Journal of Computational and Applied Mathematics 282 (2015) 262–267

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Stress strength reliability in the presence of fuzziness

Serkan Eryilmaz [a], G. Yazgı Tütüncü [b,*]

[a] Atilim University, Department of Industrial Engineering, Ankara, Turkey
[b] Izmir University of Economics, Department of Mathematics, Izmir, Turkey

ARTICLE INFO

Article history:
Received 2 May 2014
Received in revised form 8 October 2014

Keywords:
Estimation
Fuzzy reliability
System signature

ABSTRACT

This paper investigates the stress–strength reliability in the presence of fuzziness. The fuzzy membership function is defined as a function of the difference between stress and strength values, and the fuzzy reliability of single unit and multicomponent systems are calculated. The inclusion of fuzziness in the stress–strength interference enables the user to make more sensitive analysis. Illustrations are presented for various stress and strength distributions.

1. Introduction

Stress–strength analysis has attracted a great deal of attention in reliability literature. In a traditional setup of stress–strength analysis, a unit or system operates as soon as its strength exceeds the stress imposed upon it. Most of the studies in this setup focus on the computation and estimation of the reliability for various stress and strength distributions such as exponential, Weibull, normal, and gamma. A comprehensive review of the topic is presented in [1]. Some recent discussions in this direction are in [2–6].

Stress–strength reliability has also been studied under multi-component setup, i.e. the system consists of more than two components. Bhattacharya and Johnson [7] considered the k-out-of-n structure under stress strength setup. According to their definition, the system consists of n components, and functions if at least k components survive a common random stress. Hanagal [8] studied the series system reliability under stress–strength setup. The stress–strength reliability of a consecutive k-out-of-n system has been studied in [9]. Recent works on multi-component stress strength reliability are in [10–13].

In this paper, we study the stress–strength reliability in the presence of fuzziness which is attached to the difference between stress and strength values. In particular, we follow the idea of Huang [14] who investigated the reliability analysis in the presence of fuzziness attached to operating time of a system. The paper is organized as follows. In Section 2, we define and study the fuzzy stress–strength reliability for a single unit system. Section 3 extends the results to a multicomponent system having an arbitrary structure.

2. Reliability evaluation and estimation

Let X and Y denote respectively the strength and stress random variables. If X and Y are independent with respective distribution functions F_X and F_Y, then the traditional stress–strength reliability can be computed from

$$R = P\{X > Y\} = \iint_{x>y} dF_X(x)dF_Y(y). \tag{1}$$

* Corresponding author.
E-mail address: yazgi.tutuncu@ieu.edu.tr (G.Y. Tütüncü).

http://dx.doi.org/10.1016/j.cam.2014.12.047
0377-0427/

The conventional stress–strength reliability given by (1) can be rewritten as

$$R = P\{X > Y\} = \int_0^\infty \int_0^\infty C_{A(y)}(x) dF_X(x) dF_Y(y),$$

where the characteristic function for the set $A(y) = \{x : x > y\}$ is given as

$$C_{A(y)}(x) = \begin{cases} 0, & \text{if } x \le y \\ 1, & \text{if } x > y. \end{cases}$$

For the fuzzy event "X is fuzzily bigger than Y" (denoted by $X \succ Y$), let $\mu_{A(y)}(x)$ denote the corresponding membership function. Then from the definition of fuzzy probability [15], the fuzzy stress–strength reliability can be represented as

$$R_F = P\{X \succ Y\} = \int_0^\infty \int_0^\infty \mu_{A(y)}(x) dF_X(x) dF_Y(y), \tag{2}$$

where we define the appropriate membership function as

$$\mu_{A(y)}(x) = \begin{cases} 0, & \text{if } x \le y \\ h(x-y), & \text{if } x > y \end{cases}$$

for an increasing function h. Thus Eq. (2) can be rewritten as

$$R_F = P\{X \succ Y\} = \int_0^\infty \int_y^\infty h(x-y) dF_X(x) dF_Y(y). \tag{3}$$

Eq. (3) assigns a value for the reliability by considering the difference between strength and stress values. The traditional stress–strength reliability only considers the event that X is greater than Y. In the fuzzy case, the reliability considers the distance $X - Y$ when X is greater than Y. According to this new fuzzy stress–strength interference, for $X = x$ and $Y = y$, with an increase in the values of $x - y$ the system becomes more reliable. Therefore, such a consideration may enable us to make a more sensitive analysis.

