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On flexible progressive censoring

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ABSTRACT

In life testing experiment under the setup of progressive Type II censoring we consider a new flexible scheme, in which the experimenter removes r_1^* ($r_1^* \leq r_1$) units from the experiment if the first failure occurs before some predefined time t_1, and removes r_1 units when the first failure occurs after time t_1. If the second failure occurs before the time t_2, r_2^* surviving items are removed at random, otherwise r_2 surviving units are removed, ($r_2^* \leq r_2$), and so on; finally, after the mth failure, all remaining items are removed. Under this setup of the experiment we study the distributions of failure times, which are entitled as flexible progressive Type II censored (FPC) order statistics. The simulation algorithm for generating FPC samples and an illustrative example for a special case of exponential distribution are also presented. Finally, a Monte Carlo study is also conducted for the expected termination time under FCP.

1. Introduction

During the last few years, in statistical literature, an increasing interest in inference based progressively censored data can be observed. There appeared numerous research and review papers and one monograph devoted to this subject following the early work of Cohen [1], where the first description of the model was made. The model of progressive Type II right censoring is of importance in the field of reliability and life testing. Under this censoring scheme n identical units are placed on a life test; after the first failure, r_1 surviving items are removed at random from further observation; after the next failure r_2 surviving items are removed at random, and so on. This experiment terminates at the time when the mth failure is observed and the remaining $r_m = n - r_1 - r_2 - \cdots - r_{m-1} - m$ surviving units are all removed. Thus, in this type of sampling, we observe m failures and $r_1 + r_2 + \cdots + r_m$ items are progressively censored, so that $n = m + (r_1 + r_2 + \cdots + r_m)$ is the number of units. $X^{\mathbf{R}}_{1:m:n} < X^{\mathbf{R}}_{2:m:n} < \cdots < X^{\mathbf{R}}_{m:m:n}$ describe the progressively censored failure times where $\mathbf{R} = (r_1, \ldots, r_m)$ denotes the censoring scheme. The joint probability density function (p.d.f.) of the progressively censored order statistics, $X^{\mathbf{R}}_{1:m:n}, X^{\mathbf{R}}_{2:m:n}, \ldots, X^{\mathbf{R}}_{m:m:n}$ is

$$f_{X^{\mathbf{R}}_{1:m:n}X^{\mathbf{R}}_{2:m:n}\cdots X^{\mathbf{R}}_{m:m:n}}(x_1, x_2, \ldots, x_m) = C\prod_{i=1}^{k} f(x_i)[1 - F(x_i)]^{r_i} \quad 0 < x_1 < x_2 < \cdots < x_m < \infty, \tag{1}$$

where $C = n(n - r_1 - 1)(n - r_1 - r_2 - 2)\cdots(n - r_1 - \cdots - r_{m-1} - m + 1)$ is the normalizing constant. For more details on the theory of progressive Type II censoring scheme, see [2].

The papers devoted to the progressive Type II censoring mainly are concentrated on point and interval estimation of the parameters, determination of optimal progressive censoring plans, characterizations of distributions, construction of

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reliability sampling plans. See for example, [3–22]. The recent discussion in [23] provides a comprehensive review of various developments pertaining to progressive censoring.

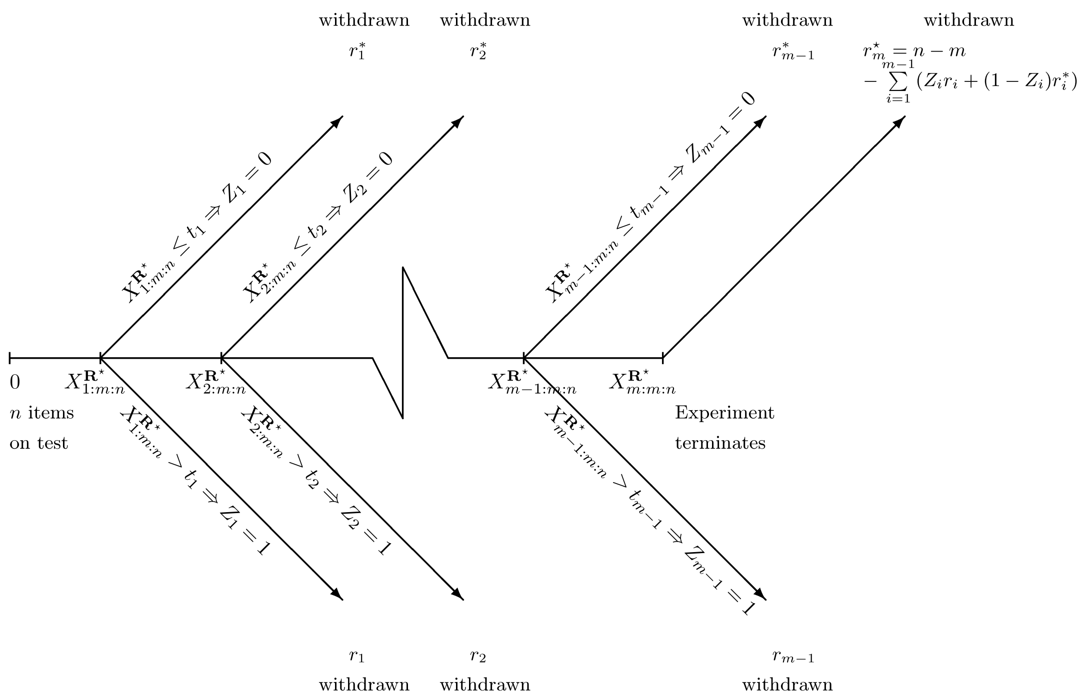
In some life testing experiments when n units are placed on a life test, depending on the length of first failure time, the experimenter may need to apply different censoring scheme. If an experimenter is not satisfied with the first failure time (for example, it comes much later than he was expecting) and desires to implement more flexible scheme and remove more units than should be removed after the first failure according to the censoring scheme, then the ordinary progressive Type II right censoring scheme will be not applicable. Therefore, for the described situation above more general and flexible censoring scheme must be considered. Now, assume that the experimenter removes r_1^* ($r_1^* \le r_1$) units from the experiment if the first failure occurs before some predefined time t_1, and removes r_1 units when the first failure occurs after time t_1. If the second failure occurs before the time t_2, r_2^* surviving items are removed at random, otherwise r_2 surviving units are removed, ($r_2^* \le r_2$), and so on; finally, after the mth failure, $r_m^* = r_m$ remaining items are removed.

