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Stability and periodicity in dynamic delay equations[☆]

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ABSTRACT

Let $\mathbb{T}$ be an arbitrary time scale that is unbounded above. By means of a variation of Lyapunov's method and contraction mapping principle this paper handles asymptotic stability of the zero solution of the completely delayed dynamic equations

$$x^{\Delta}(t) = -a(t)x(\delta(t))\delta^{\Delta}(t).$$

Moreover, if $\mathbb{T}$ is a periodic time scale, then necessary conditions are given for the existence of a unique periodic solution of the above mentioned equation.

1. Introduction and preliminaries

Let $\mathbb{T}$ be an arbitrary time scale that is unbounded above. This research considers the questions of stability and periodicity of the completely delayed equation

$$x^{\Delta}(t) = -a(t)x(\delta(t))\delta^{\Delta}(t), \tag{1.1}$$

where $\delta : [t_0, \infty) \cap \mathbb{T} \to [\delta(t_0), \infty) \cap \mathbb{T}$ is a strictly increasing and Δ-differentiable delay function having the following properties:

$$\begin{cases} \delta(t) < t \\ (\delta \circ \sigma)(t) = (\sigma \circ \delta)(t) \\ \left|\delta^{\Delta}(t)\right| < \infty \end{cases} \tag{1.2}$$

for all $t \in \mathbb{T}$. Throughout the paper, the time scale $\mathbb{T}$ should also be assumed to include $\delta(t_0)$.

In the following table, we give some particular time scales with specific delay functions and show what Eq. (1.1) turns into.

Time scale	Delay function	Eq. (1.1)
$\mathbb{T}_r = \mathbb{R}$	$\delta(t) = t - \tau, \tau \in \mathbb{R}_+$	$\frac{d}{dt}x(t) = -a(t)x(t-\tau)$
$\mathbb{T}_h = h\mathbb{Z}$	$\delta(t) = t - h\tau, \tau \in \mathbb{N}$	$\Delta_h x(t) = -a(t)x(t-\tau)$
$\mathbb{T}_q = \overline{q^{\mathbb{Z}}} = \{q^m : m \in \mathbb{Z}\} \cup \{0\}$	$\delta(t) = \frac{t}{q^\tau}, \tau \in \mathbb{N}$	$D_q x(t) = -a(t)x(\frac{t}{q^\tau})\frac{1}{q^\tau}$
$\mathbb{T}_q^i = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$	$\delta(t) = t - \tau, \tau \in \mathbb{N}$	$\Delta_q x(t) = -a(t)x(t-\tau)$

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Here, we let $0 < q < 1$ and define

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h} \quad \text{for } t \in h\mathbb{Z}, h > 0$$
$$D_q x(t) = \frac{x(qt) - x(t)}{(q-1)t} \quad \text{for } t = q^m,$$
$$\Delta_q \varphi(t) = \frac{\varphi(k - q^{m+1}) - \varphi(k - q^m)}{q^m(1-q)} \quad \text{for } t = k - q^m.$$

In [1], Raffoul investigated the stability and periodicity of the difference delay equation

$$\Delta x(t) = -a(t)x(t-\tau),$$

where Δ indicates the forward difference operator.

The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is a mapping defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The graininess function $\mu : \mathbb{T} \to \mathbb{R}$ is given by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right scattered if $\sigma(t) > t$. An isolated time scale $\mathbb{T}$ is a time scale consisting only of right scattered points. $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ are examples of isolated time scales.

Stability of dynamic delay equation (1.1) has been studied in [2]. It was proven (see [2, Theorem 3.1]) that every solution of (1.1) goes to zero at infinity if $\mathbb{T} = \mathbb{R}$ or $\mathbb{T}$ is an isolated time scale. For an arbitrary time scale, sufficient conditions for the functions $f_i(t, x(s))$; $i = 1, 2, \ldots, n$ and $g(t, s)$ were given to deduce asymptotic stability of trivial solution of nonlinear dynamic equation

$$x^{\Delta}(t) = -\int_{\delta(t)}^{t} \left(\sum_{i=1}^{n} f_i(t, x(s))\right) g^{\Delta_s}(t, s)\Delta s, \quad t \in [t_0, \infty) \cap \mathbb{T} \tag{1.3}$$

(see [2, Theorem 4.5]). It is natural to ask if (1.3) gives (1.1) in a special case. In [2], the authors gave an example (see [2, Example 4.6]) to show that the Eq. (1.1) is a special case of (1.3) whenever the time scale is *isolated*. However, they were not able to introduce convenient functions $f_i(t, x(s))$; $i = 1, 2, \ldots, n$ and $g(t, s)$ to get the same result for an *arbitrary time scale*. Hence, stability analysis of (1.1) on an arbitrary time scale is not covered by [2].

