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Reliability analysis under Marshall–Olkin run shock model

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ABSTRACT

In this paper, a new shock model called Marshall–Olkin run shock model is defined and studied. According to the model, two components are subject to shocks that may arrive from three different sources, and component i fails when it is subject to k consecutive critical shocks from source i or k consecutive critical shocks from source 3, $i = 1, 2$. Reliability and mean residual life functions of such components are studied when the times between shocks follow phase-type distribution.

1. Introduction

As is well-known, the Marshall–Olkin distribution arises from a shock model [1]. According to the model, a system consisting of two components is subject to shocks coming from three sources. These shocks are produced at a random time. A shock produced by the first source affects the first component, the shock produced by the second source affects the second component and a shock produced by the third source affects both components. Marshall–Olkin type distributions have been of great interest in recent years. Ozkut and Bayramoglu [2] introduced a Marshall–Olkin type distribution with effect of shock magnitude. Okasha and Kayid [3] introduced a new family of Marshall–Olkin extended generalized linear exponential distribution. Durante et al. [4] studied Marshall–Olkin type copulas generated by a global shock. Bayramoglu and Ozkut [5] considered coherent systems subjected to Marshall–Olkin type shocks coming at random times and destroying components of the system.

Shock models have been extensively studied in the literature. Various shock models have been defined and analyzed in the context of reliability. They can be classified as cumulative shock models [6]), extreme shock models [7], run shock models [8], delta shock models [9] and mixed shock models [10]. In a typical shock model, a system (or component) is subject to shocks of random magnitudes at random times and it fails according to the rule defined by the model. In most cases, the failure time of the system is represented by a compound random variable which appears as a function of magnitudes of shocks and times between consecutive shocks. For example, in a run shock model the system fails if it is subject to k consecutive critical shocks [8]. A critical shock is a shock which is harmful for the system. Recently, many research papers on reliability shock models have been published. Parvardeh and Balakrishnan [11] have obtained some results on reliability characteristics of a system under mixed delta shock models. Eryilmaz [12] studied the delta shock model under the assumption that shocks arrive according to a Polya process. Rafiee et al. [13] investigated reliability modeling for systems subject to dependent competing risks with generalized mixed shock models. Mercier and Pham [14] studied a bivariate failure time model with random shocks and mixed effects.

The phase-type distributions have been found to be suitable and useful for modeling times between shocks. Their mathematical tractability makes it possible to obtain interesting and useful results. For example, phase-type distributions

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are closed under some operations, which are useful in reliability analysis [15]. The phase-type distributions have been utilized for the study of reliability shock models in [16–18]. Neuts and Meier [19] used phase-type distributions in reliability modeling of systems with two components. Eryilmaz [20] proposed a method to compute optimal replacement time and mean residual life of a system defined under a particular class of shock models when the times between shocks follow phase-type distribution.

In the present paper, we define and study a shock model which combines Marshall–Olkin and run shock models. According to the new model, a system that consists of two components is subject to shocks that may arrive from three different sources. A shock that is produced by source 1 (2) only affects component 1 (2) while the shock that is produced by source 3 may affect both components. The produced shocks are classified as critical or non-critical. A component fails if it is subject to k consecutive critical shocks from the same source.

The present paper is organized as follows. In Section 2, we list some notations and acronyms, and provide some properties of phase-type distributions which will be useful in our developments. In Section 3, we define and study Marshall–Olkin type run shock model. Section 4 contains computations of mean residual life (MRL) functions under the Marshall–Olkin run shock model.

2. Definitions and preliminaries

Below, we list the notations and acronyms that will be used throughout the paper:
PH_d: Discrete phase-type distribution.
PH_c: Continuous phase-type distribution.
X_{1i}: The interarrival time between $(i-1)$th and ith shocks which is produced by source 1.
X_{2i}: The interarrival time between $(i-1)$th and ith shocks which is produced by source 2.
X_{3i}: The interarrival time between $(i-1)$th and ith shocks which is produced by source 3.
p_j: The probability that the shock produced by source j is critical
$N_j(k)$: Total number of shocks (produced by source j) until k consecutive critical shocks, $j = 1, 2, 3$
S_i: The lifetime of component i, $i = 1,2$