In this short paper we consider a new model of progressive censoring and call it Flexible Type II Progressive Censoring (FPC) scheme. We consider $\mathbf{R}_{m-1} = (r_1, r_2, \ldots, r_{m-1})$, $\mathbf{R}^*_{m-1} = (r_1^*, r_2^*, \ldots, r_{m-1}^*)$ and $\mathbf{t}_{m-1} = (t_1, t_2, \ldots, t_{m-1})$, such that $r_i^* \le r_i$ and $0 \le t_1 < t_2 < \cdots < t_{m-1}$. We need also conditions $n - \sum_{i=1}^{m-1}(r_i + 1) \ge 1$ and $n - \sum_{i=1}^{m-1}(r_i^* + 1) \ge 1$. Under FPC scheme, n units with life times having cumulative distribution function (c.d.f.) $F(x)$ and probability density function (p.d.f.) $f(x)$ are placed on a test at time zero and m failures are expected to be observed.

Define random variables $Z_1, Z_2, \ldots, Z_{m-1}$ as follows: $Z_i = 0$ if the ith failure occurs before time t_i and $Z_i = 1$ otherwise. More precisely, if m failure times are denoted by $(X^{\mathbf{R}^\star}_{1:m:n}, X^{\mathbf{R}^\star}_{2:m:n}, \ldots, X^{\mathbf{R}^\star}_{m:m:n})$, then

$$Z_i = \begin{cases} 0 & \text{if } X^{\mathbf{R}^\star}_{i:m:n} \le t_i \\ 1 & \text{if } X^{\mathbf{R}^\star}_{i:m:n} > t_i, \end{cases} \quad i = 1, 2, \ldots, m-1.$$

After the first failure $\mathbf{r}_1^\star = Z_1 r_1 + (1 - Z_1) r_1^*$ of the surviving units are randomly selected and removed. Then, immediately following the second observed failure, $\mathbf{r}_2^\star = Z_2 r_2 + (1 - Z_2) r_2^*$ of the surviving units are randomly selected and removed and so on; finally, after the mth failure, the $\mathbf{r}_m^\star = n - m - \sum_{i=1}^{m-1}[Z_i r_i + (1 - Z_i) r_i^*] = n - m - \sum_{i=1}^{m-1} \mathbf{r}_i^\star$ remaining items are withdrawn. If $r_i = r_i^*$ for $i = 1, 2, \ldots, m-1$, then the FPC reduces to the progressive Type II censoring. The FPC scheme may be described schematically as follows:



In practical applications, an experimenter may be interested to know whether the test can be completed within a specified time. This information is important for an experimenter to choose an appropriate sampling plan because the time required to complete a test is directly related to the cost. Wu et al. [15] proposed the estimated expected test time for Pareto Distribution for progressive censoring data. In this paper, we also investigate the simulated termination times for exponential distribution under FPC.

2. The model

Suppose n independent units are placed on a test with p.d.f. $f(x;\theta)$ and c.d.f. $F(x;\theta)$. Under the FPC scheme with $\mathbf{R}_{m-1}=(r_1,r_2,\ldots,r_{m-1})$, $\mathbf{R}^*_{m-1}=(r_1^*,r_2^*,\ldots,r_{m-1}^*)$ and $\mathbf{t}_{m-1}=(t_1,t_2,\ldots,t_{m-1})$, the observable data consist of $(\mathbf{X}^{\mathbf{R}^\star},\mathbf{Z},\mathbf{R}^\star)$ or $(\mathbf{X}^{\mathbf{R}^\star},\mathbf{Z})$, where $\mathbf{X}^{\mathbf{R}^\star}=(X^{\mathbf{R}^\star}_{1:m:n},X^{\mathbf{R}^\star}_{2:m:n},\ldots,X^{\mathbf{R}^\star}_{m:m:n})$ and $\mathbf{Z}=(Z_1,\ldots,Z_m)$ with $Z_i=1$ if $X^{\mathbf{R}^\star}_{i:m:n}>t_i$ and $Z_i=0$ if $X^{\mathbf{R}^\star}_{i:m:n}\le t_i$, for $i=1,2,\ldots,m-1$ and $Z_m=1$ and $\mathbf{R}^\star=(\mathbf{r}_1^\star,\ldots,\mathbf{r}_m^\star)$.

