x_t) + \sum_{i=1}^{t} H_{it}(y_{it}) + \sum_{i=t+1}^{n} B_{it}(z_{it}) \right]$$

subject to

$$x_t - \sum_{i=1}^{t} z_{ti} = y_{tt} \quad 1 \leq t \leq n, \tag{1}$$

$$(1 - \alpha_{i,t-1}) y_{i,t-1} - z_{it} = y_{it} \quad 1 \leq i < t \leq n, \tag{2}$$

$$\sum_{i=1}^{n} z_{it} = d_t \quad 1 \leq t \leq n, \tag{3}$$

$$0 \leq x_t \leq C \quad 1 \leq t \leq n, \tag{4}$$

$$y_{it} \geq 0 \quad 1 \leq i \leq t \leq n, \tag{5}$$

$$z_{it} \geq 0 \quad 1 \leq i, t \leq n. \tag{6}$$

Constraints (1) calculate the amount left from x_t at the beginning of period t after a certain level of demand up to period t is satisfied. Constraints (2) calculate the amount left from $y_{i,t-1}$ after deterioration and demand satisfaction in period t. Constraints (3) assure that demand for each period is satisfied at the end of the horizon. Constraints (4) limit the production levels by capacity and ensure nonnegativity. Constraints (5) and (6) are nonnegativity constraints.

Without constraints (4), this model represents the same problem in [1]. We combine two sets of variables in [1] under one set, and eliminate one set of constraints that calculate the amount of unsatisfied demand. We switch the age-dependent backordering cost function into a period-pair-dependent cost function in order to represent our problem as a minimum cost flow problem with flow loss. Flow loss means the incoming flow to a node is greater than or equal to the outgoing flow. In Fig. 1, we give the network flow representation for 4 periods. The only difference between this network and the one in [17] is the backlogging (backward) arcs. We assume $1 \leq i \leq t \leq n$.

- F is the source node
- N_{it} for all (i, t) pairs are inventory connection nodes
- S_i for all i are demand nodes

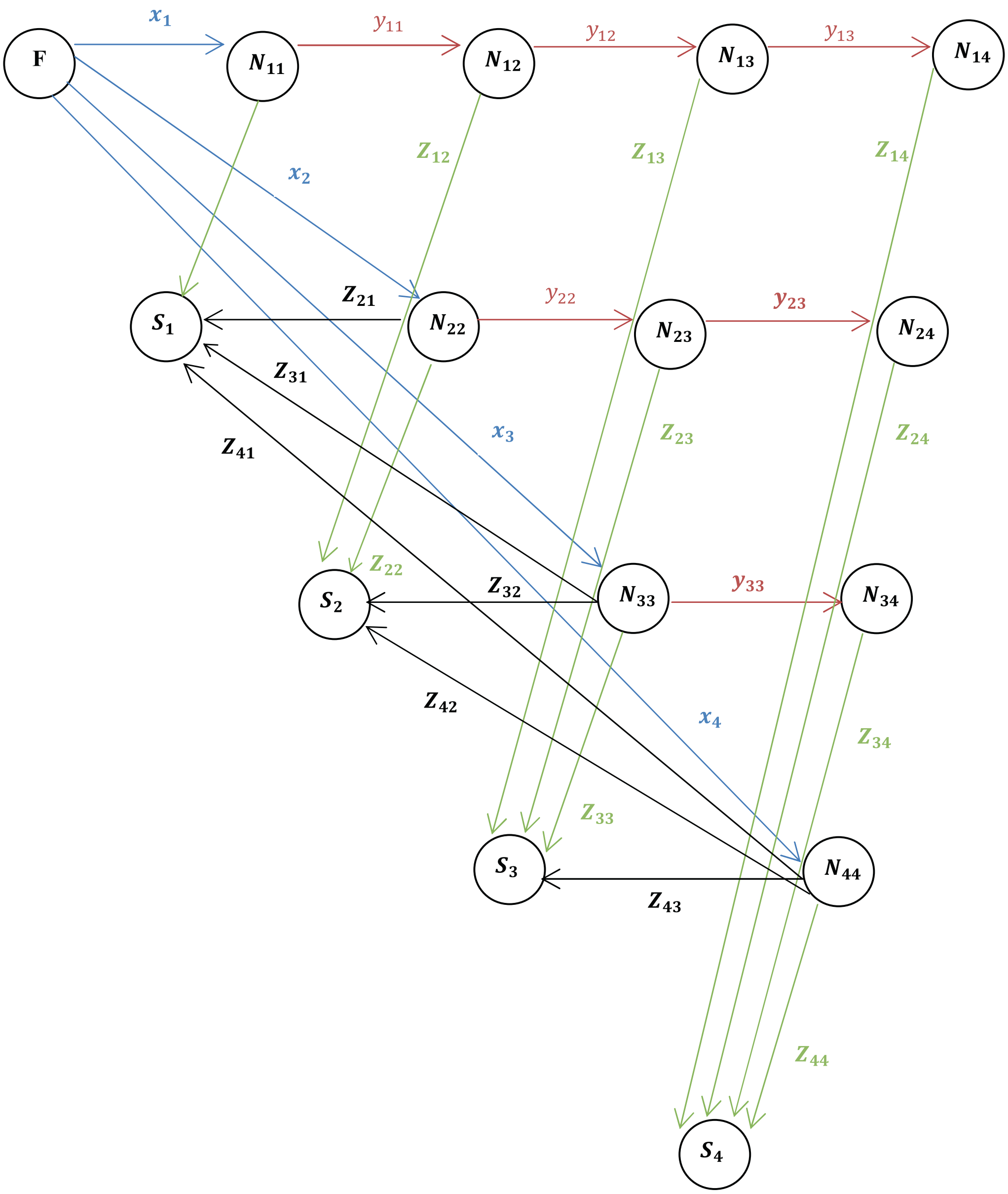

Fig. 1. Network representation of P.

- The arcs from F to N_{tt} are production arcs, namely x_t
- The arcs from N_{it} to $N_{i,t+1}$ are inventory arcs, namely y_{it}
- The arcs from N_{it} to S_t are demand satisfaction arcs via holding or production at the same period, namely z_{it}, $i \le t$
- The arcs from N_{ii} to S_t are demand satisfaction via backlogging arcs, namely z_{it}, $i > t$

Let us extend the definition of A^i_{kt} in [1], the amount needed to be kept at the beginning of period k from production at period i to satisfy one unit demand in period t. When $t < i$, only A^i_{it} is defined and equal to 1. For $1 \le i \le k < t \le n$

$$A^i_{kk} = 1 \text{ and } A^i_{kt} = \frac{1}{\prod_{l=k}^{t-1}(1-\alpha_{il})}. \tag{7}$$

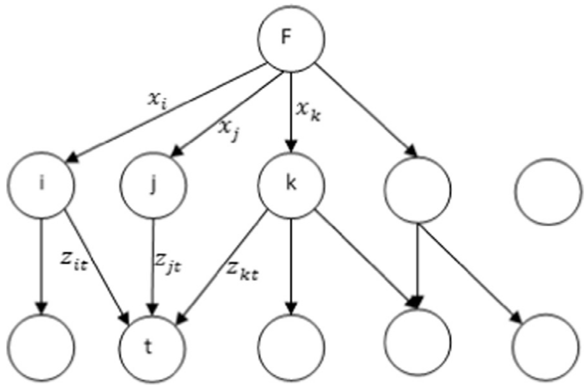

Fig. 2. Reduced network representation of P.