Below we illustrate the computation of R_F when stress and strength distributions are exponential.

Example 1. Let $F_X(x) = 1 - e^{-\lambda_1 x}, x > 0$ and $F_Y(y) = 1 - e^{-\lambda_2 y}, y > 0$. Suppose that $h(u) = 1 - e^{-ku}, u > 0$, i.e. the corresponding membership function is

$$\mu_{A(y)}(x) = \begin{cases} 0, & \text{if } x \le y \\ 1 - e^{-k(x-y)}, & \text{if } x > y. \end{cases}$$

Then

$$\begin{aligned} R_F &= \int_0^\infty \int_y^\infty (1 - e^{-k(x-y)}) \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} dx dy \\ &= \frac{\lambda_2}{\lambda_1 + \lambda_2} - \frac{\lambda_1 \lambda_2}{(\lambda_1 + k)(\lambda_1 + \lambda_2)}. \end{aligned}$$

It is easy to see that R_F tends to conventional reliability $R = \frac{\lambda_2}{\lambda_1 + \lambda_2}$ as $k \to \infty$.

The computation of R_F can be difficult for more complex distributions F_x and F_Y. In this case, it might be appropriate to get bounds for the fuzzy reliability R_F.

Let $h(u) = 1 - g(u)$ and $g(u)$ be a convex function. Then one can show that (see Appendix)

$$R_F \le R \cdot [1 - g(E(X - Y \mid X > Y))]. \tag{4}$$

The conditional expected value $E(X - Y \mid X > Y)$ can be more easily calculated than (3).

Example 2. Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. Then it can be shown that

$$E(X - Y \mid X > Y) = \mu_X - \mu_Y + \frac{1}{1 - \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)} \sqrt{\frac{\sigma_X^2 + \sigma_Y^2}{2\pi}} e^{-\frac{(\mu_X - \mu_Y)^2}{2(\sigma_X^2 + \sigma_Y^2)}},$$

and

$$R = P(X > Y) = P(X - Y > 0) = 1 - \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right),$$

where $\Phi(z)$ is the cdf of standard normal variate. Thus an upper bound for the fuzzy reliability can be obtained using the last two equations in (4). In Table 1, we compute the upper bound R_F^u for different values of k when $\mu_X = 3$, $\sigma_X = 0.5$, $\mu_Y = 1$, $\sigma_Y = 1$ and $h(u) = 1 - g(u) = 1 - e^{-ku}$. We also compute R_F by the method of computer simulation.

Table 1
Simulated value and upper bound for the fuzzy reliability.

k	R_F	R_F^u
2	0.8901	0.9485
3	0.9228	0.9614
5	0.9432	0.9632

2.1. *A simple unbiased estimator for R_F*

Eq. (3) can also be written as

$$R_F = E(h(X - Y); X > Y).$$

Thus for two independent random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ from the populations F_X and F_Y respectively, an empirical estimate of R_F can be formulated as

$$\hat{R}_F = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} h(X_i - Y_j)I(X_i > Y_j) = \frac{W}{nm},$$

where

$$W = \sum_{i=1}^{n}\sum_{j=1}^{m} h(X_i - Y_j)I(X_i > Y_j).$$

It is clear that $E(W) = nmR_F$, i.e. $\hat{R}_F$ is an unbiased estimator. The variance of $\hat{R}_F$ is derived as (see Appendix)

$$Var(\hat{R}_F) = \frac{1}{nm}\left[(n-1)(m-1)R_F^2 + u_1 + (n-1)u_2 + (m-1)u_3\right] - R_F^2, \tag{5}$$

where

$$u_1 = \int_0^\infty \int_y^\infty h^2(x - y)dF_X(x)dF_Y(y),$$

$$u_2 = \int_0^\infty \int_y^\infty \int_y^\infty h(x_1 - y)h(x_2 - y)dF_X(x_2)dF_X(x_1)dF_Y(y),$$

and

$$u_3 = \int_0^\infty \int_0^x \int_0^x h(x - y_1)h(x - y_2)dF_{Y_1}(y_1)dF_{Y_2}(y_2)dF_X(x).$$

Using Theorem 3.4.13 of Randles and Wolfe [16] we can obtain a normal approximation for $\hat{R}_F$. For $N = n + m$,

$$\sqrt{N}(\hat{R}_F - R_F)$$

has a limiting normal distribution with mean 0 and variance

$$\frac{u_2 - R_F^2}{1 - \theta} + \frac{u_3 - R_F^2}{\theta},$$

where $\lim_{N\to\infty} \frac{n}{N} = \theta \in (0, 1)$.