Let $x_0\equiv 0$, $\mathbf{x}=(x_1,x_2,\ldots,x_m)$, $\mathbf{z}=(z_1,z_2,\ldots,z_m)$, $(0,0]\equiv\emptyset$, $\mathbf{r}_i^\star=z_ir_i+(1-z_i)r_i^*$ for $i=1,2,\ldots,m$, $t_m\equiv 0$, $\bar F=1-F$. Then using the definition of FPC order statistics and repeating the proof of Balakrishnan and Aggarwala [2, Page 8], taking into account that after ith failure we remove $\mathbf{r}_i^\star$ units, it is not difficult to observe that the joint p.d.f. of $(\mathbf{X}^{\mathbf{R}^\star},\mathbf{Z})$ is:

$$f_{\mathbf{X}^{\mathbf{R}^\star},\mathbf{Z}}(\mathbf{x},\mathbf{z})=C\prod_{i=1}^{m}f(x_i)[\bar F(x_i)]^{\mathbf{r}_i^\star}\{(1-z_i)I_{(0,t_i]}(x_i)+z_iI_{(t_i,\infty)}(x_i)\}\{I_{(x_{i-1},\infty)}(x_i)\}. \tag{2}$$

C is a normalizing constant satisfying

$$\frac{1}{C}=\sum_{\mathbf{z}\in Q_{\mathbf{z}}}\int_{\mathcal{A}_1\cap\mathcal{A}_2}\prod_{i=1}^{m}f(x_i)[\bar F(x_i)]^{\mathbf{r}_i^\star}\,\mathrm{d}\mathbf{x}, \tag{3}$$

where

$$\mathcal{A}_1=\{(x_1,x_2,\ldots,x_m):0<x_1<x_2<\cdots<x_m<\infty\},$$
$$\mathcal{A}_2=\{(x_1,x_2,\ldots,x_m):x_i\in(0,t_i]\text{ for }z_i=0\text{ and }x_i\in(t_i,\infty)\text{ for }z_i=1\},$$
$$Q_{\mathbf{z}}=\{(z_1,z_2,\ldots,z_m):z_i=0\text{ or }1\text{ for }i=1,2,\ldots,m-1,z_m=1\}.$$

Theorem 1. *For a FPC model $\mathbf{t}_{m-1}=(t_1,0,\ldots,0)$ the joint p.d.f of $(\mathbf{X}^{\mathbf{R}^\star},\mathbf{Z})$ is*

$$f_{\mathbf{X}^{\mathbf{R}^\star},Z_1}(\mathbf{x},z_1)=Cf(x_1)[\bar F(x_1)]^{\mathbf{r}_1^\star}\{(1-z_1)I_{(0,t_1]}(x_1)+z_iI_{(t_1,\infty)}(x_1)\}\{I_{(0,\infty)}(x_1)\}\prod_{i=2}^{m}f(x_i)[\bar F(x_i)]^{r_i}\{I_{(x_{i-1},\infty)}(x_i)\}, \tag{4}$$

where

$$\begin{aligned}C&=\{(r_1^*+r_2+\cdots+r_m+m)(r_1+r_2+\cdots+r_m+m)\\&\quad\times(r_2+\cdots+r_m+m-1)\cdots(r_{m-1}+r_m+2)(r_m+1)\}/\{(r_1+r_2+\cdots+r_m+m)\\&\quad-(r_1+r_2+\cdots+r_m+m)\bar F^{k_1}(t_1)+(r_1^*+r_2+\cdots+r_m+m)\bar F^{k_2}(t_1)\}\\k_1&=(r_1^*+r_2+\cdots+r_m+m)(r_2+\cdots+r_m+m-1)(r_3+\cdots+r_m+m-2)\cdots(r_{m-1}+r_m+2)(r_m+1)\end{aligned} \tag{5}$$

and

$$k_2=(r_1+r_2+\cdots+r_m+m)(r_2+\cdots+r_m+m-1)\times\cdots\times(r_{m-1}+r_m+2)(r_m+1).$$

Proof. From (3), we immediately have

$$\begin{aligned}\frac{1}{C}&=\int_{t_1}^{\infty}\int_{x_1}^{\infty}\cdots\int_{x_{m-1}}^{\infty}\prod_{i=1}^{m}f(x_i)[\bar F(x_i)]^{r_i}\mathrm{d}x_m\cdots\mathrm{d}x_2\mathrm{d}x_1\\&\quad+\int_{0}^{t_1}\int_{x_1}^{\infty}\cdots\int_{x_{m-1}}^{\infty}f(x_1)[\bar F(x_1)]^{r_1^*}\prod_{i=2}^{m}f(x_i)[\bar F(x_i)]^{r_i}\mathrm{d}x_m\cdots\mathrm{d}x_2\mathrm{d}x_1.\end{aligned}$$

Upon performing the necessary integrations, we obtain (5).

