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Reliability of coherent systems with a single cold standby component

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ABSTRACT

In this paper, the influence of a cold standby component on a coherent system is studied. A method for computing the system reliability of coherent systems with a cold standby component based on signature is presented. Numerical examples are presented. Reliability and mean time to failure of different systems are computed.

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1. Introduction

There are different methods to increase system reliability. One of them is to equip the system with standby units such as warm, hot and cold. Compared to others, cold standby redundancy can be preferred when switching times are sufficiently short, since cold standby component is inactive which means it does not fail in standby. Van Gemund and Reijns [1] studied k-out-of-n system with a single standby and found an analytical way to compute the mean time to failure of the system. Eryilmaz [2] investigated various mean residual life functions for the same system. Recently, Eryilmaz [3] studied k-out-of-n system equipped with a single warm standby component.

In this paper, using system signature, conditioning on the index of the cold standby component and indices of the components failed before cold standby component is put into operation, the reliability of coherent systems having a cold standby component is derived.

Let X_i denote the lifetime of the ith component in a coherent system having lifetime T. If X_i's are s-independent and have common absolutely continuous distribution function, then the survival function can be represented as

$$P(T > t) = \sum_{i=1}^{n} p_i P(X_{i:n} > t),$$

where $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are the order statistics associated with $X_1, X_2, \dots, X_n$ and $p_i = P(T = X_{i:n})$, in other words,

$$p_i = \frac{\text{The number of orderings for which the } i\text{th failure causes the system failure}}{n!},$$

for $i = 1, 2, \dots, n$ which is well known as Samaniego's Signature [4]. The ith element of the signature vector can be easily computed from

$$p_i = \frac{r_{n-i+1}(n)}{\binom{n}{n-i+1}} - \frac{r_{n-i}(n)}{\binom{n}{n-i}}, \quad \text{for } i = 1, 2, \dots, n.$$

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[5] where $r_i(n)$ denotes thenumber of path sets including i working components in a coherent system. Since signature has vital importance to investigate the behavior of the system lifetime, recently many developments have been made in this area, see [6–13].

In a coherent system with a single cold standby, the index of the standby component as well as the indices of failed components has significant importance which makes the computation of the reliability more difficult. In Section 3, a method for computing the reliability of coherent systems is presented. Finally, comparison of the reliability and mean time to failure of some systems with and without cold standby component have been illustrated.

2. Notations

Below the notations that will be used throughout the article are provided.

n, number of components in the system;
Y, lifetime of the cold standby component;
X_i, lifetime of the component i, $1 \le i \le n$;
$X_{s:n}$, sth smallest among X_i, $1 \le i \le n$;
$X_l^{(s)}$, remaining lifetime of the components after $X_{s:n}$ fails: $X_l^{(s)} \overset{st}{=} (X_l - X_{s:n} | X_l > X_{s:n})$, $1 \le l \le n-s$;
ϕ, structure function of the system;
$T = \phi(X_1, \dots, X_n)$, lifetime of the system without cold standby component;
T^w, lifetime of the system with a cold standby component;
V_s, discrete random variable representing the index of the cold standby component when $X_{s:n}$ fails: $V_s = c \Leftrightarrow (X_c = X_{s:n} | T = X_{s:n})$, $c = 1, 2, \dots, n$;
$\mathbf{B}_{s,c} | V_s = c$, a discrete multivariate random variable representing the indices of the failed components given $V_s = c$, $s = 1, \dots, n$ and $c = 1, \dots, n$;
$(\mathbf{B}_{s,c} | V_s = c) = (B_1 = b_1, B_2 = b_2, \dots, B_{s-1} = b_{s-1} | V_s = c) \Leftrightarrow (0_{B_1} = 0_{b_1}, 0_{B_2} = 0_{b_2}, \dots, 0_{B_{s-1}} = 0_{b_{s-1}} | X_c = X_{s:n}, T = X_{s:n})$ where $\mathbf{0} = (0_{B_1}, 0_{B_2}, \dots, 0_{B_{s-1}})$ are the components which have failed before $X_{s:n}$;
$\mathbf{R}_{s,c} | V_s = c$, a discrete multivariate random variable representing the indices of the remaining components given $V_s = c$, $s = 1, \dots, n$ and $c = 1, \dots, n$: $\mathbf{R}_{s,c} = (R_1 = r_1, R_2 = r_2, \dots, R_{n-s} = r_{n-s} | V_s = c) \Leftrightarrow (X_{R_1}^{(s)} = X_{r_1}^{(s)}, X_{R_2}^{(s)} = X_{r_2}^{(s)}, \dots, X_{R_{n-s}}^{(s)} = X_{r_{n-s}}^{(s)} | X_c = X_{s:n}, T = X_{s:n})$.

3. Main results

Consider a binary coherent system with structure function ϕ. Let $T = \phi(X_1, \dots, X_n)$ denote the lifetime of a coherent system without a cold standby component and T^w denote the lifetime of the same system with a cold standby component whose lifetime is Y. Moreover, $X_1, \dots, X_n$ have a common continuous cumulative distribution function (c.d.f); F and Y have a continuous c.d.f G.

Eryilmaz [14] studied coherent systems equipped with a cold standby component which may be put into operation at the time of the first component failure in the system. In this paper, we consider the general case in which the standby component may get involved at the time of the sth component failure $s = k_\phi, \dots, z_\phi + 1$ where k_ϕ is the minimum number of failed components that cause the system failure whereas z_ϕ is the maximum number of failed components that system can still operate. It is clear that $P(T = X_{s:n}) > 0$ for $s = k_\phi, \dots, z_\phi + 1$.

