Journal of Computational and Applied Mathematics 271 (2014) 31–38

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

q-geometric and q-binomial distributions of order k

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ARTICLE INFO

Article history:
Received 24 September 2013
Received in revised form 23 January 2014

Keywords:
Distributions of order k
q-distributions
Runs

ABSTRACT

In this paper, we generalize geometric and binomial distributions of order k to q-geometric and q-binomial distributions of order k using Bernoulli trials with a geometrically varying success probability. In particular, we derive expressions for the probability mass functions of these distributions. For $q = 1$, these distributions reduce to geometric and binomial distributions of order k which have been extensively studied in the literature.

1. Introduction

The distribution of the number of trials until the first k consecutive successes in Bernoulli trials with success probability p is known to be a geometric distribution of order k. This definition is due to Philippou, Georghiou and Philippou [1]. Clearly, for $k = 1$ the geometric distribution of order k reduces to the usual geometric distribution. This distribution has been extensively studied and used in various applications including reliability and statistical process control. The distribution of the corresponding waiting time random variable has been derived also replacing the classical Bernoulli trials by different kinds of binary trials such as Markovian and exchangeable [2–5].

Much attention has been paid to the distribution of the number of runs of fixed length in a sequence of binary trials. There are various enumeration schemes for counting the number of runs. According to the nonoverlapping enumeration scheme, the distribution of the number of success runs of length k in n trials follows a Type I binomial distribution of order k which reduces to the well-known binomial distribution when $k = 1$. Type I binomial distribution of order k has been studied in [6–13].

Charalambides [14] studied discrete q-distributions on Bernoulli trials with a geometrically varying success probability. Let us consider a sequence $X_1, \ldots, X_n$ of zero (failure)–one (success) Bernoulli trials such that the trials of the subsequence after the $(i-1)$st zero until the ith zero are independent with failure probability

$$q_i = 1 - \theta q^{i-1}, \quad i = 1, 2, \ldots, 0 < \theta < 1, 0 < q \le 1. \tag{1}$$

The probability mass function of the number Z_n of successes in n trials $X_1, \ldots, X_n$ is given by

$$P\{Z_n = r\} = \begin{bmatrix} n \\ r \end{bmatrix}_q \theta^r \prod_{i=1}^{n-r} (1 - \theta q^{i-1}), \tag{2}$$

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http://dx.doi.org/10.1016/j.cam.2014.03.025
0377-0427/

for $r = 0, 1, \ldots, n$, $0 < q < 1$, where

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_{r,q}}{[r]_q!},$$

and $[x]_{k,q} = [x]_q [x-1]_q \ldots [x-k+1]_q$, $[x]_q = (1-q^x)/(1-q)$, $[x]_q! = [1]_q[2]_q \ldots [x]_q$ [14,15]. The distribution given by (2) is called a q-binomial distribution. For $q \to 1$, because

$$\begin{bmatrix} n \\ r \end{bmatrix}_q \to \binom{n}{r}$$

q-binomial distribution converges to the usual binomial distribution as $q \to 1$.

Discrete distributions of order k appear as the distributions of runs based on different enumeration schemes in binary sequences. They are widely used in various applications including statistical process control, statistical hypothesis testing and reliability. For example, a production process might be declared to be out of control when k consecutive points (charting statistics) fall outside the control limits. The number of samples or subgroups that needs to be collected before the first out of control signal is a random variable having geometric distribution of order k. Thus discrete distributions of order k are suitable models when we are interested in the number of runs of length k or the waiting time for the first run of length k. The exact probability functions of these distributions have been extensively studied in the literature under various assumptions on binary sequences including both independence and dependence. In the present paper, we study the distributions of the waiting time for the first k consecutive successes and the number of nonoverlapping success runs of length k in a sequence of independent binary trials with a geometrically varying success probability which is mentioned above. According to this model, the sequence consists of independent trials such that the subsequences after the $(i-1)$st zero until the ith zero are independent with failure probability given by (1). If the zeros (failures) represent extreme events, then the probability of getting one (success) changes after the occurrence of each extremal event. Such a stochastic model has been studied as a reliability growth model by Dubman and Sherman [16]. Investigation of discrete distributions of order k under this model is not only a mathematical generalization but also meaningful when we have binary outcomes following the abovementioned nonidentical model.