If we choose $t_1=0$ and $r_1=r_1^*$, then (4) reduces to the joint p.d.f of ordinary progressive Type II censored order statistics given in (1). □

3. Simulation algorithm for generating FPC samples

Consider the random variable X with p.d.f. $f(x;\theta)$ and c.d.f. $F(x;\theta)$. Under the FPC scheme with $\mathbf{R}_{m-1}=(r_1,r_2,\ldots,r_{m-1})$, $\mathbf{R}^*_{m-1}=(r_1^*,r_2^*,\ldots,r_{m-1}^*)$ and $\mathbf{t}_{m-1}=(t_1,t_2,\ldots,t_{m-1})$, the FPC data $(\mathbf{X}^{\mathbf{R}^\star},\mathbf{Z},\mathbf{R}^\star)$ can be generated as follows:

Step 1. Generate i.i.d. samples, $X_1,X_2,\ldots,X_n$, from $F(x;\theta)$;

Step 2. Determine the order statistics $X_{1:n},X_{2:n},\ldots,X_{n:n}$;

Step 3. Set $N=\{1,2,\ldots,n\}$ and $i=1$;

Step 4. Let $k_i = \min(N)$ and define $X^{\mathbf{R}^\star}_{i:m:n} = X_{k_i:n}$;
Step 5. If $X^{\mathbf{R}^\star}_{i:m:n} > t_i$, then $Z_i = 1$, else $Z_i = 0$;
Step 6. Let $\mathbf{r}^\star_i = Z_i r_i + (1 - Z_i) r^*_i$;
Step 7. Choose a without-replacement sample $\mathcal{R}^\star_i \subseteq N \setminus \{k_i\}$ with $|\mathcal{R}^\star_i| = \mathbf{r}^\star_i$ at random (from $N \setminus \{k_i\}$);
Step 8. Set $N = N \setminus [\{k_i\} \cup \mathcal{R}^\star_i]$ and $i = i + 1$;
Step 9. If $i \le m - 1$, then go to Step 4, else the Next;
Step 10. Let $k_m = \min(N)$ and define $X^{\mathbf{R}^\star}_{m:m:n} = X_{k_m:n}$;
Step 11. Let $\mathbf{r}^\star_m = n - m - \sum_{i=1}^{m-1} \mathbf{r}^\star_i$, $Z_m = 1$ and stop.

Evidently, the FPC data is then given by $(\mathbf{X}^{\mathbf{R}^\star}, \mathbf{Z}, \mathbf{R}^\star)$ where

$$\mathbf{X}^{\mathbf{R}^\star} = (X^{\mathbf{R}^\star}_{1:m:n}, X^{\mathbf{R}^\star}_{2:m:n}, \ldots, X^{\mathbf{R}^\star}_{m:m:n}) = (X_{k_1:n}, X_{k_2:n}, \ldots, X_{k_n:n}),$$
$$\mathbf{Z} = (Z_1, Z_2, \ldots, Z_m)$$

and

$$\mathbf{R}^\star = (\mathbf{r}^\star_1, \mathbf{r}^\star_2, \ldots, \mathbf{r}^\star_m).$$

4. Marginal distributions

Let $(\mathbf{X}^{\mathbf{R}^\star}, \mathbf{Z}, \mathbf{R}^\star)$ denote the FPC data from a continuous population with c.d.f. $F(x)$ and p.d.f. $f(x)$. It is clear that

$$f_{X^{\mathbf{R}^\star}_{1:m:n}}(x) = nf(x)[1 - F(x)]^{n-1}$$

and

$$P(Z_1 = z_1) = \begin{cases} P(R^\star_1 = r^*_1) = \displaystyle\int_0^{t_1} f_{X^{\mathbf{R}^\star}_{1:m:n}}(x)\mathrm{d}x, & \text{if } z_1 = 0 \\ P(R^\star_1 = r_1) = \displaystyle\int_{t_1}^{\infty} f_{X^{\mathbf{R}^\star}_{1:m:n}}(x)\mathrm{d}x, & \text{if } z_1 = 1. \end{cases}$$

It is not difficult to observe that the joint distribution of FPC order statistics $X^{\mathbf{R}^\star}_{1:m:n}, X^{\mathbf{R}^\star}_{2:m:n}, \ldots, X^{\mathbf{R}^\star}_{m:m:n}$, given $(Z_1 = z_1, Z_2 = z_2, \ldots, Z_{m-1} = z_{m-1})$ is the same with ordinary progressive Type II censored order statistics $X^{\mathbf{r}^\star}_{1:m:n}, X^{\mathbf{r}^\star}_{2:m:n}, \ldots, X^{\mathbf{r}^\star}_{m:m:n}$ with censoring scheme $\mathbf{r}^\star = (r^\star_1, \mathbf{r}^\star_2, \ldots, \mathbf{r}^\star_m)$, where $\mathbf{r}^\star_i = z_i r_i + (1 - z_i) r^*_i$. From the Markovian property of ordinary progressive Type II censored order statistics it follows that marginal distribution of $X^{\mathbf{r}^\star}_{i:m:n}$, $2 \le i \le m$, does not depend on $(\mathbf{r}^\star_i, \mathbf{r}^\star_{i+1}, \ldots, \mathbf{r}^\star_m)$ [2]. Upon using this fact and Kamps–Cramer representation for marginal distribution of ordinary progressive Type II censored order statistics (see [24]) we can write the conditional p.d.f. of FPC order statistic $X^{\mathbf{R}^\star}_{i:m:n}$, for $i = 2, 3, \ldots, m$ as follows

