Journal of Computational and Applied Mathematics 260 (2014) 99–102

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A new characterization of the power distribution

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ARTICLE INFO

Article history:
Received 13 February 2013
Received in revised form 10 April 2013

Keywords:
Characterization
Power distribution
Order statistics
Choquet–Deny theorem
Ranked set sample

ABSTRACT

A new characterization for the power function distribution is obtained which is based on products of order statistics. This result may be considered as a generalization of some recent results for contractions. The result is obtained by applying a new variant of the Choquet–Deny theorem. We note that in this new result the product consists of order statistics from independent samples. This characterization result may also be interpreted in terms of some special scheme of ranked set sampling.

1. Introduction

In recent years one can find many different and interesting characterization results in the literature. For some particular examples, one may refer to [1–6], among others.

The power distribution has applications in finance and economics and is used to model reliability growth of complex systems or reliability of repairable systems (see, for example, [7,8]). In this paper, a new characterization of the power distribution, based on independent order statistics, is obtained. An interesting point of this new characterization result is that it is based on order statistics from independent sampled sets. In this respect, the obtained result may be considered as one of the first characterization results obtained by using order statistics from independent samples. We also note that the proof is given by using a new variant of the Choquet–Deny theorem (see, for example, [9]).

The paper is organized as follows. In Section 2, we introduce some basic notation and related results from the literature. In Section 3, ranked set samples are briefly introduced in order to express the characterization result with an appropriate notation. Then in the next section the main result is presented.

2. Basic notation and preliminaries

Consider three independent random variables X, Y, and U, where U has some known distribution. There are several recent characterization results, which may be considered as special cases of relation (1):

$$X \stackrel{d}{=} YU. \tag{1}$$

In its most basic form U can be assumed to have a uniform distribution concentrated on $(0, 1)$. In this case, relation (1) is an example of a contraction. These type of relations have some applications like in economic modeling and reliability, for example. Some of the first results of this type were obtained, among others, by [10–12].

We will write $X \sim Pow(\alpha)$ if $F_X(x) = x^\alpha,\ \alpha > 0,\ x \in (0, 1)$, and $Y \sim Par(\alpha)$, if $F_Y(y) = 1 - y^{-\alpha},\ \alpha > 0,\ y \in (1, \infty)$. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the order statistics for random variables $X_1, X_2, \ldots, X_n$.

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0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.09.068

There are many interesting distributional relations and characterization results based on the relation given by (1). Wesołowski and Ahsanullah [3], for example, showed that $X_{i:n} \stackrel{d}{=} VX_{i-1:n}$ is a characteristic relation for the power distribution. Oncel et al. [13] investigated characterizations of the form $X_{i:n-1} \stackrel{d}{=} VX_{i:n}$, where V is a Pareto distribution. Navarro [14] obtained characterizations of $X_{i:n} \stackrel{d}{=} VX_{j:n}$, $1 \le i < j \le n$, where V is a power distribution. Martinez et al. [15] approached the same problem in a more general setting and used integral equations directly to solve the functional equations. In this way they also obtained some new characterization results.

We note that in all of the above characterizations one of the random variables is assumed to be known. In this paper, we will consider a new characterization involving products of order statistics, which may be considered to be similar to the relation (1). The main difference of this characterization from the previous results given in the literature is that there is no term with a known distribution. In this respect it may be considered as one of the first characterization results of the form

$$X \stackrel{d}{=} YZ \tag{2}$$

where all three random variables X, Y, and Z are independent random variables with unknown distribution functions.

3. Ranked set samples and independent order statistics

Before stating the main result, it will be convenient to introduce ranked set samples (RSS). Ranked set sampling is an alternative sampling design to simple random sampling when actual measurement is either difficult or expensive, but ranking a few units in a small set is relatively easy and inexpensive. This sampling design was first introduced by [16,17].

An RSS can be described as follows. Let $X_1, X_2, \ldots, X_n, \ldots$ be independent and identically distributed random variables with cdf F. Consider r independent sets of samples of sizes $n_1, n_2, \ldots, n_r$, from this distribution where $r \le n_r$. From these sets of independent samples we select r random variables as follows. From the first set of n_1 independent variables we select the smallest ($X^{(1)}_{1:n_1}$), while from the second set we select the second smallest ($X^{(2)}_{2:n_2}$). In this way we continue to select independent random variables until we have selected r representative random variables denoted here by $X_{[1,n_1]}, X_{[2,n_2]}, \ldots, X_{[n_r,n_r]}$. The notation $X_{[i,n_j]}$, $1 \le j \le r$ is used to express the fact that each ordered random variable is selected from independent sets as described. In this way a set of independent order statistics is obtained. We note here also that the basic idea in RSS is to rank the observations in each set without actual measurement. This process has been summarized as follows:

$$\begin{array}{cccccc}
\underline{X^{(1)}_{1:n_1}} & X^{(1)}_{2:n_1} & \cdots & X^{(1)}_{n_1:n_1} & \to & X_{[1,n_1]} \sim F_{1:n_1}(x) \\
X^{(2)}_{1:n_2} & \underline{X^{(2)}_{2:n_2}} & \cdots & X^{(2)}_{n_2:n_2} & \to & X_{[2,n_2]} \sim F_{2:n_2}(x) \\
\cdots & \cdots & \cdots & \cdots & \to & \cdots \\
X^{(r)}_{1:n_r} & X^{(r)}_{2:n_r} & \cdots & \underline{X^{(r)}_{n_r:n_r}} & \to & X_{[n_r,n_r]} \sim F_{n_r:n_r}(x).
\end{array}$$

If the number of elements in each set is the same, i.e. $n_i = n$ for $1 \le i \le r$, and $r \le n_r$, the obtained sample is called a balanced RSS. The joint pdf of $X_{[1,n_1]}, X_{[2,n_2]}, \ldots, X_{[n_r,n_r]}$ is given by

$$f_{[1,2,\ldots,r]}\left(x_{[1]}, x_{[2]}, \ldots, x_{[r]}\right) = \prod_{i=1}^{r} f_{i:n_i}(x_{[i]}),$$

where $f_{i:n_i}(x)$ is the pdf of the i-th order statistic for a simple random sample of size n_i. The extra information provided by the structure of the ranking process and the independence of the obtained order statistics enables RSS to improve some of the classical approaches based on simple random sampling. For more information on RSS and its applications one may refer to [18,17,19], among others.

In the following we actually will use an unbalanced RSS. In particular, we will use the following three sets of independent random variables

$$\begin{array}{ccccccccc}
X^{(1)}_{1:n} & \cdots & \underline{X^{(1)}_{k:n}} & \cdots & X^{(1)}_{n-1:n} & X^{(1)}_{n:n} & \to & X_{[k,n]} \sim F_{k:n}(x) \\
X^{(2)}_{1:n-1} & \cdots & \underline{X^{(2)}_{k:n-1}} & \cdots & X^{(2)}_{n-1:n-1} & & \to & X_{[k,n-1]} \sim F_{k:n-1}(x) \\
X^{(3)}_{1:n} & \cdots & X^{(3)}_{k:n} & \cdots & X^{(3)}_{n-1:n} & \underline{X^{(3)}_{n:n}} & \to & X_{[n,n]} \sim F_{n:n}(x)
\end{array}$$

to obtain the three independent order statistics $X_{[k,n]}, X_{[k,n-1]}$, and $X_{[n,n]}$.

4. Results

To prove the main result, the following lemma will be used. We note that this lemma can be considered as another special variant of the Choquet–Deny Theorem. For other variants and some applications of this theorem one may refer to [9,4,5], among others. This lemma can be proved by using the same idea as in the proof of Theorem 1 in [9].

Lemma 1. *Let H be a nonnegative function that is not identically equal to zero on $A = (0, 1)$. Also, let $\{\mu_x : x \in A\}$ be a family of finite measures such that for each $x \in A$, $\mu(B_x) > 0$, where $B_x = (x, 1)$. Then a continuous real-valued function H on A such that $H(x)$ has a limit as x tends to 1, satisfies*

$$\int_x^1 \left[H(x) - H\left(\frac{x}{u}\right)\right]\mu_x(du) = 0, \quad x \in (0, 1), \tag{3}$$

if and only if it is identically equal to a constant.

Using this lemma the following result can be proved.

Theorem 2. *Let $\mathbf{X}^{(1)} = \left\{X_{1:n}^{(1)}, \ldots, X_{n:n}^{(1)}\right\}$, $\mathbf{X}^{(2)} = \left\{X_{1:n-1}^{(1)}, \ldots, X_{n-1:n-1}^{(1)}\right\}$, and $\mathbf{X}^{(3)} = \left\{X_{1:n}^{(3)}, \ldots, X_{n:n}^{(3)}\right\}$ be independent sets of random variables with absolutely continuous distribution function F such that f is supported in $[0, 1]$ and $\limsup_{x\to 1} f(x) > 0$. In addition, assume that f is continuous on $(0, 1)$. Let $X_{[k,n]}$ and $X_{[k,n-1]}$ denote the k-th order statistics from the sets $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$, respectively, and let $X_{[n,n]}$ be the maximal order statistics from set $\mathbf{X}^{(3)}$. If for a fixed $1 \le k \le n-1$,*

$$X_{[k,n]} \overset{d}{=} X_{[k,n-1]}X_{[n,n]}, \tag{4}$$

then $X_i \sim Pow(\alpha)$, for some $\alpha > 0$.