After replacing the standby component with sth failed component which causes the system failure at the same time, the remaining lifetime of the system consisting of $s-1$ failed components (0's), $n-s$ functioning components, and a standby component (Y) can be represented as

$$\phi_s(0_{B_1}, 0_{B_2}, \dots, 0_{B_{s-1}}, Y_{V_s}, X_{R_1}^{(s)}, X_{R_2}^{(s)}, \dots, X_{R_{n-s}}^{(s)}).$$

When sth failure occurs which causes system failure at the same time, cold standby component gets involved to the system. At this time, there are totally $n-s+1$ functioning components in the system. The reliability of the remaining lifetime of the system is computed based on these $n-s+1$ functioning components. However, places of the $s-1$ failed components should be taken into consideration (not their lifetimes since they failed already) in the structure function of the system to calculate the main lifetime random variable T^w.

It is well known that the random variables $X_1^{(s)}, \dots, X_{n-s}^{(s)}$ are conditionally independent given $X_{s:n} = x$, and

$$P\{X_1^{(s)} > x_1, \dots, X_{n-s}^{(s)} > x_{n-s} | X_{s:n} = x\} = \prod_{l=1}^{n-s} \frac{\bar{F}(x_l + x)}{\bar{F}(x)}.$$

The main goal is to find the reliability characteristics of T^w, i.e.

$$T^w = T + \sum_{s=k_\phi}^{z_\phi+1} \phi_s(0_{B_1}, 0_{B_2}, \dots, 0_{B_{s-1}}, Y_{V_s}, X_{R_1}^{(s)}, X_{R_2}^{(s)}, \dots, X_{R_{n-s}}^{(s)}).$$

Lemma 1. *For $t > 0$ and $s = k_\phi, \dots, z_\phi + 1$;*

$$P\{\phi_s(0_{b_1}, 0_{b_2}, \dots, 0_{b_{s-1}}, Y_c, X^{(s)}_{r_1}, X^{(s)}_{r_2}, \dots, X^{(s)}_{r_{n-s}}) > t | X_{s:n} = x\}$$
$$= \frac{1}{\bar{F}^{n-s}(x)} \int \cdots \int_{\phi_s(0_{b_1}, 0_{b_2}, \dots, 0_{b_{s-1}}, y_c, x_{r_1}, x_{r_2}, \dots, x_{r_{n-s}}) > t} g(y_c) \prod_{m=1}^{n-s} f(x_{r_m} + x) dx_{r_1} dx_{r_2} \dots dx_{r_{n-s}} dy_c.$$

Proof. Due to the fact that Y and $X_1, \dots, X_n$ are independent for $s = k_\phi, \dots, z_\phi + 1$

$$P\{\phi_s(0_{b_1}, 0_{b_2}, \dots, 0_{b_{s-1}}, Y_c, X^{(s)}_{r_1}, X^{(s)}_{r_2}, \dots, X^{(s)}_{r_{n-s}}) > t | X_{s:n} = x\}$$
$$= \int \cdots \int_{\phi_s(0_{b_1}, 0_{b_2}, \dots, 0_{b_{s-1}}, y_c, x_{r_1}, x_{r_2}, \dots, x_{r_{n-s}}) > t} g(y_c) f(x_{r_1}, x_{r_2}, \dots, x_{r_{n-s}} | x_{s:n} = x) dx_{r_1} dx_{r_2} \dots dx_{r_{n-s}} dy_c.$$

Since the joint p.d.f. of $X^{(s)}_{r_1}, X^{(s)}_{r_2}, \dots, X^{(s)}_{r_{n-s}}$ given $X_{s:n} = x$ is

$$f(x_{r_1}, x_{r_2}, \dots, x_{r_{n-s}} | x_{s:n} = x) = \frac{1}{\bar{F}^{n-s}(x)} \prod_{m=1}^{n-s} f(x_{r_m} + x).$$

The proof is complete. ■

Remark 1. Due to the fact that given $X_{s:n} = x$, the random variables $X^{(s)}_1, \dots, X^{(s)}_{n-s}$ are independent for $s = k_\phi, \dots, z_\phi + 1$. So, the conditional probability given in Lemma 1 is indeed the survival function of the coherent system ϕ_s consisting of $s-1$ failed components, $n-s$ independent component having the same marginal survival function $\frac{\bar{F}(t+x)}{\bar{F}(x)}$ and the υ_sth component has the survival function $\bar{G}(t)$. Moreover given $X_{s:n} = x$ if we order the residual lifetime of the remaining $n-s$ components such that

$$X^{(s)}_{1:n-s} \le X^{(s)}_{2:n-s} \le \cdots \le X^{(s)}_{n-s:n-s}.$$

The survival function of the kth order statistics of the residual lifetime of the remaining $n-s$ components for $k = 1, 2, \dots, n-s$, can be found as

$$P(X^{(s)}_{k:n-s} > t | X_{s:n} = x) = \sum_{i=0}^{k-1} \binom{n-s}{i} \left(1 - \frac{\bar{F}(t+x)}{\bar{F}(x)}\right)^i \left(\frac{\bar{F}(t+x)}{\bar{F}(x)}\right)^{n-s-i}.$$