$$f_{X^{\mathbf{R}^\star}_{i:m:n}|Z_1,Z_2,\ldots,Z_{i-1}}(x) = f_{X^{\mathbf{R}^\star}_{i:m:n}|\mathbf{Z}_{i-1}}(x) = C_{i-1} f(x) \sum_{k=1}^{i} a_{k,i}[1 - F(x)]^{\gamma_k - 1}, \tag{6}$$

where $\mathbf{Z}_i = (Z_1, Z_2, \ldots, Z_i)$, $C_{i-1} = \prod_{j=1}^{i} \gamma_j$, $a_{k,i} = \prod_{\substack{j=1 \\ j \ne k}}^{i} (1/\gamma_j - \gamma_k)$, $1 \le k \le i \le m$, $m \ge 2$, $\gamma_j = n - \sum_{i=1}^{j-1} r^\star_i - j + 1$, $\gamma_1 = n$, $n = m + \sum_{j=1}^{m} r^\star_j$ and the empty product $\prod_\emptyset$ is defined to be 1.

Also, we have

$$P(Z_i = z_i | Z_1 = z_1, Z_2 = z_2, \ldots, Z_{i-1} = z_{i-1}) = P(Z_i = z_i | \mathbf{Z}_{i-1} = \mathbf{z}_{i-1}) = \begin{cases} \displaystyle\int_0^{t_i} f_{X^{\mathbf{R}^\star}_{i:m:n}|\mathbf{Z}_{i-1}}(x)\mathrm{d}x, & \text{if } z_i = 0 \\ \displaystyle\int_{t_i}^{\infty} f_{X^{\mathbf{R}^\star}_{i:m:n}|\mathbf{Z}_{i-1}}(x)\mathrm{d}x, & \text{if } z_i = 1. \end{cases}$$

Therefore, the p.d.f. of FPC order statistic $X^{\mathbf{R}^\star}_{i:m:n}$ for $i = 2, 3, \ldots, m$ is

$$f_{X^{\mathbf{R}^\star}_{i:m:n}}(x) = \sum_{\mathbf{z}_{i-1} \in Q_{\mathbf{Z}_{i-1}}} f_{X^{\mathbf{R}^\star}_{i:m:n}|\mathbf{Z}_{i-1}}(x) P(\mathbf{Z}_{i-1} = \mathbf{z}_{i-1}), \tag{7}$$

where

$$\begin{aligned} P(\mathbf{Z}_i = \mathbf{z}_i) &= P(Z_1 = z_1, Z_2 = z_2, \ldots, Z_i = z_i) \\ &= \prod_{j=1}^{i} P(Z_j = z_j | \mathbf{Z}_{j-1} = \mathbf{z}_{j-1}), \end{aligned}$$

with $P(Z_1 = z_1|Z_0 = z_0) = P(Z_1 = z_1)$ and

$$Q_{\mathbf{z}_i} = \{(z_1, z_2, \ldots, z_i) : z_j = 0 \text{ or } 1 \text{ for } j = 1, 2, \ldots, i\}, \quad i = 1, 2, \ldots, m-1;\ \mathbf{z} = (z_1, z_2, \ldots, z_m),\ Q_{\mathbf{z}_m} = Q_{\mathbf{z}}.$$

Illustrative example. Let X be a random variable with exponential p.d.f. $\mathrm{Exp}(\theta)$. The p.d.f. of X is

$$f(x;\theta) = \begin{matrix} \frac{1}{\theta}\mathrm{e}^{-x/\theta} & x \geq 0, \\ 0, & \text{otherwise.} \end{matrix}$$

In this special case using (6) and (7) we can obtain p.d.f.'s and expected values of FPC order statistics. It is clear that

$$f_{X^{\mathbf{R}^\star}_{1:m:n}}(x) = \frac{n}{\theta}\mathrm{e}^{-nx/\theta},$$

$$E(X^{\mathbf{R}^\star}_{1:m:n}) = \frac{\theta}{n}.$$

It is not difficult to verify that

$$P(Z_1 = z_1) = \begin{cases} P(R_1^\star = r_1^*) = 1 - \mathrm{e}^{-nt_1/\theta}, & \text{if } z_1 = 0 \\ P(R_1^\star = r_1) = \mathrm{e}^{-nt_1/\theta}, & \text{if } z_1 = 1 \end{cases}$$

and

$$E(Z_1) = \mathrm{e}^{\frac{-nt_1}{\theta}}.$$

For $i = 2, 3, \ldots, m$, we have

$$f_{X^{\mathbf{R}^\star}_{i:m:n}|\mathbf{Z}_{i-1}}(x) = \frac{C_{i-1}}{\theta}\sum_{k=1}^{i} a_{k,i}\mathrm{e}^{\frac{-\gamma_k x}{\theta}},$$

also, we have

$$P(Z_i = z_i|\mathbf{Z}_{i-1} = \mathbf{z}_{i-1}) = \begin{cases} 1 - C_{i-1}\sum_{k=1}^{i}\frac{a_{k,i}}{\gamma_k}\mathrm{e}^{\frac{-\gamma_k t_i}{\theta}}, & \text{if } z_i = 0 \\ C_{i-1}\sum_{k=1}^{i}\frac{a_{k,i}}{\gamma_k}\mathrm{e}^{\frac{-\gamma_k t_i}{\theta}}, & \text{if } z_i = 1. \end{cases}$$