The paper is organized as follows. In Sections 2 and 3, we derive expressions for the probability mass functions of the number of trials until the first k consecutive successes, and the number of nonoverlapping success runs of length k in n trials. The resulting distributions are called as q-geometric and q-binomial distributions of order k. In Section 4, we discuss the estimation of the parameters involved in these distributions.

2. *q*-geometric distribution of order *k*

We first note the following lemma which will be useful in the sequel.

Lemma 1. *For $0 < q \le 1$, define*

$$C_q(r,s) = \sum_{\substack{x_1+\cdots+x_r=s \\ 0\le x_1<k,\ldots,0\le x_r<k}} \cdots \sum q^{x_2+2x_3+\cdots+(r-1)x_r},$$

where x_i s are integers. Then $C_q(r,s)$ obeys the following recurrence relation

$$C_q(r,s) = \begin{cases} \sum_{t=0}^{k-1} q^{t(r-1)} C_q(r-1, s-t), & \text{if } r > 1 \text{ and } 0 \le s \le (k-1)r \\ 1, & \text{if } r = 1 \text{ and } 0 \le s < k \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Considering the values that x_r can take, we have

$$\begin{aligned} C_q(r,s) &= \sum_{\substack{x_1+\cdots+x_r=s \\ 0\le x_1<k,\ldots,0\le x_r<k}} \cdots \sum q^{x_2+2x_3+\cdots+(r-1)x_r} \\ &= \sum_{\substack{x_1+\cdots+x_{r-1}=s \\ 0\le x_1<k,\ldots,0\le x_{r-1}<k}} \cdots \sum q^{x_2+2x_3+\cdots+(r-2)x_{r-1}} + q^{r-1} \sum_{\substack{x_1+\cdots+x_{r-1}=s-1 \\ 0\le x_1<k,\ldots,0\le x_{r-1}<k}} \cdots \sum q^{x_2+2x_3+\cdots+(r-2)x_{r-1}} \\ &\quad + q^{2(r-1)} \sum_{\substack{x_1+\cdots+x_{r-1}=s-2 \\ 0\le x_1<k,\ldots,0\le x_{r-1}<k}} \cdots \sum q^{x_2+2x_3+\cdots+(r-2)x_{r-1}} \end{aligned}$$

$$+\cdots+q^{(k-1)(r-1)}\sum_{\substack{x_1+\cdots+x_{r-1}=s-k+1\\0\le x_1<k,\ldots,0\le x_{r-1}<k}}\cdots\sum q^{x_2+2x_3+\cdots+(r-2)x_{r-1}}$$

$$=C_q(r-1,s)+q^{r-1}C_q(r-1,s-1)+q^{2(r-1)}C_q(r-1,s-2)+\cdots+q^{(k-1)(r-1)}C_q(r-1,s-k+1),$$

for $r>1$ and $0\le s\le(k-1)r$. The other parts of the recurrence are obvious. ■

Theorem 1. *For $0<q\le 1$, the probability mass function of the number of trials until the first k consecutive successes is given by*

$$P\{T_k=x\}=\begin{cases}\sum_{i=1}^{x-k}q^{ik}\theta^{x-i}\prod_{j=1}^{i}(1-\theta q^{j-1})C_q(i,x-i-k), & \text{if } x\ge k+1\\ \theta^k, & \text{if } x=k\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Let S_x denote the total number of zeros (failures) in x binary trials. Then

$$P\{T_k=x\}=\sum_i P\{T_k=x,S_x=i\}.$$

The joint event $\{T_k=x,S_x=i\}$ can be described with the following binary sequence which consists of i zeros.

