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A note on “Stability and periodicity in dynamic delay equations” [Comput. Math. Appl. 58 (2009) 264–273]

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ABSTRACT

The purpose of this note is twofold: First we highlight the importance of an implicit assumption in [Murat Adıvar, Youssef N. Raffoul, Stability and periodicity in dynamic delay equations, Computers and Mathematics with Applications 58 (2009) 264–272]. Second we emphasize one consequence of the bijectivity assumption which enables ruling out the commutativity condition $\delta \circ \sigma = \sigma \circ \delta$ on the delay function.

1. Article outline

Let $\mathbb{T}$ be a time scale that is unbounded above and let $t_0 \in \mathbb{T}$ be a fixed point. In [1], we investigated the stability and periodicity of the completely delayed dynamic equations

$$x^{\Delta}(t) = -a(t)x(\delta(t))\delta^{\Delta}(t), \tag{1.1}$$

where $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ is a strictly increasing and Δ-differentiable delay function satisfying $\delta(t) < t$ and $\left|\delta^{\Delta}(t)\right| < \infty$, and the commutativity condition

$$\delta \circ \sigma = \sigma \circ \delta, \tag{1.2}$$

where σ is the forward jump operator defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}. \tag{1.3}$$

Note that the condition (1.2) is also required in [2, Lemma 2.2] where the time scale is restricted to $\mathbb{T} = \mathbb{R}$ or to an isolated time scale so that Eq. (1.1) can be turned into a Volterra integro-dynamic equation of the form

$$x^{\Delta}(t) = -a(\delta^{-1}(t))x(t) - \left(\int_{\delta(t)}^{t} a(\delta^{-1}(s))x(s)\Delta s\right)^{\Delta}. \tag{1.4}$$

In [2, Section 2], instead of imposing the explicit invertibility condition, the delay function δ is assumed to be delta differentiable with $\delta(t) < t$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\lim_{t\to\infty} \delta(t) = \infty$. In [1], the formula (1.4) is obtained for an arbitrary

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time scale having a strictly increasing delay function $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ satisfying $\delta(t) < t$, $\left|\delta^{\Delta}(t)\right| < \infty$, and (1.2). However, there is an implicit assumption of the invertibility of δ in the paper [1], as well. In the next section we give an example to show that a noninvertible delay function satisfying $\delta(t) < t$, $\left|\delta^{\Delta}(t)\right| < \infty$, and (1.2) on an arbitrary time scale $\mathbb{T}$ may not be strictly increasing. Note that we still keep the assumption $\delta(t_0) \in \mathbb{T}$ of [1] in this paper.

2. A clarification

In the statement of the problem in [1], the time scale $\mathbb{T}$ should be explicitly assumed to have an invertible delay function since we use the inverse of the delay function $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ throughout the paper. Evidently, the delay function δ is an injection since it is supposed to be strictly increasing. To guarantee the existence of δ^{-1} it is essential to ask whether δ maps $[t_0, \infty)_{\mathbb{T}}$ onto $[\delta(t_0), \infty)_{\mathbb{T}}$, where $[a, b)_{\mathbb{T}}$ indicates the time scale interval $[a, b) \cap \mathbb{T}$. The notation $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ may not be adequate for meaning the same thing, i.e., δ is surjective. In the case when δ is not surjective, one may easily obtain a contradiction for an arbitrary time scale. To see this we give the following example.

Example 1. Let the time scale $\widetilde{\mathbb{T}}$ be given by $\widetilde{\mathbb{T}} := (-\infty, 0] \cup [1, \infty)$. Suppose that there exists a strictly increasing and Δ-differentiable delay function $\delta : [0, \infty)_{\widetilde{\mathbb{T}}} \to [\delta(0), \infty)_{\mathbb{T}}$ on $\widetilde{\mathbb{T}}$ satisfying $\delta(t) < t$, $\left|\delta^{\Delta}(t)\right| < \infty$, and the commutativity condition (1.2). Since δ is strictly increasing we have $\delta(0) < \delta(1)$. However, from the commutativity condition we find

$$\delta(1) = \delta(\sigma(0)) = \sigma(\delta(0)) = \delta(0).$$

This shows that without the invertibility assumption on δ, commutativity condition (1.2) contradicts the condition that δ is strictly increasing.