Proof. $X_{[k,n]} \overset{d}{=} X_{[k,n-1]}X_{[n,n]}$ implies that

$$F_{k:n}(x) = F_{k:n-1}(x) + \int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)du. \tag{5}$$

Since $n\,[F_{k:n}(x) - F_{k:n-1}(x)]\,f(x) = F(x)f_{k:n}(x)$ (see, for example, [3]), we have

$$F(x)f_{k:n}(x) = nf(x)\int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)du. \tag{6}$$

By differentiating (5) with respect to x, it follows that

$$f_{k:n}(x) = \int_x^1 f_{n:n}\left(\frac{x}{u}\right)\frac{1}{u} f_{k:n-1}(u)du \tag{7}$$

From (6) and (7), we obtain

$$nf(x)\int_x^1 F_{n:n}\left(\frac{x}{u}\right) f_{k:n-1}(u)du = F(x)\int_x^1 f_{n:n}\left(\frac{x}{u}\right)\frac{1}{u} f_{k:n-1}(u)du$$

or

$$\int_x^1 \left[nf(x)F_{n:n}\left(\frac{x}{u}\right) - F(x)f_{n:n}\left(\frac{x}{u}\right)\frac{1}{u}\right] f_{k:n-1}(u)du = 0, \quad x \in (0, 1).$$

This last equation can be written as

$$\int_x^1 F^n\left(\frac{x}{u}\right)\left[\frac{xf(x)}{F(x)} - \frac{\frac{x}{u}f\left(\frac{x}{u}\right)}{F\left(\frac{x}{u}\right)}\right] f_{k:n-1}(u)du = 0, \quad x \in (0, 1), \tag{8}$$

or, defining $H(x) = \frac{xf(x)}{F(x)}$,

$$\int_x^1 F^n\left(\frac{x}{u}\right)\left[H(x) - H\left(\frac{x}{u}\right)\right] f_{k:n-1}(u)du = 0, \quad x \in (0, 1). \tag{9}$$

Now, using Lemma 1 with $\mu_x(B) = \int_{B\cap B_x} F^n\left(\frac{x}{u}\right) f_{k:n-1}(u)du$, $B_x = (x, 1)$, it follows that H is constant on $(0, 1)$;

$$H(x) = \frac{xf(x)}{F(x)} = \alpha, \quad x \in (0, 1), \tag{10}$$

for some $\alpha \in \mathbf{R}$. The solution of this separable differential equation with boundary conditions $F(0) = 0$ and $F(1) = 1$ implies that $F(x) = x^\alpha$, $x \in (0, 1)$. □

Remark 3. It should be noted that the equation

$$H(x) = \frac{xf(x)}{F(x)} = \alpha, \quad x \in (0, 1),$$

obtained in the proof actually represents a constant generalized reversed hazard rate (gRHR). A constant gRHR means that the underlying distribution is a scale-free distribution. It is known that the power distribution is the only distribution with this property among absolutely continuous distribution functions [20]. Hence, relation (4) given in the theorem can also be used to test whether the data provides evidence for a constant gRHR.

As an immediate consequence of this theorem, any order statistic $X_{k:n}$ from a power distribution can be expressed in terms of maximum order statistics from independent sets of random variables:

Corollary 4. *Let* $\mathbf{X} = \{X_1, \ldots, X_n\}$ *and* $\mathbf{X}_i = \left\{X_1^{(i)}, \ldots, X_i^{(i)}\right\}$ *be independent sets of samples of size i for* $k \leq i \leq n$*, where* $1 \leq k \leq n-1$*, from a distribution with absolutely continuous cdf F and continuous pdf f. Then* $F(x) = x^\alpha$, $x \in (0, 1)$*, that is* $F \sim Pow(\alpha)$, $\alpha > 0$*, if and only if for some fixed* $1 \leq k \leq n-1$*,*

$$X_{k:n} \stackrel{d}{=} X_{[k,k]} X_{[k+1,k+1]} \cdots X_{[n,n]}.$$

Remark 5. Note that for $k = 1$ and $\alpha = 1$, we obtain the well known representation for the uniform distribution

$$U_{1:n} \stackrel{d}{=} U_{[1,1]} U_{[2,2]} \cdots U_{[n,n]}.$$

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