This paper not only overcomes this ambiguity, but also investigates the existence of periodic solutions of (1.1).

Notable contributions of this paper can be summarized as follows:

1. to analyze stability of trivial solution of the equation
$$x^{\Delta}(t) = b(t)x(t) - a(t)x(\delta(t))\delta^{\Delta}(t),$$
by displaying a Lyapunov functional on an arbitrary time scale,
2. to analyze stability of the Eq. (1.1) for an arbitrary time scale,
3. to study the periodicity of solutions of (1.1), and
4. to extend the work of [1] to time scales (it turns out that new results concerning Eq. (1.1) in the discrete case will emerge).

In preparation for our main results we list the following results from time scale calculus. For brevity we direct the reader to [3] for an excellent theory regarding Δ-derivative and Δ-integral.

Theorem 1 (*Substitution [3, Theorem 1.98]*). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\,\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\,\tilde{\Delta} s.$$

Lemma 1 (*Integration by Parts [3, Theorem 1.77, (v)]*). *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$ then*

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f(t)g^{\Delta}(t)\Delta t.$$

Theorem 2 (*Chain Rule [3, Theorem 1.93]*). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $\omega^{\widetilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(\omega \circ \nu)^{\Delta} = (\omega^{\widetilde{\Delta}} \circ \nu)\nu^{\Delta}.$$

Theorem 3 (*Existence of Antiderivative [3, Theorem 1.74]*). *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) = \int_{t_0}^{t} f(s)\Delta s$$

is an antiderivative of f.

Definition 1. A function $h : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)h(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. The set of all regressive rd-continuous functions $h : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{h \in \mathcal{R} : 1 + \mu(t)h(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $h \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_h(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)}\mathrm{Log}(1 + \mu(z)h(z))\,\Delta z\right),$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y,\ y(s) = 1$. Other properties of the exponential function are given in the following lemma [3, Theorem 2.36].

Lemma 2. *Let $p, q \in \mathcal{R}$. Then*

i. $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
ii. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
iii. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
iv. $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
v. $e_p(t, s)e_p(s, r) = e_p(t, r)$;
vi. $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Theorem 4 (*[3, Theorem 6.1]*). *Let $y, f \in C_{rd}$ and $p \in \mathcal{R}^+$. Then*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t) \quad \text{for all } t \in \mathbb{T}$$

implies

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau$$

for all $t \in \mathbb{T}$.

Theorem 5 (*Bernoulli's Inequality [3, Theorem 6.2]*). *Let $\alpha \in \mathbb{R}$ with $\alpha \in \mathcal{R}^+$. Then*

$$e_\alpha(t, t_0) \geq 1 + \alpha(t - s) \quad \text{for all } t \geq s.$$

2. Stability

The aim of the next two subsections is to investigate the stability of delayed linear dynamic equations and to make a comparative analysis between two methods: Lyapunov's direct method and the method of fixed point theory.

2.1. Stability analysis using Lyapunov's method

First, we discuss the stability of the linear delay dynamic equation

$$x^{\Delta}(t) = b(t)x(t) - a(t)x(\delta(t))\delta^{\Delta}(t) \tag{2.1}$$

by means of Lyapunov functional, where $\mathbb{T}$ is an arbitrary time scale, τ is a positive integer, $a, b : \mathbb{T} \to \mathbb{R}$ are functions, and $b \in \mathcal{R}^+$ (positively regressive). The sole purpose is to show that, when using the Lyapunov method, many difficulties arise and severe conditions will have to be imposed on the coefficients in order to arrive at the derivative of the Lyapunov function to be less than or equal to zero along any solution. To see this we assume that for a positive real γ

$$|a(t)| \leq \gamma \quad \text{and} \quad b(t) < -\gamma \quad \text{for all } t \in \mathbb{T}. \tag{2.2}$$

If $\mu(t_0) \neq 0$ for any $t_0 \in \mathbb{T}$, then $b \in \mathcal{R}^+$ and (2.2) imply

$$|1 + \mu(t_0)b(t_0)| < 1 - \mu(t_0)\gamma, \tag{2.3}$$

and therefore,

$$b(t_0) > -\frac{2}{\mu(t_0)} + \gamma. \tag{2.4}$$

Note that stability of the delay difference equation

$$x(t+1) = b(t)x(t) - a(t)x(t - g(t)), \quad t \in \mathbb{Z}$$

has been studied in [4] by making use of Lyapunov's method.