Classification of the time scales having invertible strictly increasing delay function is the topic of another research paper. However, we can give the following sets:

$$\mathbb{T}_1 = \bigcup_{n=0}^{\infty} \left[q^{2n}, q^{2n+1}\right], \quad q > 1, \qquad \delta : [q^2, \infty)_{\mathbb{T}_1} \to [1, \infty)_{\mathbb{T}_1}, \quad \delta(t) = q^{-2}t,$$
$$\mathbb{T}_2 = [-\tau, \infty), \qquad \delta : [0, \infty) \to [-\tau, \infty), \qquad \delta(t) = t - \tau, \quad \tau > 0,$$
$$\mathbb{T}_3 = [0, \infty), \qquad \delta : [1, \infty) \to \left[\frac{1}{\tau}, \infty\right), \qquad \delta(t) = t/\tau, \quad \tau > 1,$$

to show that a time scale does not have to be periodic, isolated, or equal to $\mathbb{R}$ in order to have an invertible strictly increasing delay function.

3. An observation

In this section, we show that commutativity condition (1.2) is redundant when the delay function δ is assumed to be invertible and strictly increasing. Thus, we improve on the results of the papers [1,2] in which condition (1.2) is required besides the existence of δ^{-1}.

Hereafter, we shall suppose that $\mathbb{T}$ is a time scale having a strictly increasing and invertible delay function $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ satisfying $\delta(t) < t$ and $\left|\delta^{\Delta}(t)\right| < \infty$, where $t_0 \in \mathbb{T}$ is fixed. Denote by $\mathbb{T}_1$ and $\mathbb{T}_2$ the sets

$$\mathbb{T}_1 = [t_0, \infty)_{\mathbb{T}} \quad \text{and} \quad \mathbb{T}_2 = \delta(\mathbb{T}_1). \tag{3.1}$$

By the closedness of $\mathbb{T}$ and the real interval $[t_0, \infty)$, we know that $\mathbb{T}_1$ is closed. Since δ is strictly increasing and invertible we have $\mathbb{T}_2 = [\delta(t_0), \infty)_{\mathbb{T}}$. Hence, $\mathbb{T}_1$ and $\mathbb{T}_2$ are both closed subsets of the reals. Let σ_1 and σ_2 denote the forward jumps on the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Since

$$\mathbb{T}_1 \subset \mathbb{T}_2 \subset \mathbb{T},$$

we get

$$\sigma(t) = \sigma_2(t) \quad \text{for all } t \in \mathbb{T}_2$$

and

$$\sigma(t) = \sigma_1(t) = \sigma_2(t) \quad \text{for all } t \in \mathbb{T}_1.$$

That is, σ_1 and σ_2 are the restrictions of the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ to the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively, i.e.,

$$\sigma_1 = \sigma|_{\mathbb{T}_1} \quad \text{and} \quad \sigma_2 = \sigma|_{\mathbb{T}_2}.$$

Recall that the Hilger derivatives Δ, Δ_1, and Δ_2 on the time scales $\mathbb{T}$, $\mathbb{T}_1$, and $\mathbb{T}_2$ are defined in terms of forward jump operators σ, σ_1, and σ_2, respectively. Hence, if f is a differentiable function on $\mathbb{T}_1$, then we have

$$f^{\Delta_2}(t) = f^{\Delta_1}(t) = f^{\Delta}(t) \quad \text{for all } t \in \mathbb{T}_1.$$