Given z_{it} values, we can calculate y_{it} values (derived by using constraints (1) and (2) recursively).

$$y_{it} = \begin{cases} \prod_{k=i}^{t-1}(1-\alpha_{ik})\left(x_i - \sum_{k=1}^{i} z_{ik}\right) - \sum_{k=i+1}^{t-1}\prod_{m=k}^{t-1}(1-\alpha_{im})z_{ik} - z_{it}, & \text{if } i < t \\ x_i - \sum_{k=1}^{i} z_{ik}, & \text{if } i = t. \end{cases} \tag{8}$$

We know that

$$x_i = \sum_{k=1}^{n} A^i_{ik} z_{ik}. \tag{9}$$

By using the definition (7) and Eqs. (8) and (9), we obtain:

$$y_{it} = \sum_{k=t+1}^{n} A^i_{tk} z_{ik}. \tag{10}$$

Both y_{it} and x_t can be written as a linear combination of z_{it} values. While representing the solution, z_{it} values are enough and all x_t and y_{it} values can be driven using Eqs. (9) and (10) in $O(n^2)$ time. Our real decision variable is z_{it} and we can represent (P) in terms of z_{it}.

$$(R) \qquad \text{Minimize} \sum_{t=1}^{n}\left[c_t\left(\sum_{k=1}^{n} A^t_{tk} z_{tk}\right) + \sum_{i=1}^{t} H_{it}\left(\sum_{k=t+1}^{n} A^i_{tk} z_{ik}\right) + \sum_{i=t+1}^{n} B_{it}(z_{it})\right],$$

subject to

$$\sum_{i=1}^{n} z_{it} = d_t \quad 1 \le t \le n, \tag{11}$$

$$\sum_{k=1}^{n} A^t_{tk} z_{tk} \le C \quad 1 \le t \le n, \tag{12}$$

$$z_{it} \ge 0 \quad 1 \le i, t \le n. \tag{13}$$

Now, we represent the problem on a reduced network where the inventory arcs are eliminated. An example reduced network is given in Fig. 2, where only the arcs with positive flow are given. First level nodes are production period nodes, and the second level nodes are demand period nodes and named by their period numbers. First level arcs are production arcs, and the second level arcs are demand satisfaction arcs.

3. Structural properties of the optimal solution

Problem (P) has the objective of minimizing a concave function, and the constraints create a compact polyhedral. According to Bazaraa and Shetty [20], when the objective is minimizing a concave function and the feasible region is a compact polyhedral, the optimal solution is an extreme point in the feasible region. For a network flow problem, extreme point solution means there are no cycles of free arcs, the arcs with positive flow value less than the capacity [11].

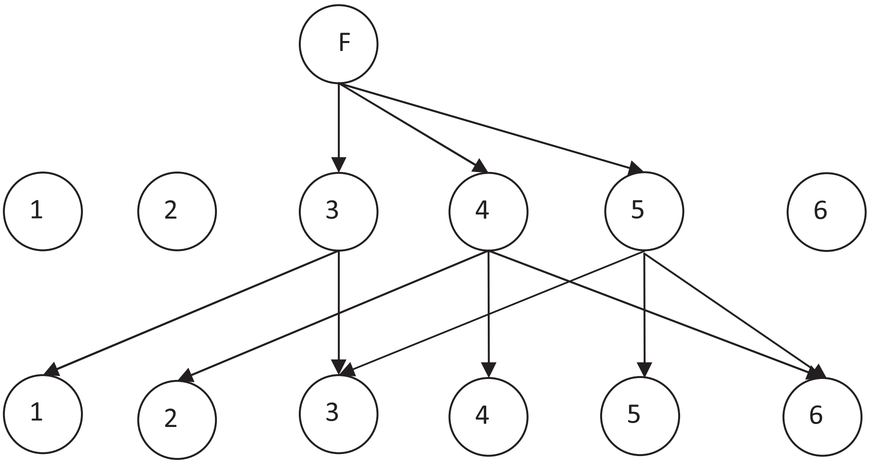

Fig. 3. The reduced network representation for the solution of Example 1.

The optimal solution of our problem (*P*) may not satisfy the basic characteristics of the general lot sizing problems. Namely, how to distribute the production quantity is not trivial. We have to decide, for which periods it is used. Even though there is inventory it may not be used to satisfy the first period that needs it. We demonstrate this with an example.

Example 1. The demand for the next six periods are 15, 10, 20, 5, 5, and 10. The values of $\alpha_{it} = 0.1$ for all $t \geq i$ and $C = 24$. We assume all cost functions as linear. Specifically, $c_t(x_t) = c_t x_t, B_{it}(z_{it}) = b_{it} z_{it}, H_{it}(y_{it}) = h_{it} y_{it}$. $c_1 = 5000$, $c_2 = 500$, $c_3 = 200$, $c_4 = 100$, $c_5 = 250$, and $c_6 = 3000$.

$$b = \begin{pmatrix} - & - & - & - & - & - \\ 100 & - & - & - & - & - \\ 200 & 196 & - & - & - & - \\ 400 & 250 & 120 & - & - & - \\ 625 & 500 & 20 & 169 & - & - \\ 700 & 600 & 350 & 256 & 196 & - \end{pmatrix} \quad h = \begin{pmatrix} 5 & 12 & 15 & 80 & 90 & 100 \\ - & 25 & 30 & 35 & 45 & 55 \\ - & - & 5 & 10 & 15 & 20 \\ - & - & - & 15 & 20 & 30 \\ - & - & - & - & 26 & 50 \\ - & - & - & - & - & 40 \end{pmatrix}$$

In the optimal solution, y_{44}=9, y_{45}=8.1, and y_{55}=3.011. The values of the other decision variables are given below.

$$z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 0 & 9 & 0 & 0 & 0 \\ 0 & 10 & 0 & 5 & 0 & 7.29 \\ 0 & 0 & 11 & 0 & 5 & 2.71 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad x = \begin{pmatrix} 0 & 0 & 24 & 24 & 19.011 & 0 \end{pmatrix}$$

The reduced network representation of the solution of Example 1 with only positive flow arcs is given in Fig. 3. The demand of period 2 is not met at period 3 even though inventory is available and the demand of period 3 is not satisfied at period 4 even though inventory is available. This observation corresponds to intersecting demand satisfaction arcs in the reduced network.

Theorem 1. *There is an optimal solution for problem (P), where for each period t, there is at most one i, where* $z_{it} > 0$ *and* $0 < x_i < C$.

Proof. Suppose that the demand of a period is satisfied by production at more than one period with free production arcs. In this case we obtain cycles of free arcs. □

Theorem 2. *There is an optimal solution, where each connected part has at most one partial production period with quantity* ϵ, *where* $0 < \epsilon < C$.

Proof. If two of the production periods have positive x values less than the capacity, we obtain at least one cycle with free arcs. Making one of these production values equal to capacity/zero, and decreasing/increasing the other one by some amount in the cycle, we obtain a better solution because of the concave cost structure. □

Let us define $\triangle f(x, \delta) = f(x + \delta) - f(x)$. We define the following assumptions on our problem parameters to compare our study with the related literature.

Assumption 1. For any $1 \leq i \leq j \leq t \leq n$, $\alpha_{jt} \leq \alpha_{it}$.

Table 1
Comparison with the related literature in terms of assumptions in the model.