Theorem 1. *Let* $\mathbf{p}$ *be the signature of a coherent system $T = \phi(X_1, \dots, X_n)$ which has a cold standby component with lifetime distribution G. Then*

$$P(T^w > t) = \sum_{s=k_\phi}^{z_\phi+1} \Bigg(p_s P(X_{s:n} > t) + p_s \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \cdots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \dots, b_{s-1})\right)$$
$$\times \int_0^t P\{\phi_s(0_{b_1}, 0_{b_2}, \dots, 0_{b_{s-1}}, Y_c, X^{(s)}_{r_1}, X^{(s)}_{r_2}, \dots, X^{(s)}_{r_{n-s}}) > t - x | X_{s:n} = x\} dF_{s:n}(x) \Bigg).$$

Proof. For a coherent system $P(T = X_{s:n}) > 0$ for $s = k_\phi, \dots, z_\phi + 1$. Any coherent system operating with n components may fail at the time of sth component failure. If the system failure is caused by the failure of the sth component, then the standby component gets involved to the system. Therefore the survival function of the coherent system with a standby component can be written as follows:

$$\begin{aligned}
P(T^w > t) &= P\{T + \phi_{k_\phi}(0_{B_1}, 0_{B_2}, \dots, 0_{B_{k_\phi - 1}}, Y_{V_{k_\phi}}, X^{(k_\phi)}_{R_1}, X^{(k_\phi)}_{R_2}, \dots, X^{(k_\phi)}_{R_{n-k_\phi}}) > t, T = X_{k_\phi:n}\} + P(T > t, T > X_{k_\phi:n}) \\
&= p_{k_\phi} P\{T + \phi_{k_\phi}(0_{B_1}, 0_{B_2}, \dots, 0_{B_{k_\phi - 1}}, Y_{V_{k_\phi}}, X^{(k_\phi)}_{R_1}, X^{(k_\phi)}_{R_2}, \dots, X^{(k_\phi)}_{R_{n-k_\phi}}) > t | T = X_{k_\phi:n}\} \\
&\quad + P(T > t, T > X_{k_\phi:n}) \\
&= p_{k_\phi} P\{T + \phi_{k_\phi}(0_{B_1}, 0_{B_2}, \dots, 0_{B_{k_\phi - 1}}, Y_{V_{k_\phi}}, X^{(k_\phi)}_{R_1}, X^{(k_\phi)}_{R_2}, \dots, X^{(k_\phi)}_{R_{n-k_\phi}}) > t | T = X_{k_\phi:n}\} \\
&\quad + p_{k_\phi+1} P\{T + \phi_{k_\phi+1}(0_{B_1}, 0_{B_2}, \dots, 0_{B_{k_\phi}}, Y_{V_{k_\phi+1}}, X^{(k_\phi+1)}_{R_1}, X^{(k_\phi+1)}_{R_2}, \dots, X^{(k_\phi+1)}_{R_{n-k_\phi-1}}) > t | T = X_{k_\phi+1:n}\} \\
&\quad + P(T > t, T > X_{k_\phi+1:n}) \\
&= p_{k_\phi} P\{T + \phi_{k_\phi}(0_{B_1}, 0_{B_2}, \dots, 0_{B_{k_\phi - 1}}, Y_{V_{k_\phi}}, X^{(k_\phi)}_{R_1}, X^{(k_\phi)}_{R_2}, \dots, X^{(k_\phi)}_{R_{n-k_\phi}}) > t | T = X_{k_\phi:n}\} \\
&\quad + p_{k_\phi+1} P\{T + \phi_{k_\phi+1}(0_{B_1}, 0_{B_2}, \dots, 0_{B_{k_\phi}}, Y_{V_{k_\phi+1}}, X^{(k_\phi+1)}_{R_1}, X^{(k_\phi+1)}_{R_2}, \dots, X^{(k_\phi+1)}_{R_{n-k_\phi-1}}) > t | T = X_{k_\phi+1:n}\} + \cdots
\end{aligned}$$

$$+ p_{z_\phi+1}P\{T + \phi_{z_\phi+1}(0_{B_1}, 0_{B_2}, \dots, 0_{B_{z_\phi}}, Y_{V_{z_\phi+1}}, X_{R_1}^{(z_\phi+1)}, X_{R_2}^{(z_\phi+1)}, \dots, X_{R_{n-z_\phi-1}}^{(z_\phi+1)}) > t | T = X_{z_\phi+1:n}\}$$
$$+ P(T > t, T > X_{z_\phi+1:n}).$$

It is obvious that $P(T > t, T > X_{z_\phi+1:n}) = 0$.
Now, for $s = k_\phi, \dots, z_\phi + 1$ consider the conditional probability