2.1. Phase-type distributions and some properties

A discrete phase type distribution is the distribution of the time to absorption in an absorbing Markov chain. A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state in one or more steps. For a discrete phase-type random variable N, the probability mass function (pmf) is represented as

$$P\{N = n\} = \mathbf{a}\mathbf{Q}^{n-1}\mathbf{u}'$$

for $n \in \mathbb{N}$, where $\mathbf{Q} = (\mathbf{q}_{ij})_{m\times m}$ is a matrix that includes the transition probabilities among the m transient states, and $\mathbf{u}' = (\mathbf{I} - \mathbf{Q})\mathbf{e}'$ is a vector which includes the transition probabilities from transient states to the absorbing state, $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_m)$ with $\sum_{i=1}^{m} a_i = 1$, and $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{Q}$ must satisfy the condition that $\mathbf{I} - \mathbf{Q}$ is nonsingular. We shall use $N \sim PH_d(\mathbf{a}, \mathbf{Q})$ to represent that the random variable N has a discrete phase-type distribution.

The distribution of a continuous random variable X is said to be phase-type if it is the distribution of the time until absorption in a finite state continuous time Markov chain with m transient states and one absorbing state. For a non-negative continuous phase-type random variable X, the cumulative distribution function (cdf) is represented as

$$P(X \le x) = 1 - \boldsymbol{\alpha}\exp(\mathbf{A}x)\mathbf{e}'$$

where the matrix $\mathbf{A}$ of dimension $m \times m$ has negative diagonal elements, and non-negative off-diagonal elements, and $\mathbf{e} = (1, \ldots, 1)_{1\times m}$. All elements of the row vector $\boldsymbol{\alpha} = (a_1, \ldots, a_m)$ are nonnegative. Exponential, Erlang, generalized Erlang, and Coxian distributions are some well-known continuous phase-type distributions [21]. We shall use $X \sim PH_c(\boldsymbol{\alpha}, \mathbf{A})$ to represent that the random variable X has a continuous phase-type distribution of order m with a PH-generator $\mathbf{A}$ and substochastic vector $\boldsymbol{\alpha}$, i.e. $\boldsymbol{\alpha}\mathbf{e}' \le 1$. The n th moment of X is given by

$$E(X^n) = (-1)^n n!\boldsymbol{\alpha}\mathbf{A}^{-n}\mathbf{e}'.$$

The class of phase-type distributions are closed under various operations. Two important closure properties that will be used in the present work are given below. Their proofs can be found in [21]. Note that $\otimes$ denotes Kronecker product.

Proposition 1. *Let $X \sim PH_c(\boldsymbol{\alpha}, \mathbf{A})$ and $Y \sim PH_c(\boldsymbol{\beta}, \mathbf{B})$ be two independent phase-type random variables. Then* $\min(X, Y) \sim PH_c(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}, \mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B})$.

Proposition 2. *Assume that $X_1, X_2, \ldots$ are independent and $X_i \sim PH_c(\boldsymbol{\alpha}, \mathbf{A})$, $i = 1, 2, \ldots$ and independently $N \sim PH_d(\mathbf{a}, \mathbf{Q})$. If $\boldsymbol{\alpha}$ and $\mathbf{a}$ are stochastic vectors, i.e. $\boldsymbol{\alpha}\mathbf{e}' = 1$, $\mathbf{a}\mathbf{e}' = 1$, then*

$$\sum_{i=1}^{N} X_i \sim PH_c(\boldsymbol{\alpha} \otimes \mathbf{a}, \mathbf{A} \otimes \mathbf{I} + (\mathbf{a}^0\boldsymbol{\alpha}) \otimes \mathbf{Q}),$$

where $\mathbf{a}^0 = -\mathbf{A}\mathbf{e}'$.

Proposition 3. *If* $X \sim PH_c(\boldsymbol{\alpha}, \mathbf{A})$, *then* $(X - t \mid X > t) \sim PH_c(\frac{\boldsymbol{\alpha}\exp(\mathbf{A}t)}{\boldsymbol{\alpha}\exp(\mathbf{A}t)\mathbf{e}'}, \mathbf{A})$.