	Hsu [17]	Hsu and Lowe [15]	Hsu [1]	Our study
Perishability	AD	–	AD	AD
Backlogging cost function	–	PPD	AD	PPD
Inventory holding cost function	AD	PPD	AD	AD
Assumption 1	✓	–	✓	✓
Assumption 2	–	–	–	✓
Assumption 3	–	–	–	✓
Assumption 4	–	–	–	✓
Assumption 5	✓	–	✓	–
Assumption 6	–	–	✓ or nondecreasing demand	–
Production capacities	∞	∞	∞	Constant

Assumption 2. For any $x, y, z \geq 0$ and $1 \leq i \leq j \leq t \leq n$

$$\triangle H_{jt}(x,z) \leq \triangle H_{it}(y,z)$$

Assumption 3. For any $x, y, z \geq 0$ and $1 \leq t < j \leq i \leq n$

$$\triangle B_{jt}(x,z) \leq \triangle B_{it}(y,z)$$

Assumption 4. For any $x, y, z \geq 0$ and $1 \leq j \leq i < t \leq n$

$$\triangle B_{ti}(x,z) \leq \triangle B_{tj}(y,z)$$

Assumption 5. For any $y \geq 0$ and $1 \leq i \leq j \leq t \leq n$

$$H_{jt}(y) \leq H_{it}(y)$$

Assumption 6. There is a nondecreasing marginal backlogging cost with respect to age of the inventory. In other words, extra unfilled units for period i is more expensive than for period j in period t, where $1 \leq i \leq j \leq t \leq n$.

Assumption 1 is a practical assumption and states that the deterioration rate at a period is more for an older inventory. Deterioration rate is equal to 1 when the age is greater than the shelf life. Assumption 2 states that there is a nondecreasing marginal holding cost with respect to the age of inventory. Assumption 3 states that for a given demand period t, there is a nondecreasing marginal cost of backlogging with respect to the production period. Assumption 4 states that for a given production period t, there is a nondecreasing marginal cost of satisfying the demand of periods by backlogging with respect to the satisfied period.

This paper includes Assumptions 1–4. Assumption 2 implies 5. Table 1 states the assumptions of the most related papers in the literature. Hsu and Lowe [15] has more restrictions on the period-pair-dependent holding and backlogging cost functions.

We first show that our problem under Assumptions 1–4 is NP-hard. In [21], it is recently shown that when the item under consideration has a deterministic expiration date (when α_{it} values are 0 or 1) under two allocation mechanisms: First Expiration, First Out (FEFO) and Last In, First Out (LIFO), (P) is NP-hard. Our model does not assume any mechanism, it finds the best solution without any defined mechanism.

Theorem 3. *Under Assumption 1, Problem (P) with zero inventory holding and backlogging costs is NP-hard.*

Proof. We convert a known NP-complete problem into (P). We use Knapsack problem as in the proof of nonperishable lot sizing problem with non-stationary capacity limits [13].

Knapsack: Given positive integers $b_1, \ldots, b_n$ and B, is there a subset $S \subset T = \{1, \ldots, n\}$ such that $\sum_{i \in S} b_i = B$?

Given any instance of Knapsack, we can obtain the following instance of problem (P), I_P in polynomial time. We have $n+1$ periods (0, 1, ..., n). We sort array b in non-decreasing order and obtain $a_1, \ldots, a_n$. We pick a $C \geq a_n$.

$$A^t_{tn} = C/a_t \quad t = 1, \ldots, n-1. \tag{14}$$

$A^t_{tn} \geq A^{t+1}_{t+1,n}$, since $a_t \leq a_{t+1}$. Moreover, in I_P, $A^t_{tn} \geq A^{t+1}_{t+1,n}$.

$$A^t_{tn} = \frac{1}{1-\alpha_{t,t}} \prod_{l=t+1}^{n-1} \frac{1}{1-\alpha_{t,l}} \geq \prod_{l=t+1}^{n-1} \frac{1}{1-\alpha_{t+1,l}} = A^{t+1}_{t+1,n},$$

due to Assumption 1 ($0 \leq \alpha_{t+1,l} \leq \alpha_{t,l} \leq 1$). We calculate all A^t_{tn} values for $t = 1, \ldots, n-1$, and given these values it takes $O(n)$ time to calculate α_{it} for all $1 \leq i \leq t \leq n-1$ using the following procedure. We start with $t = n-1$ and assign $\alpha_{i,n-1} =$

$1-\frac{a_{n-1}}{C}$ (from the solution of $A^{n-1}_{n-1,n}=\frac{1}{1-\alpha_{n-1,n-1}}$) for all $i=0,\dots,n-1$. Then we solve the below equation to find $\alpha_{n-2,n-2}$.

$$A^{n-2}_{n-2,n}=\frac{1}{(1-\alpha_{n-2,n-2})}\frac{1}{(1-\alpha_{n-2,n-1})}.$$

We assign $\alpha_{i,n-2}=\alpha_{n-2,n-2}$ for all $i=0,\dots,n-2$. We continue in this fashion until we calculate the values of α_{it} for all $0\le i\le t\le n-1$. We set A^0_{00} to 1 and calculate A^0_{0i} values for $i=1,\dots,n$.

$$A^0_{0i}=\prod_{l=0}^{i-1}\frac{1}{1-\alpha_{0l}}.$$

We create d_t values for $t=0,\dots,n-1$ such that $\sum_{i=0}^{n-1}A^0_{0i}d_i=C$ and set $d_n=B$.

$$\begin{aligned}
h_{it}(x)&=0 \quad x\ge 0;\,1\le i\le t\le n\\
b_{it}(x)&=0 \quad x\ge 0;\,1\le t<i\le n\\
c_0(x)&=0 \quad x\ge 0\\
c_t(0)&=0 \quad t=1,\dots,n\\
c_t(x)&=1+\frac{a_t-1}{C}x \quad 0<x\le C;\,t=1,\dots,n
\end{aligned}$$

This way, we create an instance of (P) obeying Assumption 1.

We claim that **Knapsack** has a solution if and only if there is a feasible production plan for I_P with total cost equals to B. Since there is no cost of production for the zeroth period and zero holding cost it uses its all capacity. The production capacity is enough to satisfy the periods $0,\dots,n-1$ since $\sum_{t=0}^{n-1}A^0_{0t}d_t=C$. We need to satisfy the demand of the last period using production in periods $1,\dots,n$.

Therefore, we can restrict our attention to the following set of solutions.

$$x_0=C,\quad \sum_{t=1}^{n}\frac{x_t}{A^t_{tn}}=B,\quad 0\le x_t\le C\quad (t=1,\dots,n)$$

The production cost functions for $t=1,\dots,n$ are concave and have the following properties.

$$\begin{aligned}
c_t(x)&>\frac{a_t}{C}x \quad 0<x<C\\
c_t(x)&=\frac{a_t}{C}x \quad x\in\{0,C\}
\end{aligned}$$

As $c_0(C)=0$ and $C=a_tA^t_{tn}$ due to expression (14), we have

$$\sum_{t=0}^{n}c_t(x_t)\ge c_0(C)+\sum_{t=1}^{n}\frac{a_t}{C}x_t=\sum_{t=1}^{n}\frac{x_t}{A^t_{tn}}=B.$$

Therefore, the total cost is at least equal to B. Total cost can be equal to B if and only if $x_t\in\{0,\ C\}$ for all $t=1,\dots,n$. Therefore if there is a $S\subset T=\{1,\dots,n\}$ that production is at full capacity such that $\sum_{i\in S}b_i=B$ then there is a solution for I_P with total cost B.