Hereafter, we develop a time scale analog of Lyapunov's method used in [4, p. 3]. Let the Lyapunov functional V be defined by

$$V(t, x(.)) = |x(t)| + \gamma \int_{\delta(t)}^{t} |x(s)|\,\Delta s. \tag{2.5}$$

Evidently, the delay function $\delta : [t_0, \infty) \cap \mathbb{T} \to [\delta(t_0), \infty) \cap \mathbb{T}$ defined in the previous section satisfies the assumptions made for the function ν in Theorem 2. Taking

$$\omega(t) = \int_a^t f(s)\Delta s$$

and using Theorems 2 and 3, it is straightforward to prove the following.

Corollary 1. *Let $f : \mathbb{T} \to \mathbb{R}$ be an rd-continuous function. Then*

$$\left[\int_a^{\delta(t)} f(s)\Delta s\right]^{\Delta} = f(\delta(t))\delta^{\Delta}(t),$$

where $a \in \mathbb{T}$ is fixed.

To differentiate the integral term in (2.5) we shall resort to the next result.

Lemma 3. *Let $f : \mathbb{T} \to \mathbb{R}$ be an rd-continuous function. Then*

$$\int_{\delta(t)}^{t} f(s)\Delta s = f(t) - f(\delta(t))\delta^{\Delta}(t),$$

where δ is as defined in the previous section.

Proof. Case 1. Let $\sigma(\delta(t)) \neq t$. Then $\sigma(\delta(t)) < t$. Thus, there exists a constant $\tau_0 \in [\delta(t), t) \cap \mathbb{T}$ such that $\sigma(\delta(t)) = \tau_0$. The result is immediate from

$$\int_{\delta(t)}^{t} f(s)\Delta t = \int_{\delta(t)}^{\tau_0} f(s)\Delta t + \int_{\tau_0}^{t} f(s)\Delta s$$

and Corollary 1.

Case 2. Let $\sigma(\delta(t)) = t$. Hence, we arrive at

$$\begin{aligned}
\left[\int_{\delta(t)}^{t} f(s)\Delta s\right]^{\Delta} &= [\mu(\delta(t)f(\delta(t)))]^{\Delta} \\
&= [(\sigma(\delta(t)) - \delta(t))f(\delta(t))]^{\Delta} \\
&= (1 - \delta^{\Delta}(t))f(\delta(t)) + (\sigma(t) - \delta(\sigma(t)))\,[f(\delta(t))]^{\Delta} \\
&= f(\delta(t)) - \delta^{\Delta}(t)f(\delta(t)) + \mu(t)\,[f(\delta(t))]^{\Delta} \\
&= f(\delta(t)) - \delta^{\Delta}(t)f(\delta(t)) + f(\delta(\sigma(t))) - f(\delta(t)) \\
&= f(t) - \delta^{\Delta}(t)f(\delta(t)),
\end{aligned}$$

where we also used the formulas

$$\mu(t)f^{\Delta}(t) = f(\sigma(t)) - f(t)$$

and

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(s)f(s)$$

(see [3, Theorem 1.16] and [3, Theorem 1.75]). □

Therefore, Lemma 3 enables us to arrive at

$$V^{\Delta}(t, x(.)) = |x(t)|^{\Delta} + \gamma \left\{ |x(t)| - |x(\delta(t))| \left|\delta^{\Delta}(t)\right| \right\}. \tag{2.6}$$

Since the product rule is given by $(fg)^{\Delta} = f^{\Delta}g^{\sigma} + fg^{\Delta}$ in time scale calculus, the derivative of $|x(t)|$ is obtained by differentiating both sides of the equation $x^2(t) = |x(t)|^2$ as follows

$$|x|^{\Delta} = \frac{x + x^{\sigma}}{|x| + |x^{\sigma}|} x^{\Delta} \quad \text{for } x \neq 0. \tag{2.7}$$