3. Marshall–Olkin run shock model

Consider a system of two components which are subject to shocks that may arrive from three different sources. According to Marshall–Olkin run shock model, component i fails when it is subject to k consecutive critical shocks from source i or k consecutive critical shocks from source 3, $i = 1, 2$. That is, for the failure of a component, k consecutive critical shocks must be produced by the same source. Then the lifetimes of components are defined respectively as

$$S_1 = \min(T_1(k), T_3(k)),$$

$$S_2 = \min(T_2(k), T_3(k)),$$

where

$$T_j(k) = \sum_{i=1}^{N_j(k)} X_{ji},$$

for $j = 1, 2, 3$. The random variable $N_j(k)$ counts the number of shocks until k consecutive critical shocks and is known to have geometric distribution of order k with phase representation $N_j(k) \sim PH_d(\mathbf{a}, \mathbf{Q}_j)$ with $\mathbf{a} = (1, 0, \dots, 0)_{1\times k}$ and

$$\mathbf{Q}_j = \begin{bmatrix} 1-p_j & p_j & 0 & \dots & 0 \\ 1-p_j & 0 & p_j & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-p_j & 0 & 0 & \dots & 0 \end{bmatrix}_{k\times k},$$

where p_j denotes the probability that the shock produced by source j is critical, $j = 1, 2, 3$ (see, e.g. [22]).

In the following theorem, we obtain the joint survival function of (S_1, S_2) when interarrival times $X_{j1}, X_{j2}, \dots$ have a common phase-type distribution with $X_{ji} \sim PH_c(\boldsymbol{\alpha}_j, \mathbf{A}_j)$, $i = 1, 2, \dots$

Theorem 4. *Let* $X_{ji} \sim PH_c(\boldsymbol{\alpha}_j, \mathbf{A}_j)$ *and* $\boldsymbol{\alpha}_j\mathbf{e}' = 1$, $\mathbf{a}\mathbf{e}' = 1$. *Then the joint survival function of* (S_1, S_2) *is*

$$\begin{aligned} P(S_1 > u_1, S_2 > u_2) &= (\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_1 \otimes \mathbf{I}) + \left((-\mathbf{A}_1\mathbf{e}')\boldsymbol{\alpha}_1 \otimes \mathbf{Q}_1\right)\right) u_1\right] \mathbf{e}' \\ &\quad \times (\boldsymbol{\alpha}_2 \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_2 \otimes \mathbf{I}) + \left((-\mathbf{A}_2\mathbf{e}')\boldsymbol{\alpha}_2 \otimes \mathbf{Q}_2\right)\right) u_2\right] \mathbf{e}' \\ &\quad \times (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\left((\mathbf{A}_3 \otimes \mathbf{I}) + \left((-\mathbf{A}_3\mathbf{e}')\boldsymbol{\alpha}_3 \otimes \mathbf{Q}_3\right)\right) \max(u_1, u_2)\right] \mathbf{e}', \end{aligned}$$

where $\otimes$ *is the Kronecker product, and* $\mathbf{I}$ *represents the identity matrix.*

Proof. By the definition of the model,

$$\begin{aligned} P(S_1 > u_1, S_2 > u_2) &= P(T_1(k) > u_1, T_2(k) > u_2, T_3(k) > \max(u_1, u_2)) \\ &= P(T_1(k) > u_1)P(T_2(k) > u_2)P(T_3(k) > \max(u_1, u_2)). \end{aligned}$$

Because $N_j(k) \sim PH_d(\mathbf{a}, \mathbf{Q}_j)$ and $X_{ji} \sim PH_c(\boldsymbol{\alpha}_j, \mathbf{A}_j)$, using Proposition 2 one obtains

$$T_j(k) \sim PH_c\left((\boldsymbol{\alpha}_j \otimes \mathbf{a}), \left(\mathbf{A}_j \otimes \mathbf{I}\right) + \left((-\mathbf{A}_j\mathbf{e}')\boldsymbol{\alpha}_j \otimes \mathbf{Q}_j\right)\right),$$

for $j = 1, 2, 3$. Therefore the survival function of $T_j(k)$ can be computed from

$$P\left(T_j(k) > t\right) = \left(\boldsymbol{\alpha}_j \otimes \mathbf{a}\right) \exp\left[\left(\left(\mathbf{A}_j \otimes \mathbf{I}\right) + \left((-\mathbf{A}_j\mathbf{e}')\boldsymbol{\alpha}_j \otimes \mathbf{Q}_j\right)\right) t\right] \mathbf{e}'.$$

Thus the proof is complete. ■

The following result is an immediate consequence of Proposition 1.