$$\begin{aligned}
&P\{T + \phi_s(0_{B_1}, 0_{B_2}, \dots, 0_{B_{s-1}}, Y_{V_s}, X_{R_1}^{(s)}, X_{R_2}^{(s)}, \dots, X_{R_{n-s}}^{(s)}) > t | T = X_{s:n}\} \\
&= \sum_{c=1}^{n} \frac{P\{X_c + \phi_s(0_{B_1}, 0_{B_2}, \dots, 0_{B_{s-1}}, Y_c, X_{R_1}^{(s)}, X_{R_2}^{(s)}, \dots, X_{R_{n-s}}^{(s)}) > t, X_{s:n} = X_c, T = X_{s:n}\}}{P(T = X_{s:n})} \\
&= \sum_{c=1}^{n} P\{X_c + \phi_s(0_{B_1}, 0_{B_2}, \dots, 0_{B_{s-1}}, Y_c, X_{R_1}^{(s)}, X_{R_2}^{(s)}, \dots, X_{R_{n-s}}^{(s)}) > t | X_{s:n} = X_c, T = X_{s:n}\} P(X_{s:n} = X_c | T = X_{s:n}) \\
&= \frac{\sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \dots < b_{s-1} \le n} P\{X_c + \phi_s(0_{b_1}, \dots, 0_{b_{s-1}}, Y_c, X_{r_1}^{(s)}, \dots, X_{r_{n-s}}^{(s)}) > t |\, 0_{B_1} = 0_{b_1}, \dots, 0_{B_{s-1}} = 0_{b_{s-1}}, X_{s:n} = X_c, T = X_{s:n}\}}{P(X_{s:n} = X_c, T = X_{s:n})} \\
&\quad \times P(0_{B_1} = 0_{b_1}, \dots, 0_{B_{s-1}} = 0_{b_{s-1}}, X_{s:n} = X_c, T = X_{s:n}) \\
&= \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \dots < b_{s-1} \le n} P(0_{B_1} = 0_{b_1}, \dots, 0_{B_{s-1}} = 0_{b_{s-1}} | X_{s:n} = X_c, T = X_{s:n}) \\
&\quad \times P\{X_c + \phi_s(0_{b_1}, \dots, 0_{b_{s-1}}, Y_c, X_{r_1}^{(s)}, \dots, X_{r_{n-s}}^{(s)}) > t |\, 0_{B_1} = 0_{b_1}, \dots, 0_{B_{s-1}} = 0_{b_{s-1}}, X_{s:n} = X_c, T = X_{s:n}\} \\
&= \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \dots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \dots, b_{s-1})\right) \int P\{\phi_s(0_{b_1}, \dots, 0_{b_{s-1}}, Y_c, X_{r_1}^{(s)}, \dots, X_{r_{n-s}}^{(s)}) > t - x | X_{s:n} = x\} dF_{s:n}(x), \\
&= \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \dots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \dots, b_{s-1})\right) \\
&\quad \times \left[\int_t^\infty dF_{s:n}(x) + \int_0^t P\{\phi_s(0_{b_1}, \dots, 0_{b_{s-1}}, Y_c, X_{r_1}^{(s)}, \dots, X_{r_{n-s}}^{(s)}) > t - x | X_{s:n} = x\} dF_{s:n}(x)\right] \\
&= \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \dots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \dots, b_{s-1})\right) \\
&\quad \times \left[\int_0^t P\{\phi_s(0_{b_1}, \dots, 0_{b_{s-1}}, Y_c, X_{r_1}^{(s)}, \dots, X_{r_{n-s}}^{(s)}) > t - x | X_{s:n} = x\} dF_{s:n}(x) + P(X_{s:n} > t)\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
P(T^w > t) = \sum_{s=k_\phi}^{z_\phi+1} \Bigg(& p_s P(X_{s:n} > t) + p_s \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \dots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \dots, b_{s-1})\right) \\
& \times \int_0^t P\{\phi_s(0_{b_1}, \dots, 0_{b_{s-1}}, Y_c, X_{r_1}^{(s)}, \dots, X_{r_{n-s}}^{(s)}) > t - x | X_{s:n} = x\} dF_{s:n}(x) \Bigg). \quad \blacksquare
\end{aligned}$$

Theorem 2. *Consider a coherent system having a signature vector* $\mathbf{p}$*, with a cold standby component having distribution function* G *while other components have common distribution function* F*. Then system reliability can be computed as follows:*

$$\begin{aligned}
P(T^w > t) = \sum_{s=k_\phi}^{z_\phi+1} \Bigg(& p_s P(X_{s:n} > t) + p_s \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \dots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \dots, b_{s-1})\right) \\
& \times \int_0^t \left[\bar{G}(t - x) \sum_{k=1}^{n-s} \bar{p}_k^{c,(b_1, b_2, \dots, b_{s-1})} P(X_{k:n-s}^{(s)} > t - x | X_{s:n} = x) \; + \bar{G}(t - x) \bar{p}_{n-s+1}^{c,(b_1, b_2, \dots, b_{s-1})} \right] dF_{s:n}(x) \Bigg),
\end{aligned}$$

where $\bar{p}_k^{c,(b_1, b_2, \dots, b_{s-1})}$ *is the number of orderings for which* k*th failure among the* $(n-s)$ *remaining components and a cold standby component cause the system to fail where* c*th component (cold standby) assumed to be functioning and components having indices* $b_1, b_2, \dots, b_{s-1}$ *have already failed. Moreover* $X_{k:n-s}^{(s)}$ *is the* k*th order statistics of the residual lifetime of the remaining* $(n-s)$ *functioning components.*

$$\bar{p}_k^{c,(b_1, b_2, \dots, b_{s-1})} = \frac{\textit{The number of orderings for which the kth failure of the remaining components causes the system to fail}}{n - s!}$$

$$k = 1, \dots, n - s,$$

and

$$\bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} = \begin{cases} 1, & \text{if the failure of the system can be caused by only the failure of the cold standby} \\ 0, & \text{if the failure of the system can be caused by the remaining components.} \end{cases}$$