So the derivative of $|x|$ depends on $\frac{x(t)}{|x(t)|}$ and $\frac{x^{\sigma}(t)}{|x^{\sigma}(t)|}$ (which indicate signs of x and x^{σ}, respectively). Given $x : \mathbb{T} \to \mathbb{R}$, let the sets $\mathbb{T}_x^+$ and $\mathbb{T}_x^-$ be defined by

$$\mathbb{T}_x^+ = \{t \in \mathbb{T} : x(t)x^{\sigma}(t) \geq 0\},$$
$$\mathbb{T}_x^- = \{t \in \mathbb{T} : x(t)x^{\sigma}(t) < 0\},$$

respectively. Note that the set $\mathbb{T}_x^-$ consists only of right scattered points of $\mathbb{T}$. Hence, if $\mathbb{T} = \mathbb{R}$, then $\mathbb{T}_x^- = \varnothing$ for all functions $x : \mathbb{R} \to \mathbb{R}$ and (2.7) turns into $|x|^{\Delta} = \frac{x}{|x|}x^{\Delta}$. However, for an arbitrary time scale (e.g. $\mathbb{T} = \mathbb{Z}$) the set $\mathbb{T}_x^-$ may not be empty. For simplicity, we need to have a formula for $|x|^{\Delta}$ which does not include x^{σ}. The next result provides a relationship between $|x|^{\Delta}$ and $\frac{x}{|x|}x^{\Delta}$. Its proof can be found in [5, Lemma 5]. Also, for more on the use of Lyapunov method in dynamic equation on time scales we refer the reader to [6].

Lemma 4 ([5, Lemma 5]). *Let $x \neq 0$ be Δ-differentiable. Then*

$$|x(t)|^{\Delta} = \begin{cases} \dfrac{x(t)}{|x(t)|}x^{\Delta}(t) & \text{if } t \in \mathbb{T}_x^+ \\ -\dfrac{2}{\mu(t)}|x(t)| - \dfrac{x(t)}{|x(t)|}x^{\Delta}(t) & \text{if } t \in \mathbb{T}_x^-. \end{cases} \tag{2.8}$$

As a consequence of (2.1)–(2.8) we get that

$$V^{\Delta}(t, x(.)) \leq \zeta(t)\,|x(t)| \quad \text{for all } t \in \mathbb{T}, \tag{2.9}$$

where

$$\zeta(t) = \begin{cases} b(t) + \gamma & \text{if } t \in \mathbb{T}_x^+ \\ -\dfrac{2}{\mu(t)} - b(t) + \gamma & \text{if } t \in \mathbb{T}_x^-. \end{cases}$$

Note that (2.2) and (2.4) imply

$$\zeta(t) < 0 \quad \text{for all } t \in \mathbb{T}.$$

This along with (2.9) shows the stability of zero solution of the Eq. (2.1) using [7, Theorem 2]. As a consequence of the discussion above, we can give the following theorem:

Theorem 6. *Let $\mathbb{T}$ be a time scale that is unbounded above. If (2.2) holds, then zero solution of Eq. (2.1) is stable.*

Observe that for the Eq. (1.1) the condition (2.2) does not hold since $b(t) = 0$ in (1.1). Hence, the use of fixed point theory is justified.

2.2. Stability analysis using fixed point theory

Consider the Eq. (1.1). For the sake of inverting Eq. (1.1) we write it in the form

$$x^{\Delta}(t) = p(t)x(t) - \left(\int_{\delta(t)}^{t} p(s)x(s)\Delta s\right)^{\Delta} \tag{2.10}$$

where

$$p(t) = -a(\delta^{-1}(t)). \tag{2.11}$$

Hereafter, we use the notation

$$I_a^b(\mathbb{T}) = [a, b) \cap \mathbb{T}$$

and assume that

$$p(t) \neq 0 \quad \text{for all } t \in I_{t_0}^{\infty}(\mathbb{T}). \tag{2.12}$$

Multiplying both sides of (2.10) by $e_{\ominus p}(\sigma(t), t_0)$, using the product rule, and integrating by parts (Lemma 1), we find

$$x(t) = x(t_0)e_p(t, t_0) - \int_{\delta(t)}^{t} p(s)x(s)\Delta s + e_p(t, t_0)\int_{\delta(t_0)}^{t_0} p(s)x(s)\Delta s + \int_{t_0}^{t} \ominus p e_p(t, s)\left(\int_{\delta(s)}^{s} p(u)x(u)\Delta u\right)\Delta s. \tag{2.13}$$