3. Reliability under coherent systems

Consider a system consisting of n independent components whose strengths are denoted by $X_1, \ldots, X_n$ with common cdf $F_X(x) = P\{X_i \le x\},\ i = 1, \ldots, n$. Suppose that these components are subject to a common random stress Y. If the system has a series structure, then the fuzzy stress–strength reliability of the system is

$$R_F^S = P\{X_{1:n} \succ Y\} = \int_0^\infty \int_y^\infty h(x - y)dF_{X_{1:n}}(x)dF_Y(y),$$

where $X_{1:n} = \min(X_1, \ldots, X_n)$. Because $F_{X_{1:n}}(x) = 1 - (1 - F_X(x))^n$,

$$R_F^S = n\int_0^\infty \int_y^\infty h(x - y)(1 - F_X(x))^{n-1}dF_X(x)dF_Y(y).$$

Similarly, if the system has a parallel structure, then

$$R_F^P = P\{X_{n:n} \succ Y\} = \int_0^\infty \int_y^\infty h(x - y)dF_{X_{n:n}}(x)dF_Y(y),$$

where $X_{n:n} = \max(X_1, \ldots, X_n)$. Because $F_{X_{n:n}}(x) = F_X^n(x)$,

$$R_F^P = n \int_0^\infty \int_y^\infty h(x-y) F_X^{n-1}(x) dF_X(x) dF_Y(y).$$

Next, consider the case when the system has $(n-i+1)$-out-of-n structure, i.e. the system functions if at least $n-i+1$ of the components survive a common random stress Y. In this case, the reliability is formulated as

$$R_F^{i,n} = P\{X_{i:n} \succ Y\} = \int_0^\infty \int_y^\infty h(x-y) dF_{X_{i:n}}(x) dF_Y(y),$$

where $X_{i:n}$ is the ith smallest among $X_1, \ldots, X_n$, and

$$dF_{X_{i:n}}(x) = \frac{1}{B(i, n-i+1)} F_X^{i-1}(x)(1-F_X(x))^{n-i} dF_X(x).$$

Now, consider the general case when the system has an arbitrary coherent structure ϕ. Then

$$R_F^\phi = P\{\phi(X_1, \ldots, X_n) \succ Y\} = \int_0^\infty \int_y^\infty h(x-y) dF_{\phi(X_1,\ldots,X_n)}(x) dF_Y(y).$$

The distribution $F_{\phi(X_1,\ldots,X_n)}(x)$ can be written in terms of system signature. According to Samaniego [17],

$$F_{\phi(X_1,\ldots,X_n)}(x) = \sum_{i=1}^n p_i F_{X_{i:n}}(x),$$

where $\mathbf{p} = (p_1, \ldots, p_n)$ is the signature of a coherent system ϕ with

$$p_i = \frac{\text{\# of orderings for which the } i\text{th failure causes system failure}}{n!},$$

$i = 1, \ldots, n$, and $\sum_{i=1}^n p_i = 1$. Thus

$$\begin{aligned} R_F^\phi &= \sum_{i=1}^n p_i \int_0^\infty \int_y^\infty h(x-y) dF_{X_{i:n}}(x) dF_Y(y) \\ &= \sum_{i=1}^n p_i P\{X_{i:n} \succ Y\} \\ &= \sum_{i=1}^n p_i E(h(X_{i:n} - Y); X_{i:n} > Y). \end{aligned}$$

Example 3. For the stress and strength distributions and the membership function in Example 1, the reliability of $(n-i+1)$-out-of-n structure can be calculated from

$$\begin{aligned} R_F^{i,n} &= \frac{1}{B(i, n-i+1)} \int_0^\infty \int_y^\infty (1-e^{-k(x-y)})(1-e^{-\lambda_1 x})^{i-1} (e^{-\lambda_1 x})^{n-i} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} dx dy \\ &= P\{X_{i:n} > Y\} - \frac{\lambda_1 \lambda_2}{B(i, n-i+1)} \int_0^\infty \int_y^\infty e^{-k(x-y)} (1-e^{-\lambda_1 x})^{i-1} (e^{-\lambda_1 x})^{n-i+1} e^{-\lambda_2 y} dx dy. \end{aligned}$$