Proof. When the cold standby component is put into operation at time x for the system to survive up to time t, the cold standby component must function between the time x and t since the failure of the cold standby component will lead to system failure with probability 1. Assuming the cold standby component functions between the time t and x system failure can be caused by the failure of the remaining components. Given $X_{s:n} = x$ residual lifetime of the remaining $(n-s)$ components are independent and identically distributed. Therefore the survival function of the coherent system $\phi_s(0_{b_1}, \ldots, 0_{b_{s-1}}, Y_c, X_{r_1}^{(s)}, \ldots, X_{r_{n-s}}^{(s)})$ having $s-1$ failed components at places $b_1, b_2, \ldots, b_{s-1}$ and a cold standby component at place c can be computed by its signature function $\bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})}$ for $k = 1, \ldots, n-s$ and $\bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} = 0$. If the failure of the system does not depend on the failure of the remaining components which means system survives until the cold standby component fails in that case $\bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})} = 0$ for $k = 1, \ldots, n-s$ and $\bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} = 1$. ■

It is known that when both active and standby components have common exponential distribution, the random variables $X_{R_1}^{(s)}, X_{R_2}^{(s)}, \ldots, X_{R_{n-s}}^{(s)}, Y_{V_s}$ are independent and have the same exponential distribution. Therefore, the structure function can be written as

$$\phi_s(0_{B_1}, \ldots, 0_{B_{s-1}}, Y_{V_s}, X_{R_1}^{(s)}, \ldots, X_{R_{n-s}}^{(s)}) \stackrel{st}{=} \phi_s(0_{B_1}, \ldots, 0_{B_{s-1}}, Y_{V_s}, X_{R_1}, \ldots, X_{R_{n-s}}).$$

Corollary 1. *Under the assumption of all components, including the cold standby component, have common exponential distribution $F(x) = 1 - e^{-\lambda x}$, $x > 0$ the reliability of coherent systems with a cold standby component turns into*

$$P(T^w > t) = \sum_{s=k_\phi}^{z_\phi+1} p_s P(X_{s:n} > t) + p_s \sum_{c=1}^{n} P(V_s = c) \sum_{1 \le b_1 < \cdots < b_{s-1} \le n} P\left(\mathbf{B}_{s,c} = (b_1, \ldots, b_{s-1})\right)$$
$$\times \int_0^t \left[\bar{F}(t-x) \sum_{k=1}^{n-s} \bar{p}_k^{c,(b_1,b_2,\ldots,b_{s-1})} P(X_{k:n-s} > t-x) + \bar{F}(t-x) \bar{p}_{n-s+1}^{c,(b_1,b_2,\ldots,b_{s-1})} \right] dF_{s:n}(x).$$

Example 1. Consider the coherent system with lifetime

$$T = \min(X_1, \max(X_2, X_3)).$$

The signature of this system is $p = (\frac{1}{3}, \frac{2}{3}, 0)$. In this system, $k_\phi = z_\phi = 1$. For $s = 1$, there are no failed components (0's). $P(V_1 = 1) = 1$, $P(V_1 = 2) = P(V_1 = 3) = 0$ which means only component 1 can be replaced by the cold standby component. The remaining lifetime of the components after the first failure is $X_2^{(1)}$ and $X_3^{(1)}$. $\bar{p}^{1,-}$ can be found as $(0, 1, 0)$ since when the cold standby component functions, the system works until both components 2 and 3 fail. For $s = 2$, all components can be cold standby component with probabilities $P(V_2 = 1) = \frac{1}{2}$, and $P(V_2 = 2) = P(V_2 = 3) = \frac{1}{4}$. Suppose component 1 is replaced with the cold standby component. Previously failed component can be 2 or 3 (0_2 or 0_3). Moreover, let component 2 (3) be replaced by the cold standby component. In this case, previously failed component is 0_3 (0_2).

$\bar{p}^{1,(2)} = \bar{p}^{1,(3)} = \bar{p}^{2,(3)} = \bar{p}^{3,(2)}$ is $(1, 0)$ because for each case the failure of the remaining component will lead to system failure.

$s = 1$	c
V_1	1
$\mathbf{B}_{1,c}$	–
$\mathbf{R}_{1,c}$	(2, 3)
$P(V_1 = c)$	1
$\bar{p}^{c,\mathbf{B}_{1,c}}$	(0, 1, 0)

$s = 2$	c			
V_2	1		2	3
$\mathbf{B}_{2,c}$	(2)	(3)	(3)	(2)
$\mathbf{R}_{2,c}$	(3)	(2)	(1)	(1)
$P(V_2 = c)$	$\frac{1}{2}$		$\frac{1}{4}$	$\frac{1}{4}$
$\bar{p}^{c,\mathbf{B}_{2,c}}$	(1, 0)	(1, 0)	(1, 0)	(1, 0)

Using Theorem 2

$$P(T^w > t) = \frac{1}{3}P(X_{1:3} > t) + \frac{2}{3}P(X_{2:3} > t) + \frac{1}{3}\int_0^t \bar{G}(t-x)\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2 + 2\frac{F(t)-F(x)}{\bar{F}(x)}\frac{\bar{F}(t)}{\bar{F}(x)}\right]dF_{1:3}(x) + \frac{2}{3}\int_0^t \bar{G}(t-x)\frac{\bar{F}(t)}{\bar{F}(x)}dF_{2:3}(x).$$

Example 2. Consider the coherent system with lifetime

$$T = \max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4)).$$

The signature of this system is $p = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. In this system $k_\phi = z_\phi = 1$.