We claim that I_P has a solution with total cost equals to B if and only if there is a solution for *Knapsack*. If we have subset $S\subset T=\{1,\dots,n\}$ with $\sum_{i\in S}b_i=B$ then if we produce at full capacity at each period $i\in S$ and do not produce at each period $i\notin S$, we obtain a plan with total cost equals to B. Otherwise, positive production quantities less than capacity gives a cost higher than B. □

We further analyze the structure of the optimal solution in terms of intersecting z arcs.

Theorem 4. *There is an optimal solution of (P), where $z_{i_1t_2}$ and $z_{i_2t_1}$ are not both positive for the following cases.*

a. $t_1\le i_1<i_2\le t_2$
b. $i_1<t_1<t_2<i_2$

Proof. Suppose that we have an optimal solution with positive $z_{i_1t_2}$ and $z_{i_2t_1}$ values. The current solution is denoted by z^+_{it},y^+_{it},x^+_t. We also use the notation $\triangle f(x,\delta)=f(x+\delta)-f(x)$ for any x.

a. We define $\delta=\min\{A^{i_1}_{i_1t_2}z^+_{i_1t_2},z^+_{i_2t_1}\}$. Without changing x_{i_1} and x_{i_2}, for any $0<z\le\delta$, suppose that we do the following changes: increase $z_{i_1t_1}$ by z and $z_{i_2t_2}$ by $\frac{z}{A^{i_1}_{i_1t_2}}$; decrease $z_{i_2t_1}$ by z and $z_{i_1t_2}$ by $\frac{z}{A^{i_1}_{i_1t_2}}$. After these changes, all demand is still satisfied and we can show that total cost does not increase. Increasing $z_{i_2t_2}$ by $\frac{z}{A^{i_1}_{i_1t_2}}$ increases the inventory held in period

$k = i_2, \ldots, t_2-1$ by $\frac{A^{i_2}_{kt_2}}{A^{i_1}_{i_1t_2}}z$. Decreasing $z_{i_1t_2}$ by $\frac{z}{A^{i_1}_{i_1t_2}}$ units decreases the inventory held in period $k = i_1, \ldots, t_2-1$ by $\frac{A^{i_1}_{kt_2}}{A^{i_1}_{i_1t_2}}z$ units. We calculate the change in cost by the following expression.

$$\sum_{k=i_1}^{t_2-1} \Delta H_{i_1k}\left(y_{i_1k}, -\frac{A^{i_1}_{kt_2}}{A^{i_1}_{i_1t_2}}z\right) + \sum_{k=i_2}^{t_2-1} \Delta H_{i_2k}\left(y_{i_2k}, \frac{A^{i_2}_{kt_2}}{A^{i_1}_{i_1t_2}}z\right) + \Delta B_{i_1t_1}(z_{i_1t_1}, z) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z).$$

Due to Assumption 3, an increase in $z_{i_2t_1}$ is more costly than the same increase in $z_{i_1t_1}$, therefore

$$\Delta B_{i_1t_1}(z_{i_1t_1}, z) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z) = \Delta B_{i_1t_1}(z_{i_1t_1}, z) - \Delta B_{i_2t_1}(z_{i_2t_1} - z, z) \le 0. \tag{15}$$

For any $0 \le z_1 \le z \le z_{i_2t_1}$

$$\begin{aligned}&\Delta B_{i_1t_1}(z_{i_1t_1}, z) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z) = \\ &\Delta B_{i_1t_1}(z_{i_1t_1}, z_1) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z_1) + \Delta B_{i_1t_1}(z_{i_1t_1}+z_1, z-z_1) + \Delta B_{i_2t_1}(z_{i_2t_1}-z_1, z_1-z),\end{aligned}$$

and by using Assumption 3 and the same argument in (15), $\Delta B_{i_1t_1}(z_{i_1t_1}, z) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z)$, $\Delta B_{i_1t_1}(z_{i_1t_1}, z_1) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z_1)$, and $\Delta B_{i_1t_1}(z_{i_1t_1}+z_1, z-z_1) + \Delta B_{i_2t_1}(z_{i_2t_1}-z_1, z_1-z)$ are negative, then we can write

$$\Delta B_{i_1t_1}(z_{i_1t_1}, z) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z) \le \Delta B_{i_1t_1}(z_{i_1t_1}, z_1) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z_1). \tag{16}$$

Therefore, left hand side of the inequality (15) is a decreasing function of z. We can reorganize the remaining change in cost as:

$$\sum_{k=i_1}^{i_2-1} \Delta H_{i_1k}\left(y_{i_1k}, -\frac{A^{i_1}_{kt_2}}{A^{i_1}_{i_1t_2}}z\right) + \sum_{k=i_2}^{t_2-1} \Delta H_{i_1k}\left(y_{i_1k}, -\frac{A^{i_1}_{kt_2}}{A^{i_1}_{i_1t_2}}z\right) + \sum_{k=i_2}^{t_2-1} \Delta H_{i_2k}\left(y_{i_2k}, \frac{A^{i_2}_{kt_2}}{A^{i_1}_{i_1t_2}}z\right).$$

The first term is negative. Since $A^{i_1}_{kt_2} \ge A^{i_2}_{kt_2}$ for any $i_2 \le k \le t_2-1$ (due to Assumption 1)

$$\begin{aligned}&\sum_{k=i_2}^{t_2-1} \Delta H_{i_1k}\left(y_{i_1k}, -\frac{A^{i_1}_{kt_2}}{A^{i_1}_{i_1t_2}}z\right) + \sum_{k=i_2}^{t_2-1} \Delta H_{i_2k}\left(y_{i_2k}, \frac{A^{i_2}_{kt_2}}{A^{i_1}_{i_1t_2}}z\right) \le \\ &\sum_{k=i_2}^{t_2-1} \Delta H_{i_1k}(y_{i_1k}, -\frac{A^{i_1}_{kt_2}}{A^{i_1}_{i_1t_2}}z) + \sum_{k=i_2}^{t_2-1} \Delta H_{i_2k}(y_{i_2k}, \frac{A^{i_1}_{kt_2}}{A^{i_1}_{i_1t_2}}z).\end{aligned} \tag{17}$$

We can show that

$$\frac{A^{i_1}_{kt_2}}{A^{i_1}_{i_1t_2}} = \frac{\prod_{l=i_1}^{t_2-1}(1-\alpha_{i_1,l})}{\prod_{l=k}^{t_2-1}(1-\alpha_{i_1,l})} = \frac{1}{A^{i_1}_{i_1k}}.$$

Now, expression (17) can be restated as below and due to Assumption 2, it is non-positive.

$$\sum_{k=i_2}^{t_2-1} \Delta H_{i_1k}\left(y_{i_1k}, -\frac{z}{A^{i_1}_{i_1k}}\right) + \sum_{k=i_2}^{t_2-1} \Delta H_{i_2k}\left(y_{i_2k}, \frac{z}{A^{i_1}_{i_1k}}\right) = -\sum_{k=i_2}^{t_2-1} \Delta H_{i_1k}\left(y_{i_1k} - \frac{z}{A^{i_1}_{i_1k}}, \frac{z}{A^{i_1}_{i_1k}}\right) + \sum_{k=i_2}^{t_2-1} \Delta H_{i_2k}\left(y_{i_2k}, \frac{z}{A^{i_1}_{i_1k}}\right) \le 0.$$