Proposition 5. *The individual lifetime random variables* S_1 *and* S_2 *have phase-type distributions with*

$$S_1 \sim PH_c\left(\mathbf{v}_1 \otimes \mathbf{v}_3, \mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right),$$

and

$$S_2 \sim PH_c\left(\mathbf{v}_2 \otimes \mathbf{v}_3, \mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right),$$

Table 1
Mean time to failure values of components.

k	p_1	p_2	p_3	$E(S_1)$	$E(S_2)$
2	0.1	0.15	0.2	19.8353	16.7872
	0.2	0.15	0.2	13.9321	16.7872
	0.1	0.25	0.2	19.8353	11.4671
	0.1	0.15	0.25	9.9870	11.5360
3	0.1	0.15	0.2	138.8134	120.6339
	0.2	0.15	0.2	97.4659	120.6339
	0.1	0.25	0.2	138.8134	75.0522
	0.1	0.15	0.25	76.1065	70.0590

where $\mathbf{v}_1 = \boldsymbol{\alpha}_1 \otimes \mathbf{a}$, $\mathbf{v}_2 = \boldsymbol{\alpha}_2 \otimes \mathbf{a}$, $\mathbf{v}_3 = \boldsymbol{\alpha}_3 \otimes \mathbf{a}$,

$$\begin{aligned}
\mathbf{Z}_1 &= (\mathbf{A}_1 \otimes \mathbf{I}) + \left((-\mathbf{A}_1\mathbf{e}')\boldsymbol{\alpha}_1 \otimes \mathbf{Q}_1\right), \\
\mathbf{Z}_2 &= (\mathbf{A}_2 \otimes \mathbf{I}) + \left((-\mathbf{A}_2\mathbf{e}')\boldsymbol{\alpha}_2 \otimes \mathbf{Q}_2\right), \\
\mathbf{Z}_3 &= (\mathbf{A}_3 \otimes \mathbf{I}) + \left((-\mathbf{A}_3\mathbf{e}')\boldsymbol{\alpha}_3 \otimes \mathbf{Q}_3\right).
\end{aligned}$$

Corollary 6. *The mean time to failure values of components can be computed from*

$$\begin{aligned}
E(S_1) &= -(\mathbf{v}_1 \otimes \mathbf{v}_3)(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)^{-1}\mathbf{e}', \\
E(S_2) &= -(\mathbf{v}_2 \otimes \mathbf{v}_3)(\mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)^{-1}\mathbf{e}'.
\end{aligned}$$

Example 7. Let the times between successive shocks produced by source j follow Erlang distribution with parameters m_j and λ_j, $j = 1, 2, 3$. That is, $X_{ji} \sim PH_c(\boldsymbol{\alpha}_j, \mathbf{A}_j)$ with $\boldsymbol{\alpha}_j = (0, \ldots, 0, 1)$, and

$$\mathbf{A}_j = \begin{bmatrix} -\lambda_j & 0 & \cdots & 0 \\ \lambda_j & -\lambda_j & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda_j & -\lambda_j \end{bmatrix}_{m_j \times m_j}.$$

Assume that the components fail when they are subject to $k = 2$ consecutive critical shocks. Then $N_j(k) \sim PH_d(\mathbf{a}, \mathbf{Q}_j)$ with $\mathbf{a} = (1, 0)$ and

$$\mathbf{Q}_j = \begin{bmatrix} 1 - p_j & p_j \\ 1 - p_j & 0 \end{bmatrix},$$

for $j = 1, 2, 3$. Let $m_j = 2$, $j = 1, 2, 3$. Then

$$\begin{aligned}
P(S_1 > u_1, S_2 > u_2) = {} & (0, 0, 1, 0)\exp\left(\mathbf{Z}_1 u_1\right)\mathbf{e}' \\
& \times (0, 0, 1, 0)\exp\left(\mathbf{Z}_2 u_2\right)\mathbf{e}' \\
& \times (0, 0, 1, 0)\exp\left(\mathbf{Z}_3 \max(u_1, u_2)\right)\mathbf{e}',
\end{aligned}$$

where

$$\mathbf{Z}_j = \begin{bmatrix} -\lambda_j & 0 & \lambda_j(1 - p_j) & \lambda_j p_j \\ 0 & -\lambda_j & \lambda_j(1 - p_j) & 0 \\ \lambda_j & 0 & -\lambda_j & 0 \\ 0 & \lambda_j & 0 & -\lambda_j \end{bmatrix},$$

$j = 1, 2, 3$.