$s = 1$	c	
V_1	2	3
$\mathbf{B}_{1,c}$	–	–
$\mathbf{R}_{1,c}$	(1, 3, 4)	(1, 2, 4)
$P(v_1 = c)$	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{p}^{c,\mathbf{B}_{1,c}}$	$(\frac{1}{3}, \frac{2}{3}, 0, 0)$	$(\frac{1}{3}, \frac{2}{3}, 0, 0)$

$s = 2$	c					
V_2	1	2		3		4
$\mathbf{B}_{2,c}$	(4)	(1)	(4)	(1)	(4)	(1)
$\mathbf{R}_{2,c}$	(2, 3)	(3, 4)	(1, 3)	(2, 4)	(1, 2)	(2, 3)
$P(V_2 = c)$	$\frac{1}{6}$	$\frac{2}{6}$		$\frac{2}{6}$		$\frac{1}{6}$
$\bar{p}^{c,\mathbf{B}_{2,c}}$	(1, 0, 0)	(1, 0, 0)	(1, 0, 0)	(1, 0, 0)	(1, 0, 0)	(1, 0, 0)

$$P(T^w > t) = \frac{1}{2}P(X_{1:4} > t) + \frac{1}{2}P(X_{2:4} > t) + \frac{1}{2}\int_0^t \bar{G}(t-x)\left[\frac{1}{3}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^3 + \frac{2}{3}\left(\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^3 + 3\frac{F(t)-F(x)}{\bar{F}(x)}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2\right)\right]dF_{1:4}(x) + \frac{1}{2}\int_0^t \bar{G}(t-x)\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2 dF_{2:4}(x).$$

Example 3. Consider the coherent system which is known as consecutive 3-out-of-5:F with lifetime

$$T = \min(\max(X_1, X_2, X_3), \max(X_2, X_3, X_4), \max(X_3, X_4, X_5)).$$

The signature of this system is $p = (0, 0, \frac{3}{10}, \frac{1}{2}, \frac{2}{10})$. In this system $k_\phi = 3$ and $z_\phi = 4$.

$s = 3$	c								
V_3	1	2		3			4		5
$\mathbf{B}_{3,c}$	(2, 3)	(1, 3)	(3, 4)	(1, 2)	(2, 4)	(4, 5)	(3, 5)	(2, 3)	(3, 4)
$\mathbf{R}_{3,c}$	(4, 5)	(4, 5)	(1, 5)	(4, 5)	(1, 5)	(1, 2)	(1, 2)	(1, 5)	(1, 2)
$P(V_3 = c)$	$\frac{1}{9}$	$\frac{2}{9}$		$\frac{3}{9}$			$\frac{2}{9}$		$\frac{1}{9}$
$\bar{p}^{c,\mathbf{B}_{3,c}}$	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2}, 0)$	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2}, 0)$

$s = 4$	c									
V_4	1	2		3				4		5
$\mathbf{B}_{4,c}$	(2, 3, 5)	(1, 3, 4)	(1, 3, 5)	(1, 2, 4)	(1, 2, 5)	(1, 4, 5)	(2, 4, 5)	(2, 3, 5)	(1, 3, 5)	(1, 3, 4)
$\mathbf{R}_{4,c}$	(4)	(5)	(4)	(5)	(4)	(2)	(1)	(1)	(2)	(2)
$P(V_4 = c)$	$\frac{1}{10}$	$\frac{2}{10}$		$\frac{4}{10}$				$\frac{2}{10}$		$\frac{1}{10}$
$\bar{p}^{c,\mathbf{B}_{4,c}}$	(1, 0)	(1, 0)	(1, 0)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)	(1, 0)	(1, 0)

$s=5$	c
V_5	3
$\mathbf{B}_{5,c}$	(1, 2, 4, 5)
$\mathbf{R}_{5,c}$	–
$P(V_5=c)$	1
$\bar{p}^{c,\mathbf{B}_{5,c}}$	(1)

$$
\begin{aligned}
P(T^w>t) &= \frac{3}{10}P(X_{3:5}>t)+\frac{1}{2}P(X_{4:5}>t)+\frac{2}{10}P(X_{5:5}>t)\\
&\quad+\frac{3}{10}\frac{4}{9}\int_0^t \bar{G}(t-x)\left[\frac{1}{2}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+\frac{1}{2}\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+2\frac{F(t)-F(x)}{\bar{F}(x)}\frac{\bar{F}(t)}{\bar{F}(x)}\right]\right]dF_{3:5}(x)\\
&\quad+\frac{3}{10}\frac{2}{9}\int_0^t \bar{G}(t-x)\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+2\frac{F(t)-F(x)}{\bar{F}(x)}\frac{\bar{F}(t)}{\bar{F}(x)}\right]dF_{3:5}(x)+\frac{3}{10}\frac{3}{9}\int_0^t \bar{G}(t-x)dF_{3:5}(x)\\
&\quad+\frac{1}{2}\frac{6}{10}\int_0^t \bar{G}(t-x)\frac{\bar{F}(t)}{\bar{F}(x)}dF_{4:5}(x)+\frac{1}{2}\frac{4}{10}\int_0^t \bar{G}(t-x)dF_{4:5}(x)+\frac{2}{10}\int_0^t \bar{G}(t-x)dF_{5:5}(x).
\end{aligned}
$$

Example 4. Consider the coherent system with lifetime

$$T=\min(X_1,\max(X_2,X_3),\max(X_3,X_4)).$$

The signature of this system is $p=(\frac{1}{4},\frac{7}{12},\frac{1}{6},0)$. In this system $k_\phi=1$ and $z_\phi=2$.