$$\underbrace{1\ldots1}_{0\le x_1<k}0\underbrace{1\ldots1}_{0\le x_2<k}0\ldots0\underbrace{1\ldots1}_{0\le x_i<k}0\underbrace{1\ldots1}_{k},$$

where $x_1+\cdots+x_i=x-k-i$. Thus for $x\ge k+1$,

$$P\{T_k=x\}=\sum_i\sum_{\substack{x_1+\cdots+x_i=x-k-i\\0\le x_1<k,\ldots,0\le x_i<k}}\cdots\sum(\theta q^0)^{x_1}(1-\theta q^0)$$

$$\times(\theta q)^{x_2}(1-\theta q)(\theta q^2)^{x_3}(1-\theta q^2)\ldots(\theta q^{i-1})^{x_i}(1-\theta q^{i-1})(\theta q^i)^k$$

$$=\sum_{i=1}^{x-k}q^{ik}\theta^{x-i}\prod_{j=1}^{i}(1-\theta q^{j-1})\sum_{\substack{x_1+\cdots+x_i=x-k-i\\0\le x_1<k,\ldots,0\le x_i<k}}\cdots\sum q^{x_2+2x_3+\cdots+(i-1)x_i}$$

$$=\sum_{i=1}^{x-k}q^{ik}\theta^{x-i}\prod_{j=1}^{i}(1-\theta q^{j-1})C_q(i,x-i-k).$$

The proof for $x=k$ is obvious and hence omitted. ■

Remark 1. For $q=1$ in Theorem 1, the quantity $C_1(i,x-i-k)$ corresponds to the number of integer solutions to the equation $x_1+\cdots+x_i=x-k-i$ such that $0\le x_1<k,\ldots,0\le x_i<k$, and it is known to be

$$C_1(i,x-i-k)=\sum_{j=0}^{x-i-k}(-1)^j\binom{x-k-i}{j}\binom{x-k(j+1)-1}{x-k-i-1}$$

[17]. Therefore from Theorem 1, the probability mass function of the number of trials until the first k consecutive successes in Bernoulli trials with the success probability θ is obtained as

$$P\{T_k=x\}=\begin{cases}\sum_{i=1}^{x-k}\theta^{x-i}(1-\theta)^iC_1(i,x-i-k), & \text{if } x\ge k+1\\ \theta^k, & \text{if } x=k\\ 0, & \text{otherwise}\end{cases}$$

(see, e.g. [7]).

In Table 1, we compute the probability mass function of T_2 for some values of θ and q. Table 2 contains the expected value of T_k when $k=2,3,5$.

Table 1
$P\{T_2 = x\}$, $x = 2, 3, \ldots, 10$.

x	$\theta = 0.5, q = 0.5$	$\theta = 0.5, q = 0.8$	$\theta = 0.9, q = 0.5$
2	0.25000	0.25000	0.81000
3	0.03125	0.08000	0.02025
4	0.02148	0.07072	0.02101
5	0.00568	0.04102	0.00430
6	0.00215	0.02882	0.00210
7	0.00063	0.01916	0.00061
8	0.00019	0.01313	0.00021
9	0.00005	0.00894	0.00006
10	0.00001	0.00609	0.00001

Table 2
Expected value of T_k.

k	$\theta = 0.5, q = 0.5$	$\theta = 0.5, q = 0.8$	$\theta = 0.9, q = 0.5$
2	2.3244	3.7201	2.0820
3	3.1478	4.2125	3.0909
5	5.1026	6.0220	5.0265

3. *q*-Binomial distribution of order *k*

Lemma 2. *For* $0 < q \leq 1$, *define*

$$A_q(r, s, t) = \sum_{\substack{y_1+\cdots+y_r=s \\ \left[\frac{y_1}{k}\right]+\cdots+\left[\frac{y_r}{k}\right]=t \\ y_1\geq 0,\ldots,y_r\geq 0}} \cdots \sum q^{y_2+2y_3+\cdots+(r-1)y_r},$$

where $[x]$ *denotes the integer part of* x *and* y_i*s are integers. Then* $A_q(r, s, t)$ *obeys the following recurrence relation*

$$A_q(r, s, t) = \begin{cases} \sum_{j=0}^{s} q^{(r-1)j} A_q\left(r-1, s-j, t-\left[\frac{j}{k}\right]\right) & \text{if } r > 1, s \geq 0, t \geq 0 \\ 1 & \text{if } r = 1, s \geq 0, \left[\frac{s}{k}\right] = t \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Considering the values that y_r can take, we have