Let $\mathcal{C}$ be the Banach space of bounded functions $\phi : I_{\delta(t_0)}^{\infty}(\mathbb{T}) \to \mathbb{R}$ endowed with the norm

$$\|\phi\| = \sup_{t \in I_{\delta(t_0)}^{\infty}(\mathbb{T})} |\phi(t)|.$$

Let $\psi : I_{\delta(t_0)}^{t_0}(\mathbb{T}) \to \mathbb{R}$ be a given initial function.

Hereafter, we say that $x(t) := x(t, t_0, \psi)$ is a solution of (1.1) if

$$x(t) = \psi(t) \quad \text{on } I_{\delta(t_0)}^{t_0}(\mathbb{T})$$

and x satisfies (2.13) on $I_{t_0}^{\infty}(\mathbb{T})$.

Let the subset $S \subset \mathcal{C}$ be defined by

$$S = \left\{\varphi \in \mathcal{C} : \varphi(t) = \psi(t) \text{ if } t \in I_{\delta(t_0)}^{t_0}(\mathbb{T}) \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\right\}.$$

Obviously, S is a complete metric space.

For the next theorem we employ the following conditions

$$e_p(t, t_0) \to 0 \quad \text{as } t \to \infty \tag{2.14}$$

and

$$\int_{\delta(t)}^{t} |p(s)|\,\Delta s + \int_{t_0}^{t} |\ominus p(s)|\, e_p(t, s)\left(\int_{\delta(s)}^{s} |p(u)|\,\Delta u\right)\quad \Delta s \leq \alpha < 1. \tag{2.15}$$

The next example shows that there may be a function p so that the condition (2.14) holds.

Example 1. If there exists a positive constant M such that $-M \in \mathcal{R}^+$ and

$$|p(t)| \leq M \quad \text{for all } t \in \mathbb{T}, \tag{2.16}$$

then the condition (2.14) holds.

Proof. If $-M$ is positively regressive, from (2.16) p is positively regressive, and hence, $e_p(t, t_0) > 0$. Since

$$\begin{aligned} e_p^{\Delta}(t, t_0) &= p(t)e_p(t, t_0) \\ &\leq Me_p(t, t_0) \\ &\leq \frac{M}{1 - M\mu(t)} e_p(t, t_0) \\ &= \ominus(-M)e_p(t, t_0), \end{aligned}$$

we get by Theorem 4 that

$$0 < e_p(t, t_0) \leq e_{\ominus(-M)}(t, t_0).$$

By Bernoulli's inequality (Theorem 5) we know that

$$e_{\ominus(-M)}(t, t_0) \leq \frac{1}{1 - Mt} \to 0 \quad \text{as } t \to \infty.$$

The proof is complete. □

Theorem 7. *Suppose that (2.12), (2.14)–(2.15) hold. Then every solution $x(t) := x(t, t_0, \psi)$ of (1.1) is bounded and*

$$\lim_{t \to \infty} x(t) = 0.$$

Proof. For $\varphi \in S$, define the mapping P by

$$(P\varphi)(t) = \psi(t) \quad \text{for } t \in I_{\delta(t_0)}^{t_0}(\mathbb{T})$$

and

$$(P\varphi)(t) = \psi(t_0)e_p(t,t_0) - \int_{\delta(t)}^{t} p(s)\varphi(s)\Delta s + e_p(t,t_0)\int_{\delta(t_0)}^{t_0} p(s)\psi(s)\Delta s + \int_{t_0}^{t} \ominus p e_p(t,s)\left(\int_{\delta(s)}^{s} p(u)\varphi(u)\Delta u\right)\Delta s$$

for $t \in I_{t_0}^{\infty}(\mathbb{T})$. Continuity of P on S is evident. From (2.14) we can find a constant Q such that

$$e_p(t,t_0) \le Q \quad \text{for } t \in I_{t_0}^{\infty}(\mathbb{T}).$$

If $\varphi \in S$, then there exists a K such that $\|\varphi\| \le K$. Let ψ be a small given initial value function with $|\psi| < \gamma$, $\gamma > 0$. Then by making use of (2.15) we arrive at