Since this change decreases as z increases (due to similar discussion in inequality (16)), we obtain the best solution at $z = \delta$. At $z = \delta$, at least one of the variables $z_{i_1t_2}$ and $z_{i_2t_1}$ becomes zero.

b. We define $\delta = \min\{z^+_{i_1t_2}, z^+_{i_2t_1}\}$. We show that for any $0 < z \le \delta$, increasing $z_{i_1t_1}$ and $z_{i_2t_2}$ values by z; and decreasing $z_{i_1t_2}$ and $z_{i_2t_1}$ by z without changing x_{i_1} and x_{i_2} values, does not increase the total cost. Increasing $z_{i_1t_1}$ by z increases the inventory held in period $k = i_1, \ldots, t_1-1$ by $A^{i_1}_{kt_1}z$. Decreasing $z_{i_1t_2}$ by z units decreases the inventory held in period $k = i_1, \ldots, t_2-1$ by $A^{i_1}_{kt_2}z$ units. We calculate the change in cost by the following expression.

$$\sum_{k=i_1}^{t_2-1} \Delta H_{i_1k}(y_{i_1k}, -A^{i_1}_{kt_2}z) + \sum_{k=i_1}^{t_1-1} \Delta H_{i_1k}(y_{i_1k}, A^{i_1}_{kt_1}z) + \Delta B_{i_2t_2}(z_{i_2t_2}, z) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z).$$

We can reorganize the terms as below.

$$\sum_{k=i_1}^{t_1-1} \Delta H_{i_1k}(y_{i_1k}, (A^{i_1}_{kt_1} - A^{i_1}_{kt_2})z) + \sum_{k=t_1}^{t_2-1} \Delta H_{i_1k}(y_{i_1k}, -A^{i_1}_{kt_2}z) + \Delta B_{i_2t_2}(z_{i_2t_2}, z) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z).$$

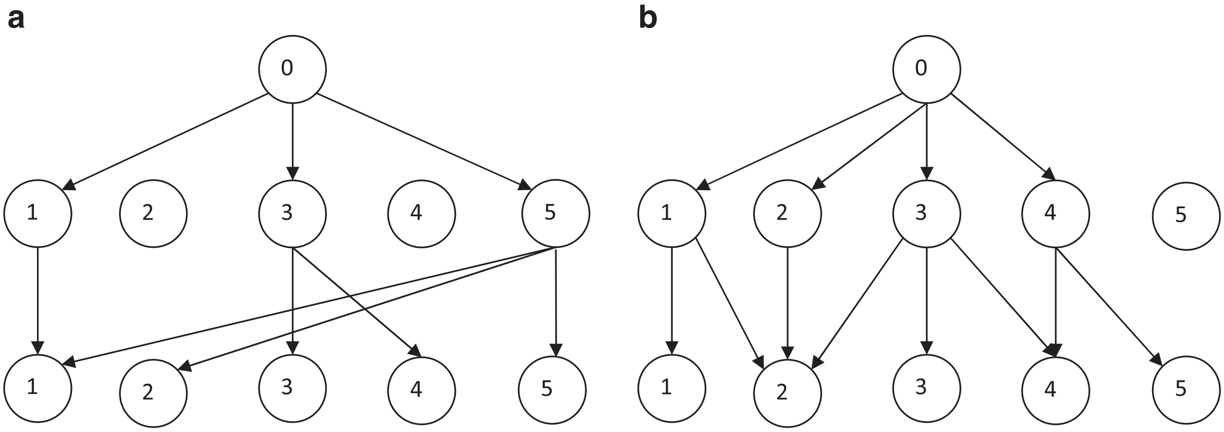

Fig. 4. Two examples for subplan (1, 5, 1, 5). (a) A disconnected subplan, (b) A connected subplan.

Since $A^{i_1}_{kt_1} \le A^{i_1}_{kt_2}$ for any $i_1 \le k \le t_1 - 1$,

$$\sum_{k=i_1}^{t_1-1} \Delta H_{i_1k}(y_{i_1k}, (A^{i_1}_{kt_1} - A^{i_1}_{kt_2})z) + \sum_{k=t_1}^{t_2-1} \Delta H_{i_1k}(y_{i_1k}, -A^{i_1}_{kt_2}z) \le 0,$$

and this change decreases as z increases since H_{i_1k} is an increasing function. Moreover, because of Assumption 4

$$\Delta B_{i_2t_2}(z_{i_2t_2}, z) + \Delta B_{i_2t_1}(z_{i_2t_1}, -z) = \Delta B_{i_2t_2}(z_{i_2t_2}, z) - \Delta B_{i_2t_1}(z_{i_2t_1} - z, z) \le 0$$

and this change decreases as z increases with a similar argument in (16). Therefore, we obtain the best solution at $z = \delta$. At $z = \delta$, at least one of the variables $z_{i_1t_2}$ and $z_{i_2t_1}$ becomes zero. □

4. Solution approach

We summarize the properties of an optimal solution of an instance with parameters obeying Assumptions 1–4 as below as a consequence of Theorems 3.1–3.4.

Property 1. *If a period is satisfied by multiple periods at most one of these periods can have partial production.*

Property 2. *Each connected component (composed of positive flow arcs) has at most one partial production period.*

Property 3. *If production period j satisfies period $k > j$ then any period t ($j < t < k$) cannot be satisfied by a period later than k and cannot satisfy a period $t \le j$.*

Let us define subplan (s, m, p, r) as the production periods from s to m, $s \le m$ and the demand periods from p to r, $p \le r$, where the demand of periods from p to r are fully satisfied by the production periods from s to m, where m is a positive production period. This means that, outside the subplan there is no incoming inventory/backlog (z arcs) from production periods nor outgoing inventory/backlog (z arcs) to the demand periods. Moreover, there is no passing through z arc, i.e. z_{ij} with $i \notin [s, m]$ and $j \notin [p, r]$. The next subplan starts with production period $m + 1$ and ends with a positive production period. An optimal solution is composed of at least one subplan.

There are two types of subplans: connected and disconnected. A connected subplan means all demand nodes are connected via positive flow arcs. In Fig. 4, we picture the reduced network for two cases of subplan (1, 5, 1, 5) with only positive flow arcs. In Fig. 4(b), we consider a connected case for subplan (1, 5, 1, 5). There is a undirected path of positive flow arcs between all pairs of demand nodes. In Fig. 4(a), we picture a disconnected subplan with two connected parts. First part is composed of demand nodes 1, 2, and 5 (satisfied by periods 1 and 5) and the second part is composed of demand periods 3 and 4 (satisfied by only period 3).

An optimal solution is composed of at least one subplan. This structure enables us to use Dynamic Programing (DP) approach in our solution procedure. We define

- $V(m, r)$ as the minimum cost of satisfying the demand of periods $1, \dots, r$ by the production at periods $1, \dots, m$.
- $P(s, m, p, r)$ as the minimum possible cost for subplan (s, m, p, r).