$$\begin{aligned} A_q(r, s, t) = & \sum_{\substack{y_1+\cdots+y_{r-1}=s \\ \left[\frac{y_1}{k}\right]+\cdots+\left[\frac{y_{r-1}}{k}\right]=t \\ y_1\geq 0,\ldots,y_{r-1}\geq 0}} \cdots \sum q^{y_2+2y_3+\cdots+(r-2)y_{r-1}} + q^{r-1} \sum_{\substack{y_1+\cdots+y_{r-1}=s-1 \\ \left[\frac{y_1}{k}\right]+\cdots+\left[\frac{y_{r-1}}{k}\right]=t-\left[\frac{1}{k}\right] \\ y_1\geq 0,\ldots,y_{r-1}\geq 0}} \cdots \sum q^{y_2+2y_3+\cdots+(r-2)y_{r-1}} \\ & + q^{2(r-1)} \sum_{\substack{y_1+\cdots+y_{r-1}=s-2 \\ \left[\frac{y_1}{k}\right]+\cdots+\left[\frac{y_{r-1}}{k}\right]=t-\left[\frac{2}{k}\right] \\ y_1\geq 0,\ldots,y_{r-1}\geq 0}} \cdots \sum q^{y_2+2y_3+\cdots+(r-2)y_{r-1}} + \cdots + q^{s(r-1)} \sum_{\substack{y_1+\cdots+y_{r-1}=0 \\ \left[\frac{y_1}{k}\right]+\cdots+\left[\frac{y_{r-1}}{k}\right]=t-\left[\frac{s}{k}\right] \\ y_1\geq 0,\ldots,y_{r-1}\geq 0}} \cdots \sum q^{y_2+2y_3+\cdots+(r-2)y_{r-1}} \\ = & A_q(r-1, s, t) + q^{r-1} A_q\left(r-1, s-1, t-\left[\frac{1}{k}\right]\right) \\ & + q^{2(r-1)} A_q\left(r-1, s-2, t-\left[\frac{2}{k}\right]\right) + \cdots + q^{s(r-1)} A_q\left(r-1, 0, t-\left[\frac{s}{k}\right]\right), \end{aligned}$$

for $r > 1$. The other parts of the recurrence are obvious. ■

Note that if $\left[\frac{y_1}{k}\right] + \cdots + \left[\frac{y_r}{k}\right] = 0$, then $0 \leq y_1 < k, \ldots, 0 \leq y_r < k$ so that $A_q(r, s, 0) = C_q(r, s)$.

Let $N_{n,k}$ denote the total number of nonoverlapping success runs of length k in n trials. In the following we obtain the probability mass function of $N_{n,k}$.

Theorem 2. *For $0 < q \leq 1$, the probability mass function of the number of nonoverlapping success runs of length k in n trials is given by*

$$P\left\{N_{n,k} = x\right\} = \sum_{i=0}^{n-kx} \theta^{n-i} \prod_{j=1}^{i}(1-\theta q^{j-1}) A_q(i+1, n-i, x),$$

$x = 0, 1, \ldots, \left[\frac{n}{k}\right]$.

Proof. Let S_n denote the total number of zeros (failures) in n binary trials. Then

$$P\left\{N_{n,k} = x\right\} = \sum_{i} P\left\{N_{n,k} = x, S_n = i\right\}.$$

The joint event $\left\{N_{n,k} = x, S_n = i\right\}$ can be described with the following binary sequence which consists of i zeros.

$$\underbrace{1\ldots1}_{y_1}0\underbrace{1\ldots1}_{y_2}0\ldots0\underbrace{1\ldots1}_{y_i}0\underbrace{1\ldots1}_{y_{i+1}},$$

where

$$\begin{aligned} & y_1 + \cdots + y_{i+1} = n - i \\ & \text{s.t} \\ & \left[\frac{y_1}{k}\right] + \cdots + \left[\frac{y_{i+1}}{k}\right] = x \\ & y_j \geq 0, \quad j = 1, \ldots, i+1. \end{aligned} \tag{3}$$