Thus the p.d.f. of FPC order statistic $X^{\mathbf{R}^\star}_{i:m:n}$ for $i = 2, 3, \ldots, m$ is

$$f_{X^{\mathbf{R}^\star}_{i:m:n}}(x) = \sum_{\mathbf{z}_{i-1}\in Q_{\mathbf{z}_{i-1}}} f_{X^{\mathbf{R}^\star}_{i:m:n}|\mathbf{Z}_{i-1}}(x)P(\mathbf{Z}_{i-1} = \mathbf{z}_{i-1}),$$

where

$$\begin{aligned} P(Z_1 = z_1, Z_2 = z_2, \ldots, Z_i = z_i) &= P(\mathbf{Z}_i = \mathbf{z}_i) \\ &= \prod_{j=1}^{i} P(Z_j = z_j|\mathbf{Z}_{j-1} = \mathbf{z}_{j-1}), \end{aligned}$$

with $P(Z_1 = z_1|Z_0 = z_0) = P(Z_1 = z_1)$.

For the expected value of $X^{\mathbf{R}^\star}_{i:m:n}$ we have

$$E(X^{\mathbf{R}^\star}_{i:m:n}|\mathbf{Z}_{i-1} = \mathbf{z}_{i-1}) = C_{i-1}\sum_{k=1}^{i}\frac{a_{k,i}}{\gamma_k},$$

$$E(X^{\mathbf{R}^\star}_{i:m:n}) = \sum_{\mathbf{z}_{i-1}\in Q_{\mathbf{z}_{i-1}}} P(\mathbf{Z}_{i-1} = \mathbf{z}_{i-1})C_{i-1}\sum_{k=1}^{i}\frac{a_{k,i}}{\gamma_k}.$$

The conditional expected value of Z_i given $\mathbf{Z}_{i-1} = \mathbf{z}_{i-1}$ is obtained as

$$E(Z_i|\mathbf{Z}_{i-1} = \mathbf{z}_{i-1}) = C_{i-1}\sum_{k=1}^{i}\frac{a_{k,i}}{\gamma_k}\mathrm{e}^{\frac{-\gamma_k t_i}{\theta}}$$

and the expected value of Z_i is

$$E(Z_i) = \sum_{\mathbf{z}_{i-1}\in Q_{\mathbf{z}_{i-1}}} P(\mathbf{Z}_{i-1} = \mathbf{z}_{i-1})C_{i-1}\sum_{k=1}^{i}\frac{a_{k,i}}{\gamma_k}\mathrm{e}^{\frac{-\gamma_k t_i}{\theta}}.$$

Table 1
FPC sample ($m = 10, n = 35$).

i	1	2	3	4	5	6	7	8	9	10
z_i	1	0	1	1	0	0	0	0	0	1
$\mathbf{r}_i^\star$	4	0	0	2	2	1	2	3	1	10
$x_{i:m:n}^{\mathbf{R}^\star}$	0.237	0.263	0.826	0.943	1.312	1.392	1.425	1.740	2.073	2.405
$x_{i:m:n}^{\mathbf{R}^*}$	0.066	0.283	0.361	0.450	0.685	1.031	1.210	1.530	1.674	1.979
$x_{i:m:n}^{\mathbf{R}}$	0.383	0.526	1.776	1.783	1.814	2.045	2.692	3.201	3.272	4.352

Table 2
FPC sample ($m = 8, n = 20$).

i	1	2	3	4	5	6	7	8
z_i	0	1	1	1	1	1	1	1
$\mathbf{r}_i^\star$	2	2	0	1	0	2	1	4
$x_{i:m:n}^{\mathbf{R}^\star} = x_{i:m:n}^{\mathbf{R}^*}$	0.075	0.174	0.413	0.537	0.857	0.975	1.045	1.442
$x_{i:m:n}^{\mathbf{R}}$	0.260	0.384	0.484	1.040	1.132	2.267	2.611	2.832

Table 3
The simulated termination times for some small, moderate and large values of n and m and different combinations of $\mathbf{R}_{m-1}$; $\mathbf{R}^*_{m-1}$ and $\mathbf{t}_{m-1}$ when X has an exponential distribution with parameter $\theta = 5$.