We write the following backward recursion. Without loss of generality, we consider the subplans (s, m, p, r) such that $1 \le s \le m$, $1 \le p \le r$.

$$V(m, r) = \min_{1 \le s \le m, 1 \le p \le r} \{V(s-1, p-1) + P(s, m, p, r)\}$$

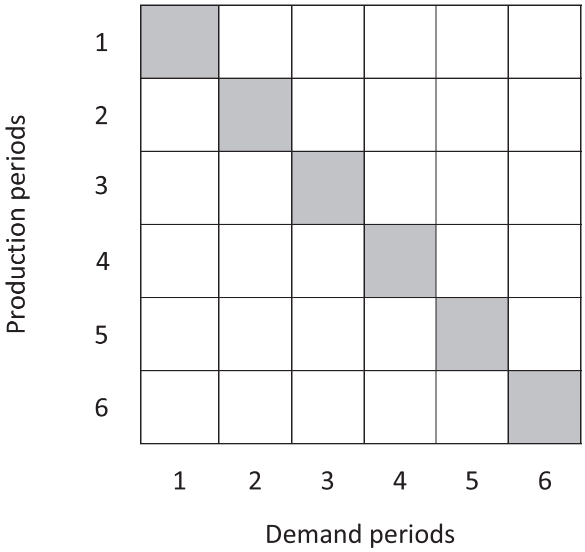

Fig. 5. The matrix for subplan (1, 6, 1, 6).

We are looking for $\min_i V(i, n)$ as the optimal solution value of (P). DP algorithm runs in $O(n^4)$ time if we know the minimum cost of each subplan. Because this Dynamic Programing formulation corresponds to a shortest path problem on an acyclic network with $O(n^4)$ arcs, it can be solved in $O(n^4)$ time [22]. To apply DP algorithm we need to compute all $P(s, m, p, r)$ values.

4.1. Calculating the cost of a subplan

We analyze the production quantities in the optimal solution via theorems. We focus on subplans with at most one partial production. The number of full production periods β is at least $\lfloor \frac{\sum_{k=p}^{r} d_k}{C} \rfloor$ because of the deteriorating inventory.

Let us define *combination of a subplan* (s, m, p, r) as the subplan with its full and partial production periods information. A combination is represented by an array of production indicators (PI) for each production period. The possible values are 0, 1, and 2, and means no production, full production, and partial production, respectively.

For a feasible subplan (s, m, p, r) we have $\binom{m-s}{\beta}(\beta+1)$ possible combinations. This value takes its maximum value at $\beta = \lceil (m-s)/2 \rceil$. One period is chosen as the partial production period, period m and other $\beta - 1$ periods are chosen as the full production periods.

The partial production quantity depends on its period, it is not fixed as in classical ELS. Let r_t be the demand of period t that is satisfied from partial production period. The partial production quantity if produced in period i, ϵ_i, is calculated based on r_t values.

$$\epsilon_i = \sum_{t=p}^{r} A_{it}^{i} r_t.$$

Example 2. Consider a 6-period problem. A possible combination for subplan (1,6,1,6) is [011210]. Periods 2, 3, and 5 are full production periods. Period 4 is a partial production period and there is no production at other periods.

There are $O(n^4)$ subplans. For each subplan, we check the feasibility. We first check if $\lfloor \frac{\sum_{k=p}^{r} d_k}{C} \rfloor < m - s + 1$. This is a necessary condition but it is not enough. It is also important to check whether all demand can be met if we have full production in each period, and consume it on a “first produced first consumed” basis. Since each subplan has exponential number of combinations to choose from, we find a good solution for the given combination.

4.2. Calculating the cost of a subplan’s combination

The following section presents a method to find a good solution for a given subplan and its combination, which is similar to Northwest corner method is used to find an initial basic feasible solution to the transportation problem.

We fill out a square matrix of size $\max\{m, r\} - \min\{s, p\}$ (see Fig. 5). The element (i, j) of this matrix represents z_{ij}, where i is the production period and j is the demand period. We assign zero demand and production to the periods not initially in the subplan. For a given combination of a subplan, finding the optimal z values is easier since a good solution has z positive values close the diagonal to avoid backlogging and holding costs, and we have to distribute all production quantity.

We aim to find z values, where the sum of the i^{th} column is equal to d_i and sum product of the j^{th} row with A^j_{ji} values is x_j. In order to find the best solution for a subplan and its combination, we start with the case in which there is at most one partial production period and try to improve the solution by disconnecting subplans, thus creating new partial production periods.

While filling out the matrix, we try to obey the properties of the optimal solution.

- A production period first satisfies its own period, it is the least costly demand satisfaction for a production period. This corresponds to moving through the diagonal when production period changes. The diagonal is shaded with gray in Fig. 5.
- When we know the production periods, we first distribute from the full production periods and then use the partial period for the remaining demand.
- Because of Property 3, we escape from the moves crossing the diagonal with an angle $0 < \alpha < 90$ to the x-axis.
- While distributing the units from the full production periods
  - when going down satisfy the demand from left to right to finish all on hand
  - when going up satisfy right to left to use all on hand.

The detailed steps of our heuristic for finding a good solution for a combination of a subplan is given below. In our heuristic, we start at the top left corner of the matrix and move. While moving in the matrix, x and y represent the current location on the x-axis and y-axis, respectively. For example in Fig. 5, $x, y = 1, \ldots, 6$. The remaining production at period x for period y is calculated as below.

$$\left(\text{Total production at } x - \sum_{k=1}^{n} A^x_{xk} z_{xk}\right) / A^x_{xy}.$$

FIND COST ((s, m, p, r), combination)

0. Feasibility check.
   (a) If $\beta = \lfloor \frac{\sum_{k=p}^{r} d_k}{C} \rfloor < m - s + 1$, go to (b).
   (b) Assume we have full production at each period and consume it in first produced first consumed manner. If all demand can be satisfied then go to step 1.
1. Forward move
   (a) Create the square subplan (min{s, p}, max{m, r}, min{s, p}, max{m, r}).
   (b) $x = y = \min\{s, p\}$. If $PI[x] = 2$ go to step 2.
   (c) Set z_{xy} to the minimum of the *remaining production at period* x and unsatisfied demand at y and update these values.
   (d) If extra production remains then set y to first unsatisfied period. Go to (c).
   (e) $x = x + 1,\ y = x$. If $PI[x] \neq 2$ go to (c).
2. Backward move
   (a) $x = y = \max\{m, r\}$. If $PI[x] = 2$ go to step 3.
   (b) Set z_{xy} to the minimum of the *remaining production at period* x and unsatisfied demand at y and update these values.
   (c) If there is any unsatisfied demand, $x = x - 1$ and go to (b).
   (d) If $y > x$ and unused production remains then $y = y - 1$.
   (e) Set z_{xy} to the minimum of the *remaining production at period* x and unsatisfied demand at y and update these values. Go to (d).
   (f) $x = x - 1,\ y = x$. If $PI[x] \neq 2$ go to (b).
3. Distribute the remaining production amount
   (a) $x = \max\{m, r\}$.
   (b) If the *remaining production at period* x is positive and $PI[x] = 1$ then find the closest unsatisfied period y and go to (c), else go to step (d).
   (c) Set z_{xy} to the minimum of the *remaining production at* x and unsatisfied demand at y and update these values. Go to (b).
   (d) $x = x - 1$. If $x \geq \min\{m, r\}$ go to (b).
4. Determine the value of ϵ using unsatisfied demand of all periods.
5. *Improvement 1:* for each diagonal element, we try to find a rectangle represented by its corners, where the diagonal element is the right top corner. There should be at least two positive elements at opposite corners. If shifting one unit around this rectangle is cost saving, apply it for the maximum possible value.
6. *Improvement 2:* disconnect a connected subplan by removing a middle period if it saves cost.
   (a) If period e, where $PI[e] = 2$ does not satisfy all of its own demand, find a later period (denoted by k) that satisfies period e. Otherwise, stop.
   (b) If k satisfies a period (denoted by t) earlier than e, define $\rho = z_{ke}$. Increase z_{ee} by ρ, set z_{ke} to 0, increase z_{kt} by ρ, decrease z_{et} by min{ρ, z_{et}}, and for any full production period $l < e$, where $z_{lt} > 0$ decrease z_{lt} by $\max\{\rho - z_{et}, 0\}$.