Under the model (1),

$$\begin{aligned} P\left\{N_{n,k} = x\right\} &= \sum_{i} \sum_{\substack{y_1+\cdots+y_{i+1}=n-i \\ \left[\frac{y_1}{k}\right]+\cdots+\left[\frac{y_{i+1}}{k}\right]=x}} \cdots \sum (\theta q^0)^{y_1}(1-\theta q^0)(\theta q)^{y_2}(1-\theta q)\ldots(\theta q^{i-1})^{y_i}(1-\theta q^{i-1})(\theta q^i)^{y_{i+1}} \\ &= \sum_{i=0}^{n-kx} \theta^{n-i} \prod_{j=1}^{i}(1-\theta q^{j-1}) \sum_{\substack{y_1+\cdots+y_{i+1}=n-i \\ \left[\frac{y_1}{k}\right]+\cdots+\left[\frac{y_{i+1}}{k}\right]=x}} \cdots \sum q^{y_2+2y_3+\cdots+(i-1)y_i+iy_{i+1}}. \end{aligned}$$

Thus the proof is completed. ■

Corollary 1. *Let L_n be the length of the longest success run in n binary trials. Then*

$$P\{L_n < k\} = P\left\{N_{n,k} = 0\right\} = \sum_{i=0}^{n} \theta^{n-i} \prod_{j=1}^{i}(1-\theta q^{j-1}) A_q(i+1, n-i, 0).$$

Remark 2. For $q = 1$ in Theorem 2, the quantity $A_1\,(i+1, n-i, x)$ corresponds to the number of integer solutions to the system (3) and it is known to be

$$A_1\,(i+1, n-i, x) = \binom{x+i}{x} \sum_{j=0}^{\min\left(i+1, \left[\frac{n-i-kx}{k}\right]\right)} (-1)^j \binom{i+1}{j} \binom{n-kx-jk}{i}.$$

Therefore from Theorem 2, the probability mass function of binomial distribution of order k is obtained as

$$P\left\{N_{n,k} = x\right\} = \sum_{i=0}^{n-kx} \theta^{n-i}(1-\theta)^i A_1(i+1, n-i, x),$$

for $x = 0, 1, \ldots, \left[\frac{n}{k}\right]$ [7].

Table 3 displays the distribution of $N_{10,2}$ for selected values of the parameters θ and q. In Table 4, we compute the expected value of $N_{n,k}$ for different choices of k, n and the parameters θ and q. $E(N_{n,k})$ is increasing in both θ and q.

Table 3
Probability mass function of $N_{10,2}$.

x	$\theta = 0.5, q = 0.5$	$\theta = 0.5, q = 0.8$	$\theta = 0.9, q = 0.5$
0	0.68854	0.48212	0.14144
1	0.22953	0.33201	0.15284
2	0.06126	0.13519	0.13273
3	0.01566	0.04077	0.11327
4	0.00403	0.00894	0.11104
5	0.00098	0.00097	0.34868

Table 4
Expected value of $N_{n,k}$.

n	k	$\theta = 0.5, q = 0.5$	$\theta = 0.5, q = 0.8$	$\theta = 0.9, q = 0.5$
10	2	0.4200	0.7653	2.9457
	3	0.1605	0.2668	1.7036
20	2	0.4211	0.8262	4.0150
	3	0.1610	0.2798	2.3796
	5	0.0333	0.0476	1.2820

4. Estimation

In this section we discuss how to calculate the point estimator of the parameter θ in the distribution of T_k using the method proposed by Balakrishnan and Koutras [18, p. 34]. Let $T_{k,1}, T_{k,2}, \ldots, T_{k,N}$ be a random sample of size N from the q-geometric distribution of order k with a known q value. Assume that the entire sequences of binary trials leading to the realizations of $T_{k,1}, T_{k,2}, \ldots, T_{k,N}$ are available. That is, the individual sequences of binary trials are summarized by the pairs $(S_1, F_1), (S_2, F_2), \ldots, (S_N, F_N)$, where S_i and F_i represent respectively the number of successes and failures observed corresponding to the realization of $T_{k,i}$ with $T_{k,i} = S_i + F_i$, $i = 1, 2, \ldots, N$.