$$\begin{aligned}|P\varphi(t)| &\le (\gamma+\gamma\alpha)\,Q + \int_{\delta(t)}^{t} |p(s)|\,|\varphi(s)|\,\Delta s + \int_{t_0}^{t} |\ominus p(s)|\,e_p(t,s)\left(\int_{\delta(s)}^{s} |p(u)|\,|\varphi(u)|\,\Delta u\right)\Delta s \\ &\le (\gamma+\gamma\alpha)\,Q + \left\{\int_{\delta(t)}^{t} |p(s)|\,\Delta s + \int_{t_0}^{t} |\ominus p(s)|\,e_p(t,s)\left(\int_{\delta(s)}^{s} |p(u)|\,\Delta u\right)\Delta s\right\}\|\varphi\| \\ &\le (\gamma+\gamma\alpha)\,Q + \alpha K \quad \text{for } t \in I_{t_0}^{\infty}(\mathbb{T}). \end{aligned} \tag{2.17}$$

Thus, $P\varphi$ is bounded. It remains to show that $P\varphi(t) \to 0$ as $t \to \infty$. By (2.14) and (2.15), we know that the first and third terms on the right-hand side of $P\varphi(t)$ tends to zero as $t \to \infty$. Since $\varphi \in S$, we have $\varphi(t) \to 0$ as $t \to \infty$. From continuity of norm we obtain

$$\|\varphi\|_{I_{\delta(t)}^{t}(\mathbb{T})} \to 0 \quad \text{as } t \to \infty,$$

where

$$\|\phi\|_{I_{\delta(a)}^{b}(\mathbb{T})} = \sup_{t \in I_{\delta(a)}^{b}(\mathbb{T})} |\phi(t)|\,.$$

Hence, we get by (2.15) that

$$\int_{\delta(t)}^{t} |p(s)|\,|\varphi(s)|\,\Delta s \le \|\varphi\|_{I_{\delta(t)}^{t}(\mathbb{T})}\int_{\delta(t)}^{t} |p(s)| \to 0 \quad \text{as } t \to \infty.$$

That is, the second term in $P\varphi(t)$ goes to zero. Let us denote by $J(t,t_0)$ the last term of $P\varphi(t)$, i.e.,

$$J(t,t_0) = \int_{t_0}^{t} |\ominus p(s)|\,e_p(t,s)\left(\int_{\delta(s)}^{s} |p(u)|\,|\varphi(u)|\,\Delta u\right)\Delta s.$$

Given $\varphi \in S$ and $\varepsilon > 0$, we can choose a sufficiently large $t^* \in I_{t_0}^{t}(\mathbb{T})$ such that

$$\alpha\,\|\varphi\|\,e_p(t,t^*) < \frac{\varepsilon}{2}$$

and

$$\alpha\,\|\varphi\|_{I_{\delta(t^*)}^{\infty}(\mathbb{T})} < \frac{\varepsilon}{2}.$$

Therefore, we find

$$\begin{aligned} J(t,t_0) &\le e_p(t,t^*)\int_{t_0}^{t^*} |p(s)|\,e_p(t^*,\sigma(s))\left(\int_{\delta(s)}^{s} |p(u)|\,|\varphi(u)|\,\Delta u\right)\Delta s \\ &\quad + \int_{t^*}^{t} |\ominus p(s)|\,e_p(t,s)\left(\int_{\delta(s)}^{s} |p(u)|\,|\varphi(u)|\,\Delta u\right)\Delta s \\ &\le \alpha\,\|\varphi\|\,e_p(t,t^*) + \alpha\,\|\varphi\|_{I_{\delta(t^*)}^{\infty}(\mathbb{T})} < \varepsilon. \end{aligned}$$

This yields

$$(P\varphi)(t) \to 0 \quad \text{as } t \to \infty,$$

and therefore, $P : S \to S$. On the other hand, P is a contraction since

$$\begin{aligned} |(P\eta)(t) - (P\xi)(t)| &\leq \left\{ \int_{\delta(t)}^{t} |p(s)|\,\Delta s + \int_{t_0}^{t} |\ominus p(s)|\, e_p(t,s) \left(\int_{\delta(s)}^{s} |p(u)|\,\Delta u \right) \Delta s \right\} \|\eta - \xi\| \\ &\leq \alpha \|\eta - \xi\| \end{aligned}$$

for any φ and ψ in S. Thus, by the contraction mapping principle P has a unique fixed point in S which solves the Eq. (2.10), bounded and tends to zero as $t \to \infty$. □

3. Periodicity

For clarity, we restate the following definitions and introductory examples which can be found in [8,9].