$s=1$	c
V_1	1
$\mathbf{B}_{1,c}$	–
$\mathbf{R}_{1,c}$	(2, 3, 4)
$P(V_1=c)$	1
$\bar{p}^{c,\mathbf{B}_{1,c}}$	$(0,\frac{2}{3},\frac{1}{3},0)$

$s=2$	c						
V_2	1			2	3		4
$\mathbf{B}_{2,c}$	(2)	(3)	(4)	(3)	(2)	(4)	(3)
$\mathbf{R}_{2,c}$	(3, 4)	(2, 4)	(2, 3)	(1, 4)	(1, 4)	(1, 2)	(1, 2)
$P(V_2=c)$	$\frac{3}{7}$			$\frac{1}{7}$	$\frac{2}{7}$		$\frac{1}{7}$
$\bar{p}^{c,\mathbf{B}_{2,c}}$	$(\frac{1}{2},\frac{1}{2},0)$	(1, 0, 0)	$(\frac{1}{2},\frac{1}{2},0)$	(1, 0, 0)	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$	(1, 0, 0)

$s=3$	c	
V_3	1	3
$\mathbf{B}_{3,c}$	(2, 4)	(2, 4)
$\mathbf{R}_{3,c}$	(3)	(1)
$P(V_3=c)$	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{p}^{c,\mathbf{B}_{3,c}}$	(1, 0)	(1, 0)

$$
\begin{aligned}
P(T^w>t) &= \frac{1}{4}P(X_{1:4}>t)+\frac{7}{12}P(X_{2:4}>t)+\frac{1}{6}P(X_{3:4}>t)\\
&\quad+\frac{1}{4}\int_0^t \bar{G}(t-x)\left[\begin{array}{c}\frac{2}{3}\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^3+3\frac{F(t)-F(x)}{\bar{F}(x)}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2\right]\\ +\frac{1}{3}\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^3+3\frac{F(t)-F(x)}{\bar{F}(x)}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+3\left(\frac{F(t)-F(x)}{\bar{F}(x)}\right)^2\frac{\bar{F}(t)}{\bar{F}(x)}\right]\end{array}\right]dF_{1:4}(x)\\
&\quad+\frac{7}{12}\frac{3}{7}\int_0^t \bar{G}(t-x)\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2 dF_{2:4}(x)
\end{aligned}
$$

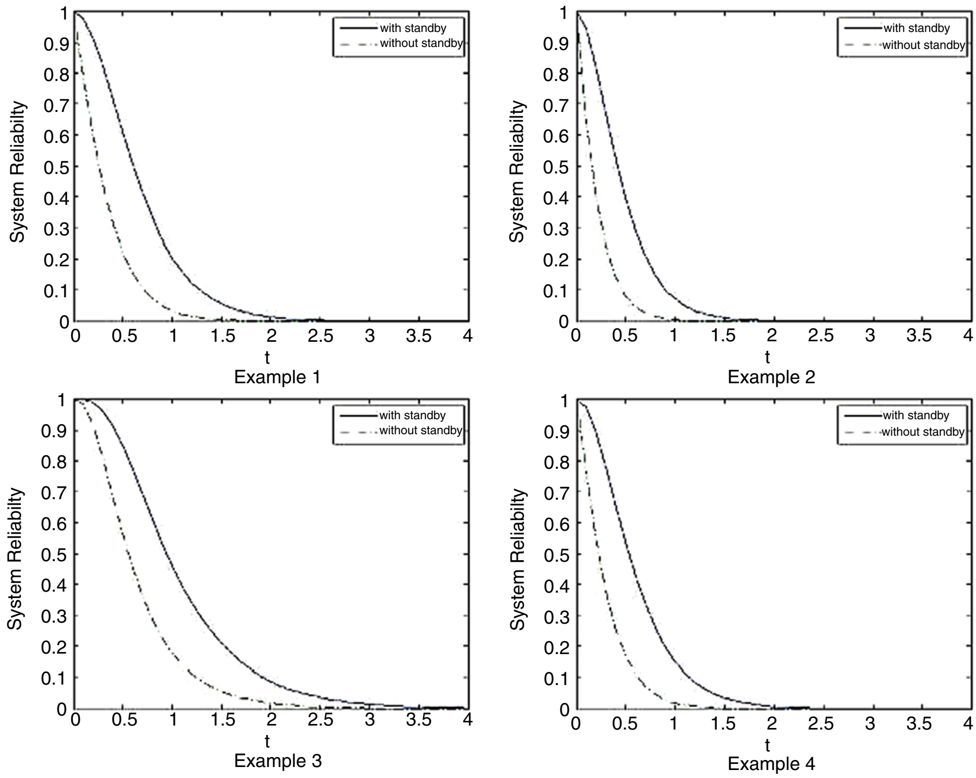

Fig. 1. Reliability function of systems with and without standby.

$$+\frac{7}{12}\frac{4}{7}\int_0^t \bar{G}(t-x)\left[\frac{1}{2}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+\frac{1}{2}\left[\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right)^2+\left(\frac{F(t)-F(x)}{\bar{F}(x)}\right)\frac{\bar{F}(t)}{\bar{F}(x)}\right]\right]dF_{2:4}(x)$$
$$+\frac{1}{6}\int_0^t \bar{G}(t-x)\frac{\bar{F}(t)}{\bar{F}(x)}dF_{3:4}(x).$$

As it can be seen from the examples it is hard to compute $P(T^w > t)$ for systems having complex structures even if they have few components. However for some particular systems of order n it can be computed easily.