Example 3. Now, we consider the case when $n = 6$ and $C = 20$. The demand for the next six periods are 15, 10, 20, 5, 1, and 10. The values of $\alpha_{it} = 0.1$ for all $t \geq i$.

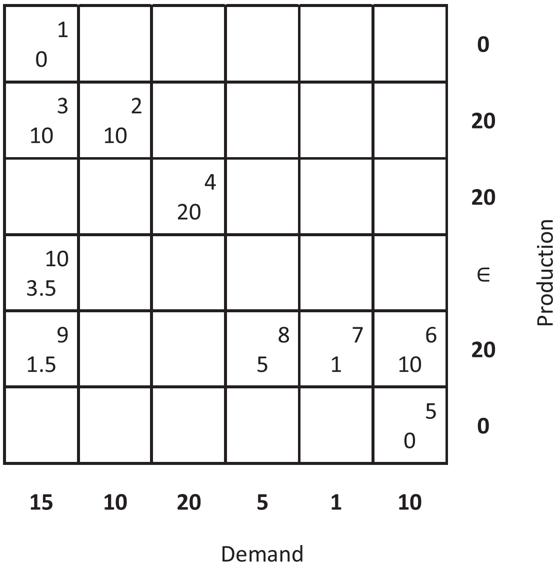

Fig. 6. Algorithm steps and solution before improvement steps.

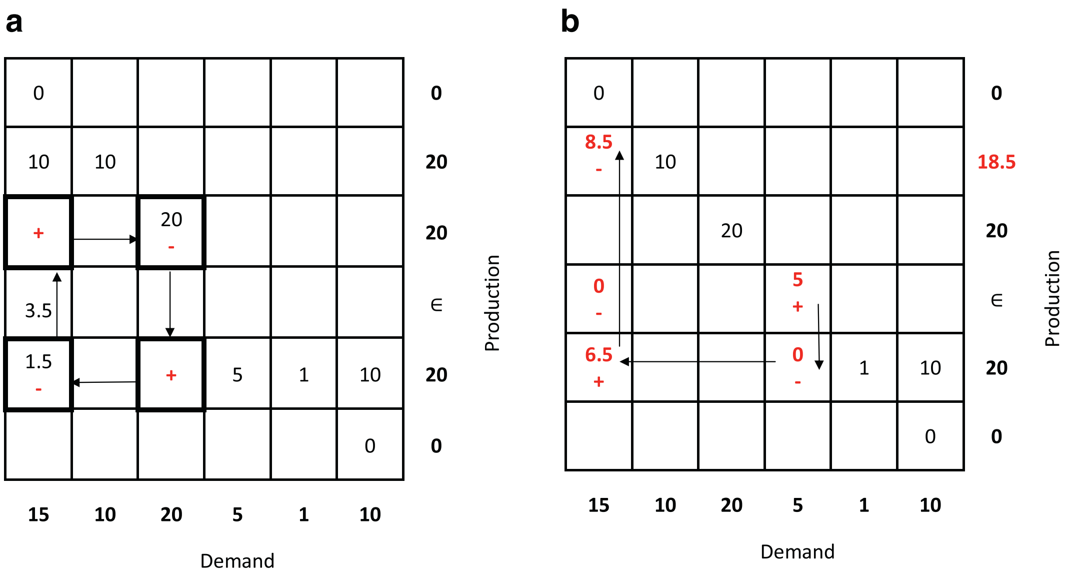

Fig. 7. (a) Improvement step 1 (b) Improvement step 2.

We apply FIND COST heuristic for the subplan (1,6,1,6) and its combination represented by [011210]. In Fig. 6, we apply the steps 0 to 4, and assign z values according to the heuristic (given in the center of the cell) and the order of this assignment on the top right corner of the cell.

An example for improvement step 1 is shown in Fig. 7(a). We choose period 3 and find a lower rectangle with two positive values at opposite corners. If this move decreases the total cost, we decrease z_{33} and z_{51} by 1.5 units, and increase z_{53} and z_{31} by the same amount.

An example for improvement step 2 is shown in Fig. 7(b). In spite of being a production period, period 4 does not satisfy any of its demand. Period 5 satisfies all of its demand, therefore we supply these 5 units from period 4 instead of period 5. We decrease the supply of period 2 by 1.5 units and increase the supply of period 4 by 1.5 units. This move is along a trapezoid.

Theorem 5. *Dynamic Programing based heuristic runs in* $O(n^6)$ *time.*

Proof. FIND COST runs for at most two combinations for a given subplan. For each candidate combination, feasibility check takes $O(n^2)$ time. The matrix has $O(n^2)$ cells to fill in and each of them can be visited at most once. For the improvement

Table 2
Experimental results.

d_t	n	N_F	$\%N_0$	$\%N_1$	$A(G_1)$	$M(G_1)$	$S(G_1)$
Random	5	694	68.87	81.84	0.69	10.26	1.58
	10	7899	46.29	60.69	1.76	52.04	3.23
	20	54,688	18.35	32.82	3.34	23.97	3.67
Constant	5	795	74.96	85.28	0.50	7.10	1.22
	10	8806	44.17	56.67	2.49	55.70	4.36
	20	122,642	14.96	23.92	4.54	44.90	4.68
Increasing	5	645	78.90	88.83	0.3	6.81	0.89
	10	9691	56.40	72.60	0.98	52.62	2.14
	20	35,659	47.94	66.76	1.2	62.08	2.58

steps, starting from a diagonal element finding a cell (as a corner of rectangle or trapezoid) takes $O(n)$ time since it requires scanning a column as a side of rectangle or trapezoid. Improvement step is applied for all $O(n)$ diagonal elements. Therefore, the heuristic runs in $O(n^6)$ time. □

5. Experimental results

We test our FIND COST heuristic on different problem sizes. In this section, we discuss our results. We assume fixed-plus-linear production cost function and linear backlogging and holding cost functions.

$$C_t(x_t) = F_t + c_t x_t, B_{it}(z_{it}) = b_{it} z_{it}, H_{it}(y_{it}) = h_{it} y_{it}$$

We have 9 problem sets for different problem sizes ($n = 5, 10, 20$) and three types of demand generation schemes. We generate 10 random instances based on the following procedure. We represent discrete uniform random variable in interval $[a, b]$ by $U[a, b]$.