The mean time to failure is an important reliability characteristic which provides useful information for a design engineer. In Table 1, we compute the mean time to failure values of components under the assumption of Example 7 when $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. As expected, an increase in k leads to an increase in mean time to failure values. There is a considerable difference between the mean time to failure values when k is changed 3 from 2. If the probability of having a critical shock increases, then mean time to failure values decreases.

4. Mean residual life functions

From Propositions 3 and 5, we readily obtain the following expressions for the mean residual life functions of S_1 and S_2:

$$E(S_1 - t | S_1 > t) = -\frac{(\mathbf{v}_1 \otimes \mathbf{v}_3)\exp\left((\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)\,t\right)}{(\mathbf{v}_1 \otimes \mathbf{v}_3)\exp\left((\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)\,t\right)\mathbf{e}'}(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)^{-1}\mathbf{e}',$$

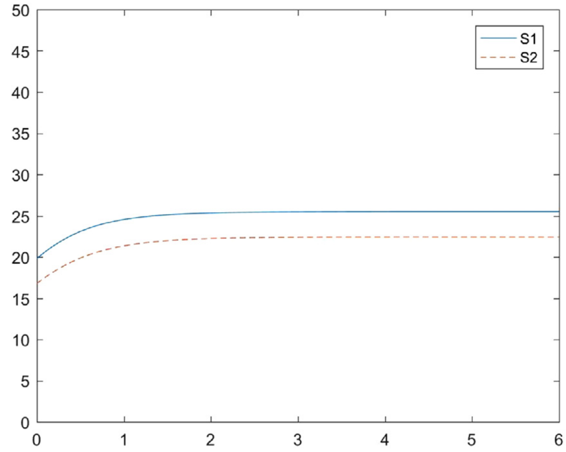

Fig. 1. MRL functions of components when $k = 2$ and times between shocks follow Erlang distribution.

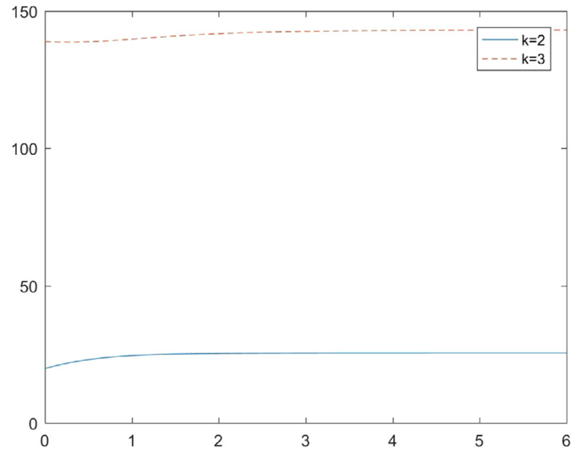

Fig. 2. MRL function of component 1 for selected values of k.

and,

$$E(S_2 - t|S_2 > t) = -\frac{\mathbf{v}_2 \otimes \mathbf{v}_3 \exp\left((\mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)\, t\right)}{\left(\mathbf{v}_2 \otimes \mathbf{v}_3 \exp\left((\mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)\, t\right)\right)\mathbf{e}'} (\mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)^{-1}\mathbf{e}'.$$

Fig. 1 plots MRL functions corresponding to S_1 and S_2 when $k = 2$ and the times between shocks follow Erlang distribution with parameters $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. The parameters p_1, p_2, and p_3 are chosen to be $p_1 = 0.1$, $p_2 = 0.15$ and $p_3 = 0.2$. The MRL functions are nondecreasing and component 1 has a larger MRL than the component 2.

In Fig. 2, we plot MRL function of component 1 under the Marshall–Olkin run shock model for different values of k when the times between shocks have Erlang distribution with $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. From the figures, we observe that the MRL function is increasing in k.