It is clear that

$$\begin{aligned} &\int_0^\infty \int_y^\infty e^{-k(x-y)} (1-e^{-\lambda_1 x})^{i-1} (e^{-\lambda_1 x})^{n-i+1} e^{-\lambda_2 y} dx dy \\ &= \sum_{m=0}^{i-1} (-1)^m \binom{i-1}{m} \int_0^\infty \int_y^\infty e^{-x(\lambda_1(m+n-i+1)+k)} e^{-y(\lambda_2-k)} dx dy \\ &= \sum_{m=0}^{i-1} (-1)^m \binom{i-1}{m} \frac{1}{(\lambda_1(m+n-i+1)+k)(\lambda_2+\lambda_1(m+n-i+1))}, \end{aligned}$$

and

$$P\{X_{i:n} > Y\} = \frac{1}{B(i, n-i+1)} \sum_{m=0}^{i-1} (-1)^m \binom{i-1}{m} \frac{1}{(m+n-i+1)\left(1+\frac{\lambda_1}{\lambda_2}(m+n-i+1)\right)}.$$

Table 2
Fuzzy stress–strength reliability of all coherent systems with three components when $\lambda_1 = 1$ and $\lambda_2 = 5$.

k	R_F^S	$R_F^{\phi_1}$	$R_F^{2,3}$	$R_F^{\phi_2}$	R_F^P
1	0.1563	0.3199	0.4018	0.4985	0.6920
2	0.2500	0.4643	0.5714	0.6627	0.8452
3	0.3125	0.5446	0.6607	0.7411	0.9018
4	0.3571	0.5952	0.7143	0.7857	0.9286
5	0.3906	0.6298	0.7494	0.8140	0.9433
10	0.4808	0.7097	0.8242	0.8720	0.9678

Therefore

$$R_F^{i,n} = \frac{1}{B(i, n-i+1)} \sum_{m=0}^{i-1} (-1)^m \binom{i-1}{m} \frac{1}{(m+n-i+1)\left(1+\frac{\lambda_1}{\lambda_2}(m+n-i+1)\right)}$$
$$- \frac{\lambda_1\lambda_2}{B(i, n-i+1)} \sum_{m=0}^{i-1} (-1)^m \binom{i-1}{m} \frac{1}{(\lambda_1(m+n-i+1)+k)(\lambda_2+\lambda_1(m+n-i+1))},$$

for $i = 1, \dots, n$.

Example 4. For the stress and strength distributions and the membership function in Example 1, consider the reliability of the coherent system with structure function

$$\phi_1(x_1, x_2, x_3) = \min(x_1, \max(x_2, x_3)).$$

The signature of this structure is $\mathbf{p} = (\frac{1}{3}, \frac{2}{3}, 0)$ (see, e.g. Table 1 of Eryilmaz [18]). Therefore

$$R_F^{\phi_1} = \frac{1}{3} R_F^{1,3} + \frac{2}{3} R_F^{2,3},$$

where $R_F^{1,3}$ and $R_F^{2,3}$ can be calculated from Example 2. Similarly, for the structure

$$\phi_2(x_1, x_2, x_3) = \max(x_1, \min(x_2, x_3)),$$

the signature is $\mathbf{p} = (0, \frac{2}{3}, \frac{1}{3})$, and hence

$$R_F^{\phi_2} = \frac{2}{3} R_F^{2,3} + \frac{1}{3} R_F^{3,3}.$$

In Table 2, we compute fuzzy stress–strength reliability of all coherent systems with $n = 3$ three components, i.e. the series and parallel structures, 2-out-of-3 structure, and the structures defined by ϕ_1 and ϕ_2.

Appendix

Proof of (4). If $h(u) = 1 - g(u)$, then from (3) we have

$$\begin{aligned} R_F &= R - \int_0^\infty \int_y^\infty g(x-y)dF_X(x)dF_Y(y) \\ &= R - E(g(X-Y); X > Y) \\ &= R - R \cdot E(g(X-Y) \mid X > Y) \\ &= R \cdot [1 - E(g(X-Y) \mid X > Y)]. \end{aligned}$$