Example 5. Consider the coherent system with lifetime

$$T = \min(X_1, \max(X_2, X_3, \dots, X_n)).$$

The signature of this system is $p = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}, \frac{2}{n}, 0)$. In this system $k_\phi = 1$ and $z_\phi = n-2$. For $s = 1, 2, \dots, n-2$, $P(V_s = 1) = 1$ and $P(V_s = c) = 0$, $c = 2, \dots, n$. For $s = n-1$ $P(V_{n-1} = 1) = \frac{1}{2}$ and $P(V_{n-1} = c) = \frac{1}{2(n-1)}$ $c = 2, 3 \dots, n$. Furthermore $\bar{p}^{c,\mathbf{B}_{s,c}} = (\underbrace{0, 0, \dots, 0,}_{n-s-1} 1, 0)$ for all s and c. Therefore

$$P(T^w > t) = \sum_{s=1}^{n-1}\left(p_s P(X_{s:n} > t) + p_s\int_0^t \bar{G}(t-x)P(X^{(s)}_{n-s:n-s} > t-x|X_{s:n} = x)dF_{s:n}(x)\right).$$

In general, the computation of $P(T^w > t)$ is not easy even when components have exponential lifetime distributions. In Fig. 1 one can see the reliability function of the four examples given above with and without a standby when $F(t) = G(t) = 1 - e^{-2t}$, $t > 0$.

In Table 1, the mean time to failure of different coherent systems with a standby unit ($E(T^w)$) and without a standby unit ($E(T)$) having independent and identical exponentially distributed components with mean 1, have been computed.

It can be seen clearly from Table 1 that the mean time to failure of coherent systems is nearly doubled by adding a cold standby component which shows the effect of the cold standby to coherent systems.

Table 1
Mean time to failure of systems with and without standby.

	T	$\mathbf{p}$	$E(T)$	$E(T^w)$
1	$\min(X_1, \max(X_2, X_3))$	$(\frac{1}{3}, \frac{2}{3}, 0)$	0.6667	1.2222
2	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	0.4167	0.7917
3	$\min(\max(X_1, X_2, X_3), \max(X_2, X_3, X_4), \max(X_3, X_4, X_5))$	$(0, 0, \frac{3}{10}, \frac{1}{2}, \frac{2}{10})$	1.3333	2.0944
4	$\min(X_1, \max(X_2, X_3), \max(X_2, X_4))$	$(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$	0.5833	1.0625
5	$\min(\max(X_1, X_2), \max(X_2, X_3), \max(X_3, X_4))$	$(0, \frac{1}{2}, \frac{1}{2}, 0)$	0.8333	1.3611
6	$\min(\max(X_1, X_2), \max(X_1, X_3), \max(X_1, X_4))$	$(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$	1.0833	1.9167
7	$\min(\max(X_1, X_2), \max(X_2, X_3), \max(X_3, X_4), \max(X_4, X_5))$	$(0, \frac{4}{10}, \frac{5}{10}, \frac{1}{10}, 0)$	0.7000	1.1417

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References

[1] A.J.C. Van Gemund, G.L. Reijns, Reliability analysis of k-out-of-n systems with single cold standby using Pearson distributions, IEEE Trans. Reliab. 61 (2012) 526–532.
[2] S. Eryilmaz, On the mean residual life of a k-out-of-n:G system with a single cold standby component, European J. Oper. Res. 222 (2012) 273–277.
[3] S. Eryilmaz, On reliability of a k-out-of-n system equipped with a single warm standby component, IEEE Trans. Reliab. 62 (2013) 499–503.
[4] S. Kochar, H. Mukerjee, F. Samaniego, The ''signature'' of a coherent system and its application to comparison among systems, Naval Res. Logist. 46 (1999) 507–523.
[5] P.J. Boland, Signatures of indirect majority systems, J. Appl. Probab. 38 (2001) 597–603.
[6] W. Kuo, M.J. Zuo, Optimal Reliability Modeling, Principles and Applications, John Wiley & Sons, Inc., New Jersey, 2003.
[7] J. Navarro, R. Rubio, Comparisons of coherent systems using stochastic precedence, TEST 19 (2010) 469–486.
[8] J. Navarro, T. Rychlik, Comparisons and bounds for expected lifetime of reliability systems, European J. Oper. Res. 207 (2010) 309–317.
[9] M. Mahmoudi, M. Asadi, The dynamic signature of coherent systems, IEEE Trans. Reliab. 60 (2011) 817–822.
[10] J. Navarro, F.J. Samaniego, N. Balakrishnan, Signature-based representations for the reliability of systems with heterogeneous components, J. Appl. Probab. 48 (2011) 856–867.
[11] S. Eryilmaz, A new look at dynamic behavior of binary coherent system from a state-level perspective, Ann. Oper. Res. 212 (2012) 115–125.
[12] M. Asadi, A. Berred, On the number of failed components in a coherent operating system, Statist. Probab. Lett. 82 (2012) 2156–2163.
[13] S. Eryilmaz, Computing reliability indices of repairable systems via signature, J. Comput. Appl. Math. 260 (2014) 229–235.
[14] S. Eryilmaz, A study on reliability of coherent systems equipped with a cold standby component, Metrika 77 (2013) 349–359.