Definition 2. A time scale $\mathbb{T}$ is said to be *periodic* if there exists a $\lambda > 0$ such that $t \pm \lambda \in \mathbb{T}$ for all $t \in \mathbb{T}$. If $\mathbb{T} \neq \mathbb{R}$, the smallest positive λ is called the *period* of the time scale.

Example 2. The following time scales are periodic.

1. $\mathbb{T} = \mathbb{Z}$ has period $\lambda = 1$,
2. $\mathbb{T} = h\mathbb{Z}$ has period $\lambda = h$,
3. $\mathbb{T} = \mathbb{R}$,
4. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [(2i-1)h, 2ih]$, $h > 0$ has period $\lambda = 2h$,
5. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $\lambda = 1$.

Remark 1. All periodic time scales are unbounded above and below.

Definition 3. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period λ. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = n\lambda$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Let $\mathbb{T}$ be a periodic time scale. Suppose that there is a positive real T such that

$$a(t+T) = a(t) \tag{3.1}$$

for all $t \in \mathbb{T}$. It follows from (2.11) and (3.1) that

$$p(t+T) = p(t).$$

In addition to assumptions in (1.2), we also assume that the delay function $\delta : \mathbb{T} \to \mathbb{T}$ satisfies

$$\delta(t \pm T) = \delta(t) \pm T. \tag{3.2}$$

Hereafter, we suppose that

$$1 - e_p(t, t-T) \neq 0. \tag{3.3}$$

We multiply the Eq. (2.10) by $e_{\ominus p}(\sigma(t), t_0)$, use the product rule, and integrate the obtained equation from $t - T$ to t to find

$$\begin{aligned} x(t) = (1 - e_p(t, t-T))^{-1} \bigg[& e_p(t, t-T) \int_{\delta(t-T)}^{t-T} p(s)x(s)\Delta s - \int_{\delta(t)}^{t} p(s)x(s)\Delta s \\ & + \int_{t-T}^{t} \ominus p(s) e_p(t,s) \left(\int_{\delta(s)}^{s} p(u)x(u)\Delta u \right) \Delta s \bigg]. \end{aligned}$$

For $x \in P_T$ define the mapping H by

$$\begin{aligned} Hx(t) = (1 - e_p(t, t-T))^{-1} \bigg[& e_p(t, t-T) \int_{\delta(t-T)}^{t-T} p(s)x(s)\Delta s - \int_{\delta(t)}^{t} p(s)x(s)\Delta s \\ & + \int_{t-T}^{t} \ominus p(s) e_p(t,s) \left(\int_{\delta(s)}^{s} p(r)x(r)\Delta r \right) \Delta s \bigg]. \end{aligned}$$

It can be easily shown by making use of the substitution $u = s + T$, (2.11), Theorem 1, and the equalities

$$e_p(t+T, s+T) = e_p(t,s), \qquad e_p(t+T, t) = e_p(t, t-T)$$

that

$$(Hx)(t+T) = (Hx)(t).$$

Let P_T be the set of all T-periodic bounded functions $x : \mathbb{T} \to \mathbb{R}$. Then $(P_T, \|.\|)$ is a Banach space endowed with the norm

$$\|x\| = \max_{t\in\mathbb{T}} |x(t)| = \max_{t\in[0,T]\cap\mathbb{T}} |x(t)| .$$

Thus,

$$H : P_T \to P_T .$$

Similar to that of Theorem 7 we can prove the following theorem.

Theorem 8. *Assume that (3.1)–(3.3) hold for all $t \in \mathbb{T}$. If there is an $\alpha > 0$ such that*

$$\left|(1 - e_p(t, t - T))^{-1}\right| \left[\int_{\delta(t)}^{t} |p(s)|\, \Delta s + e_p(t, t - T) \int_{\delta(t)-T}^{t-T} |p(s)|\, \Delta s \right.$$
$$\left. + \int_{t-T}^{t} |\ominus p(s)|\, e_p(t, s) \int_{\delta(s)}^{s} |p(r)|\, \Delta r \Delta s \right] \le \alpha < 1,$$

then the Eq. (2.10) has a unique T-periodic solution.

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