In Fig. 3, we plot MRL functions of S_1 and S_2 when the times between successive shocks produced by source j follow mixture of exponential distributions, i.e. $X_{ji} \sim PH_c(\boldsymbol{\alpha}_j, \mathbf{A}_j)$ with $\boldsymbol{\alpha}_j = (c_1^{(j)}, c_2^{(j)})$, and

$$\mathbf{A}_j = \begin{bmatrix} -\beta_j & 0 \\ 0 & -\gamma_j \end{bmatrix},$$

for $j = 1, 2, 3$. The parameter values are chosen to be $c_1^{(1)} = 0.5$, $c_2^{(1)} = 0.5$, $c_1^{(2)} = 0.4$, $c_2^{(2)} = 0.6$, $c_1^{(3)} = 0.25$, $c_2^{(3)} = 0.75$, $\beta_1 = 1\gamma_1, = 2$, $\beta_2 = 2$, $\gamma_2 = 2$, $\beta_3 = 2$, $\gamma_3 = 1$, $k = 2$, $p_1 = 0.1$, $p_2 = 0.15$, $p_3 = 0.2$.

Based on Figs. 1–3, we observe that the MRL functions approach constant for large t. This is consistent with the fact that the hazard function of phase type distributions converges to a constant for large t (see, e.g. [23]).

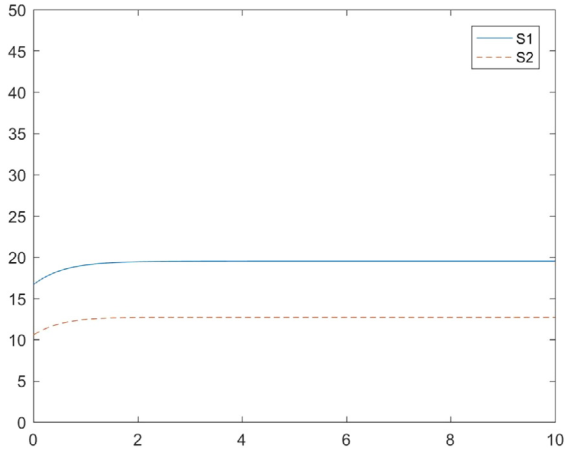

Fig. 3. MRL functions of components when $k = 2$ and times between shocks follow mixture of exponential distributions.

Lemma 8.

$$E(S_1 - t_1|S_1 > t_1, S_2 > t_2) = \begin{cases} \int_0^{\infty} \frac{P(T_1(k) > x+t_1)P(T_3(k) > x+t_1)}{P(T_1(k) > t_1)\,P(T_3(k) > t_1)}dx & , \text{ if } t_1 > t_2 \\ \int_0^{t_2-t_1} \frac{P(T_1(k) > x+t_1)}{P(T_1(k) > t_1)}dx + \int_{t_2-t_1}^{\infty} \frac{P(T_1(k) > x+t_1)P(T_3(k) > x+t_1)}{P(T_1(k) > t_1)\,P(T_3(k) > t_2)}dx & , \text{ if } t_1 < t_2 \end{cases}$$

Proof. Consider

$$P(S_1 - t_1 > x|S_1 > t_1, S_2 > t_2) = \frac{P(S_1 > x+t_1, S_1 > t_1, S_2 > t_2)}{P(S_1 > t_1, S_2 > t_2)} = \frac{P(S_1 > x+t_1, S_2 > t_2)}{P(S_1 > t_1, S_2 > t_2)}$$

Because $S_1 = \min(T_1(k), T_3(k))$,

$$\frac{P(S_1 > x+t_1, S_2 > t_2)}{P(S_1 > t_1, S_2 > t_2)} = \frac{P(T_1(k) > x+t_1, T_3(k) > x+t_1, T_2(k) > t_2, T_3(k) > t_2)}{P(T_1(k) > t_1, T_3(k) > t_1, T_2(k) > t_2, T_3(k) > t_2)}$$
$$= \frac{P(T_1(k) > x+t_1, T_2(k) > t_2, T_3(k) > \max(x+t_1, t_2))}{P(T_1(k) > t_1, T_2(k) > t_2, T_3(k) > \max(t_1, t_2))}$$
$$= \frac{P(T_1(k) > x+t_1)\,P(T_3(k) > \max(x+t_1, t_2))}{P(T_1(k) > t_1)\,P(T_3(k) > \max(t_1, t_2))}$$