- C is 20
- c_t is $U[10, 50]$
- d_t is random ($U[5, 20]$ for each period), constant ($U[5, 15]$ and used for all periods), or increasing ($d_t = U(3, 7) + (t-1) * U(0, 3)$)
- F_t is $U[100, 500]$
- For all i, $b_{ii} = 0$ and $b_{i0} = b_{i-1,0} + U[5, 15]$
- For all $i > j > 0$, $b_{ij} = U[b_{i-1,j} + 1, b_{i,j-1} - 1]$
- For all i, $h_{ii} = U[1, 10]$
- For all $i \geq j > 0$, $h_{ji} = h_{j-1,i} + U[5, 10]$
- For all i, α_{ii} is $U[0, 0.1]$
- For all $i \geq j > 0$, $\alpha_{ji} = \alpha_{j-1,i} + U[0, 0.05]$

We measure the performance of FIND COST heuristic on finding subplan combination pair cost by comparing it with the optimal cost found by CPLEX. We have the following notation.

- N_F is the number of feasible subplan combination pairs considered
- z_h is the cost of the subplan combination pair found by FIND COST heuristic
- z^* is the optimal cost of the subplan combination pair
- N_0 is the number of times the heuristic finds the optimal solution out of N_F
- $\%N_0$ is the percentage of hitting the optimal solution for subplan combination pair and calculated as $\frac{N_0}{N_F} * 100$
- G_1 is the percentage deviation from the optimal solution for a subplan combination pair

$$G_1 = \frac{z_h - z^*}{z^*} * 100$$

- $\%N_1$ is the percentage of cases when G_1 is less than 1%
- $A(x)$ is the average of the values of x
- $M(x)$ is the maximum value of x
- $S(x)$ is the standard deviation of the values of x

For each n and d_t interval, we create 10 problem instances and we report our results in Table 2. The heuristic performs best when the demand is increasing through the time. The percentage of cases where FIND COST heuristic finds the optimal solution is 78.90% and less than one percent is 88.83% when $n = 5$. These values drop to 47.94% and 66.76% when $n = 20$. However, the average deviation is at most 1.2% when we have increasing demand through the time. We observe the worst performance of the heuristic when the demand values are constant. The average deviation is 4.54% and the maximum deviation is 55.70%.

An example solution. For a problem instance, when $n = 10$ and d_t is random, the optimal solution by CPLEX 12.4 is given below. The solution is composed of four connected subplans (1, 1, 1, 1), (2, 2, 2, 2), (3, 7, 3, 7) and (8, 10, 8, 10). All subplans have one partial production period.

$$Z^* = \left(\begin{array}{c|c|ccccc|ccc} 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 15 & 4.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.1 & 8 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 18 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 9 \end{array}\right)$$

6. Conclusion and future research directions

In this study, we present a model for ELS problem for a perishable item, where finite production capacities present. We assume non-decreasing concave cost functions (holding, backlogging, and production). Deterioration rates and holding cost functions are age-dependent, and backlogging cost functions are period-pair dependent.

We show the structural properties of the optimal solution, prove that it is composed of independent subplans, and therefore we propose a Dynamic Programing based heuristic. We suggest an approximation algorithm (FIND COST) to find a good solution for each subplan and combination pair. We discuss the performance of FIND COST heuristic. In [1], a DP algorithm that finds the optimal solution in $O(n^4)$ time is proposed. We show that a more realistic version of Hsu [1] with finite production capacities is NP-hard and we offer a good solution when the subplan and its positive production periods are given. As managerial insights, it is not economical to satisfy the demand in period t from a period k in two cases: when a period later than k (including k) is satisfied by a period between t and k (including t); and when a period between t and k is satisfied from a period before t.

As a future research, the cost structures and parameters such that cost calculations for all subplans can be handled in polynomial time, can be investigated. In that case, we have a polynomial time Dynamic Programing-based algorithm.

Acknowledgments

The authors would like to thank two anonymous reviewers and the Associate Editor for their comments on earlier versions of our paper.

References

[1] V.N. Hsu, An economic lot size model for perishable products with age-dependent inventory and backorder costs, IIE Trans. 35 (2003) 775–780.
[2] S.K. Goyal, B.C. Giri, Recent trends in modeling of deteriorating inventory, Eur. J. Oper. Res. 134 (1) (2001) 1–16.
[3] M. Bakker, J. Riezebos, R.H. Teunter, Review of inventory systems with deterioration since 2001, Eur. J. Oper. Res. 221 (2) (2012) 275–284.
[4] A.A. Taleizadeh, M. Nematollahi, An inventory control problem for deteriorating items with back-ordering and financial considerations, Appl. Math. Model. 38 (1) (2014) 93–109.
[5] S.S. Sanni, W.I.E. Chukwu, An economic order quantity model for items with three-parameter weibull distribution deterioration, ramp-type demand and shortages, Appl. Math. Model. 37 (23) (2013) 9698–9706.
[6] H.-M. Wee, Y.-S. Shum, Model development for deteriorating inventory in material requirement planning systems, Comput. Ind. Eng. 36 (1999) 219–225.
[7] S. Nahmias, Perishable Inventory Systems, Springer, New York, USA, 2011.
[8] H.M. Wagner, T.M. Whitin, Dynamic version of the economic lot size model, Manag. Sci. 5 (1958) 89–96.
[9] W.I. Zangwill, A backlogging model and a dynamic economic lot size production system – a network approach, Manag. Sci. 15 (1969) 506–527.
[10] G.R. Bitran, H.H. Yanasse, Computational complexity of the capacitated lot size problem, Manag. Sci. 28 (10) (1982) 1174–1186.
[11] W.I. Zangwill, Minimum concave cost flows in certain networks, Manag. Sci. 14 (1968) 429–450.
[12] L.M.A. Chan, A. Muriel, Z.J. Shen, D. Simchi-Levi, On the effectiveness of zero-inventory-ordering policies for the economic lot-sizing model with a class of piecewise linear cost structure, Oper. Res. 50 (6) (2002) 1058–1067.
[13] M. Florian, J.K. Lenstra, A.H.G. Rinnooy Kan, Deterministic production planning: algorithms and complexity, Manag. Sci. 26 (7) (1980) 669–679.
[14] M. Florian, M. Klein, Deterministic production planning with concave costs and capacity constraints, Manag. Sci. 18 (1) (1971) 12–22.
[15] V.N. Hsu, T.J. Lowe, Dynamic economic lot size models with period-pair-dependent backorder and inventory costs, Oper. Res. 49 (2) (2001) 316–321.
[16] Y. Friedman, Y. Hoch, A dynamic lot-size model with inventory deterioration, INFOR 16 (1978) 183–188.
[17] V.N. Hsu, Dynamic economic lot size model with perishable inventory, Manag. Sci. 46 (8) (2000) 1159–1169.
[18] L.Y. Chu, V.N. Hsu, Z.J.M. Shen, An economic lot-sizing problem with perishable inventory and economies of scale costs: approximation solutions and worst case analysis, Naval Res. Logist. 52 (6) (2005) 536–548.
[19] G.Q. Bai, Y.Z. Zhang, G.L. Dong, A note on an economic lot-sizing problem with perishable inventory and economies of scale costs: approximation solutions and worst case analysis, Int. J. Autom. Comput. 7 (2010) 132–136.
[20] M.S. Bazaraa, C.M. Shetty, Nonlinear Programming: Theory and Algorithms, John Wiley & Sons, Inc., New York, 1979.
[21] M. Önal, H.E. Romeijn, A. Sapra, W. van den Heuvel, The economic lot-sizing problem with perishable items and consumption order preference, Eur. J. Oper. Res. 244 (3) (2015) 881–891.
[22] R.K. Ahuja, T.L. Magnanti, J.B. Orlin, Network Flows: Theory, Algorithms and Applications, Prentice Hall, New Jersey, 1993.