n	m	$\mathbf{R}_{m-1}$; $\mathbf{R}^*_{m-1}$; $\mathbf{t}_{m-1}$	$\overset{\approx}{X}{}^{\mathbf{R}^*}_{m:m:n}$	$\overset{\approx}{X}{}^{\mathbf{R}^\star}_{m:m:n}$	$\overset{\approx}{X}{}^{\mathbf{R}}_{m:m:n}$
20	5	(15, 0, 0, 0); (10, 0, 0, 0); (0.2, 0, 0, 0)	2.966	6.503	10.728
		(15, 0, 0, 0); (10, 0, 0, 0); (0.5, 0, 0, 0)	2.963	4.168	10.804
		(15, 0, 0, 0); (7, 0, 0, 0); (0.3, 0, 0, 0)	2.193	4.934	10.750
		(7, 0, 8, 0); (3, 0, 4, 0); (0.1, 0, 1, 0)	1.940	4.778	8.723
		(7, 0, 8, 0); (3, 0, 4, 0); (0.25, 0, 1, 0)	1.986	3.888	8.724
		(0, 0, 15, 0); (0, 0, 7, 0); (0, 0, 0.8, 0)	1.830	4.747	8.291
		(0, 0, 15, 0); (0, 0, 7, 0); (0, 0, 1.2, 0)	1.862	2.939	8.504
		(0, 0, 0, 12); (0, 0, 0, 5); (0, 0, 0, 1.3)	1.496	1.756	2.297
		(0, 0, 0, 12); (0, 0, 0, 5); (0, 0, 0, 1.7)	1.517	1.609	2.360
	10	(5, 5, 0, …, 0); (2, 3, 0, …, 0); (0.2, 0.5, 0, …, 0)	5.065	8.639	14.243
		(5, 5, 0, …, 0); (2, 3, 0, …, 0); (0.3, 0.8, 0, …, 0)	4.971	6.990	14.510
		(0, …, 0, 2, 3, 5); (0, …, 0, 1, 2, 2); (0, …, 0.2, 1.5, 3)	3.804	6.344	8.101
	15	(5, 0, …, 0); (3, 0, …, 0); (0.15, 0, …, 0)	9.779	13.408	16.625
50	20	(30, 0, …, 0); (15, 0, …, 0); (0.15, 0, …, 0)	4.091	7.461	17.990
		(30, 0, …, 0); (15, 0, …, 0); (0.5, 0, …, 0)	4.049	4.155	17.420
		(10, 10, 10, 0, …, 0); (5, 5, 5, 0, …, 0); (0.1, 0.25, 0.3, 0, …, 0)	4.001	7.858	17.627
	25	(25, 0, …, 0); (15, 0, …, 0); (0.15, 0, …, 0)	6.096	9.003	19.078
		(25, 0, …, 0); (15, 0, …, 0); (0.25, 0, …, 0)	6.062	7.093	18.945
		(10, 5, 10, 0, …, 0); (5, 2, 5, 0, …, 0); (0.1, 0.15, 0.25, 0, …, 0)	5.155	10.406	18.533
100	30	(70, 0, …, 0); (30, 0, …, 0); (0.1, 0, …, 0)	2.741	5.431	19.708
	50	(40, 0, …, 0); (15, 0, …, 0); (0.1, 0, …, 0)	4.389	5.113	8.725
	70	(25, 0, …, 0); (10, 0, …, 0); (0.1, 0, …, 0)	7.413	8.305	13.039

Example 1. By using statistical software **R** the data are generated from Exp(5) distribution. For $m = 10, n = 35$, providing a flexible progressive Type II censoring scheme $\mathbf{R}_{m-1} = (4, 1, 0, 2, 5, 3, 2, 4, 2)$, $\mathbf{R}^*_{m-1} = (1, 0, 0, 0, 2, 1, 2, 3, 1)$ and $\mathbf{t}_{m-1} = (0.2, 0.5, 0.8, 0.9, 1.5, 2, 2.1, 2.6, 3)$ by using simulation algorithm given in Section 3, the FPC data are obtained and presented in Table 1.

The FPC data can be used in the model (2) for construction estimators of a parameter θ of underlying distribution based on FPC sample and also can be used for other inferential procedures.

Example 2. The data are generated from Exp(2) distribution and for $m = 8, n = 20$, $\mathbf{R}_{m-1} = (5, 2, 0, 1, 0, 2, 1)$, $\mathbf{R}^*_{m-1} = (2, 2, 0, 1, 0, 2, 1)$ and $\mathbf{t}_{m-1} = (0.1, 0, \dots, 0)$ the FPC data are obtained. The FPC data are presented in Table 2.

The FPC data can be used in the model (4) for construction estimators for a parameter θ of underlying distribution based on FPC sample and also can be used for other inferential procedures.

5. Expected test time

In practical applications, it is often useful to have an idea of the termination time of the whole test. The information is important for an experimenter to choose an appropriate sampling; because the time required to complete an experiment has direct implication on the cost. For progressively Type II censoring and FPC sampling, the termination point for the experiment

Table 4
The simulated termination times for some small, moderate and large values of n and m and different combinations of $\mathbf{R}_{m-1}$; $\mathbf{R}^*_{m-1}$ and $\mathbf{t}_{m-1}$ when X has an exponential distribution with parameter $\theta = 10$.