Hence,

$$E(S_1 - t_1|S_1 > t_1, S_2 > t_2) = \int_0^{\infty} \frac{P(T_1(k) > x+t_1)\,P(T_3(k) > \max(x+t_1, t_2))}{P(T_1(k) > t_1)\,P(T_3(k) > \max(t_1, t_2))}dx$$

If $t_1 > t_2 \Rightarrow t_1 + x > t_2$ and $x > t_2 - t_1$, then

$$E(S_1 - t_1|S_1 > t_1, S_2 > t_2) = \int_0^{\infty} \frac{P(T_1(k) > x+t_1)P(T_3(k) > x+t_1)}{P(T_1(k) > t_1)\,P(T_3(k) > t_1)}dx.$$

If $t_1 < t_2 \Rightarrow t_1 + x < t_2$(that is $x < t_2 - t_1$) or $t_1 + x > t_2$ (that is $x > t_2 - t_1$), then

$$E(S_1 - t_1|S_1 > t_1, S_2 > t_2) = \int_0^{t_2-t_1} \frac{P(T_1(k) > x+t_1)}{P(T_1(k) > t_1)}dx + \int_{t_2-t_1}^{\infty} \frac{P(T_1(k) > x+t_1)P(T_3(k) > x+t_1)}{P(T_1(k) > t_1)\,P(T_3(k) > t_2)}dx.$$

The proof now follows considering the two cases $t_1 < t_2$ and $t_1 > t_2$. ■

Theorem 9. *For* $t_1 > t_2$,

$$\begin{aligned}&E\left(S_1 - t_1 | S_1 > t_1, S_2 > t_2\right)\\ &= \frac{((\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \otimes (\boldsymbol{\alpha}_3 \otimes \mathbf{a}))\left(-\left(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right)\right)^{-1} \exp\left(\left(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right) t_1\right) \left(\mathbf{e}' \otimes \mathbf{e}'\right)}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 t_1\right] \mathbf{e}'},\end{aligned}$$

and for $t_1 < t_2$,

$$\begin{aligned}&E\left(S_1 - t_1 | S_1 > t_1, S_2 > t_2\right)\\ &= \frac{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \mathbf{Z}_1^{-1} \exp\left[\mathbf{Z}_1 t_2\right] \mathbf{e}' - (\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \mathbf{Z}_1^{-1} \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}'}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}'}\\ &\quad + \frac{((\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \otimes (\boldsymbol{\alpha}_3 \otimes \mathbf{a}))\left(-\left(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right)\right)^{-1} \exp\left(\left(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right) t_2\right) \left(\mathbf{e}' \otimes \mathbf{e}'\right)}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 t_2\right] \mathbf{e}'}\end{aligned}$$

where $\mathbf{Z}_j = \left(\left(\mathbf{A}_j \otimes \mathbf{I}\right) + \left((-\mathbf{A}_j \mathbf{e}')\boldsymbol{\alpha}_j \otimes \mathbf{Q}_j\right)\right), j = 1, 2, 3.$

Proof. The proof is based on Lemma 8 and phase representations of the random variables $T_j(k)$. Indeed, for $t_1 > t_2$,

$$\begin{aligned}&\int_0^\infty \frac{P(T_1(k) > x + t_1)P(T_3(k) > x + t_1)}{P\left(T_1(k) > t_1\right) P\left(T_3(k) > t_1\right)} dx\\ &= \frac{1}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 t_1\right] \mathbf{e}'}\\ &\quad \times \int_0^\infty (\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 (x + t_1)\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 (x + t_1)\right] \mathbf{e}' dx\\ &= \frac{((\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \otimes (\boldsymbol{\alpha}_3 \otimes \mathbf{a}))\left(-\left(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right)\right)^{-1} \exp\left(\left(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right) t_1\right) \left(\mathbf{e}' \otimes \mathbf{e}'\right)}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 t_1\right] \mathbf{e}'}\end{aligned}$$