Because g is convex,

$$E(g(X-Y) \mid X > Y) \geq g(E(X-Y \mid X > Y))$$

and hence the proof is complete. ■

Proof of (5). It is clear that

$$Var(\hat{R}_F) = \frac{1}{(nm)^2} E(W^2) - R_F^2. \quad (6)$$

The second moment of W can be computed from

$$
\begin{aligned}
E(W^2) = E\Bigg(&\sum_{i_1\neq i_2}\sum_{j_1\neq j_2} h(X_{i_1}-Y_{j_1})I(X_{i_1}>Y_{j_1})h(X_{i_2}-Y_{j_2})I(X_{i_2}>Y_{j_2}) + \sum_{i_1=i_2}\sum_{j_1=j_2} h^2(X_{i_1}-Y_{j_1})I(X_{i_1}>Y_{j_1}) \\
&+ \sum_{i_1\neq i_2}\sum_{j_1=j_2} h(X_{i_1}-Y_{j_1})I(X_{i_1}>Y_{j_1})h(X_{i_2}-Y_{j_2})I(X_{i_2}>Y_{j_2}) \\
&+ \sum_{i_1=i_2}\sum_{j_1\neq j_2} h(X_{i_1}-Y_{j_1})I(X_{i_1}>Y_{j_1})h(X_{i_2}-Y_{j_2})I(X_{i_2}>Y_{j_2})\Bigg) \\
= nm\Bigg[&(n-1)(m-1)R_F^2 + \iint_{x>y} h^2(x-y)dF_X(x)dF_Y(y) \\
&+ (n-1)\iiint_{x_1>y,x_2>y} h(x_1-y)h(x_2-y)dF_X(x_1)dF_X(x_2)dF_Y(y) \\
&+ (m-1)\iiint_{x>y_1,x>y_2} h(x-y_1)h(x-y_2)dF_{Y_1}(y_1)dF_{Y_2}(y_2)dF_X(x)\Bigg].
\end{aligned}
$$

Thus the proof of (5) is immediate from (6).

References

[1] S. Kotz, Y. Lumelskii, M. Pensky, The Stress Stength Model and its Generalizations, World Scientific, Singapore, 2003.
[2] A. Chaturvedia, V. Sharmaa, A note on the estimation of $P(Y > X)$ in two-parameter exponential distributions, Statistics 44 (2010) 73–75.
[3] K. Krishnamoorthy, Y. Lin, Confidence limits for stress–strength reliability involving Weibull models, J. Statist. Plann. Inference 140 (2010) 1754–1764.
[4] S. Rezaei, R. Tahmasbi, M. Mahmoodi, Estimation of $P(Y < X)$ for generalized Pareto distribution, J. Statist. Plann. Inference 140 (2010) 480–494.
[5] S. Eryilmaz, A new perspective to stress–strength models, Ann. Inst. Statist. Math. 63 (2011) 101–115.
[6] K. Huang, J. Mi, Z. Wang, Inference about reliability parameter with gamma strength and stress, J. Statist. Plann. Inference 142 (2012) 848–854.
[7] G.K. Bhattacharya, R.A. Johnson, Estimation of reliability in a multicomponent stress–strength model, J. Amer. Statist. Assoc. 69 (1974) 966–970.
[8] D. Hanagal, Estimation of system reliability in multicomponent series stress strength models, J. Indian Statist. Assoc. 41 (2003) 1–7.
[9] S. Eryilmaz, Consecutive k-out-of-n: G system in stress–strength setup, Comm. Statist. Simulation Comput. 37 (2008) 579–589.
[10] S. Eryilmaz, On system reliability in stress–strength setup, Statist. Probab. Lett. 80 (2010) 834–839.
[11] N.A. Mokhlisa, S.K. Khames, Reliability of multi-component stress–strength models, J. Egyptian Math. Soc. 19 (2011) 106–111.
[12] I. Bairamov, S. Gurler, B. Ucer, On the mean remaining strength of the k-out-of-n: F system with exchangeable components, Comm. Statist. Simulation Comput. 44 (2015) 1–13.
[13] M. Ozkut, I. Bayramoglu, On Marshall–Olkin type distribution with effect of shock magnitude, J. Comput. Appl. Math. 271 (2015) 150–162.
[14] H.Z. Huang, Reliability analysis method in the presence of fuzziness attached to operating time, Microelectron. Reliab. 35 (1995) 1483–1487.
[15] L.A. Zadeh, Probability measures of fuzzy events, J. Math. Anal. Appl. 10 (1968) 421–427.
[16] R.H. Randles, D.A. Wolfe, Introduction to the Theory of Nonparametric Statistics, John Wiley, New York, 1979.
[17] F.J. Samaniego, System Signatures and their Applications in Engineering Reliability, Springer, New York, 2007.
[18] S. Eryilmaz, Computing reliability indices of repairable systems via signature, J. Comput. Appl. Math. 260 (2014) 229–235.