n	m	$\mathbf{R}_{m-1}$; $\mathbf{R}^*_{m-1}$; $\mathbf{t}_{m-1}$	$\overset{\approx}{X}{}^{\mathbf{R}^*}_{m:m:n}$	$\overset{\approx}{X}{}^{\mathbf{R}^\star}_{m:m:n}$	$\overset{\approx}{X}{}^{\mathbf{R}}_{m:m:n}$
20	5	(15, 0, 0, 0); (10, 0, 0, 0); (0.9, 0, 0, 0)	5.980	8.627	21.243
		(15, 0, 0, 0); (10, 0, 0, 0); (1.1, 0, 0, 0)	6.069	7.613	21.072
		(15, 0, 0, 0); (7, 0, 0, 0); (0.3, 0, 0, 0)	4.228	13.815	21.322
		(7, 0, 8, 0); (3, 0, 4, 0); (0.8, 0, 2, 0)	3.869	6.394	17.189
		(7, 0, 8, 0); (3, 0, 4, 0); (0.9, 0, 2.5, 0)	3.876	5.561	17.391
		(0, 0, 15, 0); (0, 0, 7, 0); (0, 0, 1, 0)	3.672	12.870	16.526
		(0, 0, 15, 0); (0, 0, 7, 0); (0, 0, 1.3, 0)	3.717	10.571	15.694
		(0, 0, 0, 12); (0, 0, 0, 5); (0, 0, 0, 2.5)	3.071	3.534	4.510
		(0, 0, 0, 12); (0, 0, 0, 5); (0, 0, 0, 3.8)	3.103	3.209	4.647
	10	(5, 5, 0, …, 0); (2, 3, 0, …, 0); (0.2, 1.5, 0, …, 0)	9.922	17.328	28.375
		(5, 5, 0, …, 0); (2, 3, 0, …, 0); (0.9, 1.8, 0, …, 0)	10.095	12.640	28.420
		(0, …, 0, 2, 3, 5); (0, …, 0, 1, 2, 2); (0, …, 1.8, 3.5, 4)	8.559	15.485	22.739
	15	(5, 0, …, 0); (3, 0, …, 0); (0.7, 0, …, 0)	19.540	22.442	33.191
50	20	(30, 0, …, 0); (15, 0, …, 0); (0.4, 0, …, 0)	8.219	11.601	35.804
		(30, 0, …, 0); (15, 0, …, 0); (0.6, 0, …, 0)	8.202	9.144	35.638
		(10, 10, 10, 0, …, 0); (5, 5, 5, 0, …, 0); (0.3, 0.6, 1, 0, …, 0)	8.108	11.637	35.424
	25	(25, 0, …, 0); (15, 0, …, 0); (0.3, 0, …, 0)	12.244	18.612	38.450
		(25, 0, …, 0); (15, 0, …, 0); (0.5, 0, …, 0)	12.086	13.593	37.954
		(10, 5, 10, 0, …, 0); (5, 2, 5, 0, …, 0); (0.3, 0.5, 0.8, 0, …, 0)	10.508	16.234	38.774
100	30	(70, 0, …, 0); (30, 0, …, 0); (0.25, 0, …, 0)	5.525	8.488	39.718
	50	(40, 0, …, 0); (15, 0, …, 0); (0.15, 0, …, 0)	8.768	10.924	17.385
	70	(25, 0, …, 0); (10, 0, …, 0); (0.1, 0, …, 0)	14.830	18.952	26.188

is given by the expectation of the mth order statistic ($E(X^{\mathbf{R}}_{m:m:n})$ and $E(X^{\mathbf{R}^*}_{m:m:n})$ for ordinary progressively Type II censoring and $E(X^{\mathbf{R}^\star}_{m:m:n})$ for FPC in this paper). To compare $E(X^{\mathbf{R}}_{m:m:n})$, $E(X^{\mathbf{R}^*}_{m:m:n})$ and $E(X^{\mathbf{R}^\star}_{m:m:n})$, we compute the simulated termination times for two ordinary progressively Type II censoring schemes ($\mathbf{R} = (r_1, r_2, \ldots, r_m)$ and $\mathbf{R}^* = (r_1^*, r_2^*, \ldots, r_m^*)$) and FPC scheme ($\mathbf{R}^\star = (r_1^\star, r_2^\star, \ldots, r_m^\star)$). We use the $\overset{\approx}{X}{}^{\mathbf{R}}_{m:m:n}$, $\overset{\approx}{X}{}^{\mathbf{R}^*}_{m:m:n}$ and $\overset{\approx}{X}{}^{\mathbf{R}^\star}_{m:m:n}$ notations for the simulated termination times, respectively. Thus, we simulate 1000 samples from both ordinary progressively Type II censoring and FCP with exponential p.d.f. Exp(θ), for different combinations of n; m; θ; $\mathbf{R}_{m-1}$; $\mathbf{R}^*_{m-1}$ and $\mathbf{t}_{m-1}$ and obtain the simulated termination times. The simulation results are also shown in Tables 3 and 4. From Tables 3 and 4, it is easy to see that $\overset{\approx}{X}{}^{\mathbf{R}^*}_{m:m:n} \le \overset{\approx}{X}{}^{\mathbf{R}^\star}_{m:m:n} \le \overset{\approx}{X}{}^{\mathbf{R}}_{m:m:n}$; from Table 3, for example, for X having exponential distribution with parameter $\theta = 5$ and $n = 50$, $m = 25$, if we choose the censoring schemes as $\mathbf{R}_{m-1} = (25, 0, \ldots, 0)$ and $\mathbf{R}^*_{m-1} = (15, 0, \ldots, 0)$, then the simulated termination times will be 6.096 and 19.078, respectively. In the case of the FPC scheme, the simulated termination time will be 9.003. From Table 4, with X having exponential distribution with parameter $\theta = 10$ and $n = 20$, $m = 10$, if the censoring schemes are selected as $\mathbf{R}_{m-1} = (5, 5, 0, \ldots, 0)$ and $\mathbf{R}^*_{m-1} = (2, 3, 0, \ldots, 0)$, then we observe that the simulated termination times will be 9.922 and 28.375, respectively. In the case of the FPC scheme, the simulated termination time will be 17.328. Thus, FPC scheme, balances this situation and results in the simulated termination times and these results provide important information to practitioners planning a life test.

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