For $t_1 < t_2$,

$$\begin{aligned}&\int_0^{t_2 - t_1} \frac{P(T_1(k) > x + t_1)}{P(T_1(k) > t_1)} dx\\ &= \frac{1}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}'} \int_{t_1}^{t_2} (\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 u\right] \mathbf{e}' du\\ &= \frac{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \mathbf{Z}_1^{-1} \exp\left[\mathbf{Z}_1 t_2\right] \mathbf{e}' - (\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \mathbf{Z}_1^{-1} \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}'}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}'},\end{aligned}$$

and

$$\begin{aligned}&\int_{t_2 - t_1}^\infty \frac{P(T_1(k) > x + t_1)P(T_3(k) > x + t_1)}{P\left(T_1(k) > t_1\right) P\left(T_3(k) > t_2\right)} dx\\ &= \frac{1}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 t_2\right] \mathbf{e}'}\\ &\quad \times \int_{t_2}^\infty (\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 u\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 u\right] \mathbf{e}' du\\ &= \frac{((\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \otimes (\boldsymbol{\alpha}_3 \otimes \mathbf{a}))\left(-\left(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right)\right)^{-1} \exp\left(\left(\mathbf{Z}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right) t_2\right) \left(\mathbf{e}' \otimes \mathbf{e}'\right)}{(\boldsymbol{\alpha}_1 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_1 t_1\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 t_2\right] \mathbf{e}'} \quad \blacksquare\end{aligned}$$

The MRL function $E(S_2 - t_2 | S_1 > t_1, S_2 > t_2)$ can be similarly computed. Its matrix-based representation is given by the following theorem.

Theorem 10. *For* $t_2 > t_1$,

$$\begin{aligned}&E(S_2 - t_2 | S_1 > t_1, S_2 > t_2)\\ &= \frac{((\boldsymbol{\alpha}_2 \otimes \mathbf{a}) \otimes (\boldsymbol{\alpha}_3 \otimes \mathbf{a}))\left(-\left(\mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right)\right)^{-1} \exp\left[\left(\mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3\right) t_2\right] \left(\mathbf{e}' \otimes \mathbf{e}'\right)}{(\boldsymbol{\alpha}_2 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_2 t_2\right] \mathbf{e}' \, (\boldsymbol{\alpha}_3 \otimes \mathbf{a}) \exp\left[\mathbf{Z}_3 t_2\right] \mathbf{e}'}\end{aligned}$$

and for $t_2 < t_1$,

$$E(S_2 - t_2|S_1 > t_1, S_2 > t_2) = \frac{(\boldsymbol{\alpha}_2 \otimes \mathbf{a})\,\mathbf{Z}_2^{-1}\exp[\mathbf{Z}_2 t_1]\,\mathbf{e}' - (\boldsymbol{\alpha}_2 \otimes \mathbf{a})\,\mathbf{Z}_2^{-1}\exp[\mathbf{Z}_2 t_2]\,\mathbf{e}'}{(\boldsymbol{\alpha}_2 \otimes \mathbf{a})\exp[\mathbf{Z}_2 t_2]\,\mathbf{e}'} + \frac{((\boldsymbol{\alpha}_2 \otimes \mathbf{a})(\boldsymbol{\alpha}_3 \otimes \mathbf{a}))\,(-(\mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3))^{-1}\exp[(\mathbf{Z}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{Z}_3)\,t_1]\left(\mathbf{e}' \otimes \mathbf{e}'\right)}{(\boldsymbol{\alpha}_2 \otimes \mathbf{a})\exp[\mathbf{Z}_2 t_2]\,\mathbf{e}'\,(\boldsymbol{\alpha}_3 \otimes \mathbf{a})\exp[\mathbf{Z}_3 t_1]\,\mathbf{e}'}.$$

5. Conclusions

In this paper, we have studied a generalized version of the Marshall–Olkin shock model by incorporating run shock model. Times between shocks were assumed to follow phase-type distributions which have various advantages. One of the main advantages of using phase type distributions is the mathematical simplicity. They enable us to express distributions and their moments in a matrix form which can be easily calculated and evaluated using mathematical software. We were able to compute mean residual lifetimes without taking integration since the usage of phase type distributions transforms integrations into matrix calculations (see, e.g. Theorems 9–10).

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