The contribution of each $T_{k,i}$ to the log-likelihood function for θ is

$$\begin{aligned} l_i(\theta; T_{k,i}) &= S_i \ln\theta + \ln \prod_{j=1}^{F_i} (1 - \theta q^{j-1}) \\ &= S_i \ln\theta + \sum_{j=1}^{F_i} \ln(1 - \theta q^{j-1}). \end{aligned}$$

Therefore the log-likelihood function for θ is

$$\begin{aligned} l(\theta; T_{k,1}, \ldots, T_{k,N}) &= \sum_{i=1}^{N} l_i(\theta; T_{k,i}) \\ &= \ln\theta \sum_{i=1}^{N} S_i + \sum_{i=1}^{N} \sum_{j=1}^{F_i} \ln(1 - \theta q^{j-1}). \end{aligned}$$

The derivative of the log-likelihood function with respect to θ is

$$\frac{\partial l(\theta; T_{k,1}, \ldots, T_{k,N})}{\partial\theta} = \frac{1}{\theta} \sum_{i=1}^{N} S_i - \sum_{i=1}^{N} \sum_{j=1}^{F_i} \frac{q^{j-1}}{1 - \theta q^{j-1}}.$$

Thus we need to solve the equation

$$h(\theta) = \frac{1}{\theta} \sum_{i=1}^{N} S_i - \sum_{i=1}^{N} \sum_{j=1}^{F_i} \frac{q^{j-1}}{1 - \theta q^{j-1}} = 0.$$

According to the Newton–Raphson method, the maximum likelihood estimate of θ can be obtained iteratively as

$$\theta_{m+1} = \theta_m - \frac{h(\theta_m)}{h'(\theta_m)}, \tag{4}$$

where

$$h'(\theta) = -\frac{1}{\theta^2} \sum_{i=1}^{N} S_i - \sum_{i=1}^{N} \sum_{j=1}^{F_i} \frac{q^{2j-2}}{(1 - \theta q^{j-1})^2}.$$

Table 5
Simulated data for $k = 5, \theta = 0.8$ and $q = 0.9$.

i	Sequence	S_i	F_i
1	01110010111011111	12	5
2	10111011110110111011111	18	5
3	1111011111	9	1
4	10101111011111	11	3
5	11111	5	0
6	111011011011111	12	3
7	11100011111	8	3
8	011111	5	1
9	0101111000011111	10	6
10	110011110111001000101100011111	18	12
11	1110011001111011111	14	5
12	111101110011111	12	3
13	11011111	7	1
14	110110110111011011111	16	5
15	101110011111	9	3
16	11001110101011111	12	5
17	11110011111	9	2
18	01101011011011010010011011111	18	11
19	011111	5	1
20	11110011010000011111	12	8
Total		222	83

An approximate $100(1-\alpha)\%$ confidence interval for θ is estimated as

$$\hat{\theta} \pm z_{\alpha/2}/\sqrt{I(\hat{\theta})},$$

where $I(\hat{\theta}) = -h'(\hat{\theta})$ denotes the observed Fisher information. For $q = 1$, the maximum likelihood estimate of θ is obtained as

$$\hat{\theta} = \frac{\sum_{i=1}^{N} S_i}{\sum_{i=1}^{N} S_i + \sum_{i=1}^{N} F_i} = \frac{\sum_{i=1}^{N} S_i}{\sum_{i=1}^{N} T_{k,i}}.$$

For an illustration, we generate $N = 20$ samples for $k = 5, \theta = 0.8$ and $q = 0.9$. Table 5 contains simulated data.

For the data presented in Table 5 the maximum likelihood estimate of θ is found to be $\widehat{\theta} = 0.8064$. Approximate 95% confidence interval for θ is (0.7484, 0.8644).

Next, consider the estimation of the parameter θ in the q-binomial distribution of order k. Let $N_{n,k}^{(1)}, N_{n,k}^{(2)}, \ldots, N_{n,k}^{(N)}$ be a random sample of size N from the q-binomial distribution of order k with a known q value. If this random sample is the only information on hand, then the maximum likelihood estimate of θ may be obtained iteratively using (4) when

$$h(\theta) = \sum_{i=1}^{N} \frac{1}{f(N_{n,k}^{(i)}; \theta)} \frac{\partial}{\partial \theta} f(N_{n,k}^{(i)}; \theta),$$

where $f(\cdot; \theta)$ is the probability mass function of $N_{n,k}$ which is given in Theorem 2. Such a method has been proposed by Aki and Hirano [19] for estimating the parameter of the binomial distribution of order k.

Acknowledgment

The authors would like to express gratitude to the referee for a thorough review and valuable comments.

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