Similarly, if $a, b \in \mathbb{T}_2$ are two points with $a < b$ and f is an rd-continuous function on the interval $[a, b)_{\mathbb{T}_2}$, then

$$\int_a^b f(s)\Delta_2 s = \int_a^b f(s)\Delta s.$$

Lemma 1. *Let $\mathbb{T}$ be a time scale that is unbounded above. If $\mathbb{T}$ has a strictly increasing and invertible delay function $\delta : \mathbb{T}_1 \to \mathbb{T}_2$ satisfying $\delta(t) < t$, then δ preserves the structure of every point in $\mathbb{T}_1$, i.e.,*

$$\sigma_1(t) = t \quad \textit{implies } \sigma_2(\delta(t)) = \delta(t) \textit{ for all } t \in \mathbb{T}_1$$

and

$$\sigma_1(t) > t \quad \textit{implies } \sigma_2(\delta(t)) > \delta(t) \textit{ for all } t \in \mathbb{T}_1.$$

Proof. It is follows from (1.3) that $\sigma_1(t) \geq t$ for all $t \in \mathbb{T}_1$. Thus,

$$\delta(\sigma_1(t)) \geq \delta(t).$$

Since $\sigma_2(\delta(t))$ is the smallest element satisfying

$$\sigma_2(\delta(t)) \geq \delta(t),$$

we get

$$\delta(\sigma_1(t)) \geq \sigma_2(\delta(t)) \quad \text{for all } t \in \mathbb{T}_1. \tag{3.2}$$

First, if $t^* \in \mathbb{T}_1$ is right dense, i.e., $\sigma_1(t^*) = t^*$, then we get

$$\delta(t^*) = \delta(\sigma_1(t^*)) \geq \sigma_2(\delta(t^*))$$

by (3.2). That is,

$$\delta(t^*) = \sigma_2(\delta(t^*)).$$

Second, if $t^* \in \mathbb{T}_1$ is right scattered, i.e., $\sigma_1(t^*) > t^*$, then

$$(t^*, \sigma_1(t^*))_{\mathbb{T}_1} = (t^*, \sigma_1(t^*))_{\mathbb{T}} = \varnothing$$

and

$$\delta(\sigma_1(t^*)) > \delta(t^*).$$

Suppose to the contrary that $\delta(t^*)$ is right dense, i.e., $\sigma_2(\delta(t^*)) = \delta(t^*)$. This along with (3.2) implies

$$(\delta(t^*), \delta(\sigma_1(t^*)))_{\mathbb{T}_2} \neq \varnothing.$$

Pick one element $s \in (\delta(t^*), \delta(\sigma_1(t^*)))_{\mathbb{T}_2}$. Since δ is strictly increasing and invertible there should be an element $t \in (t^*, \sigma_1(t^*))_{\mathbb{T}_1}$ such that $\delta(t) = s$. This leads to a contradiction. Hence, $\delta(t^*)$ must be right scattered. $\square$

Conclusion 1. *Let $\mathbb{T}$ be a time scale having a strictly increasing and invertible delay function $\delta : \mathbb{T}_1 \to \mathbb{T}_2$ satisfying $\delta(t) < t$. Then*

$$\delta \circ \sigma_1(t) = \sigma_2 \circ \delta(t) \quad \textit{for all } t \in \mathbb{T}_1.$$

That is,

$$\delta \circ \sigma(t) = \sigma \circ \delta(t) \quad \textit{for all } t \in \mathbb{T}_1.$$

Proof. If $t^* \in \mathbb{T}_1$ is right dense then the proof is trivial from the previous lemma. Suppose that $t^* \in \mathbb{T}_1$ is right scattered. Then from the second part of the proof of Lemma 1,

$$(t^*, \sigma_1(t^*))_{\mathbb{T}_1} = \varnothing \quad \text{implies } (\delta(t^*), \delta(\sigma_1(t^*)))_{\mathbb{T}_2} = \varnothing.$$

This shows that $\delta(\sigma_1(t^*))$ cannot be greater than $\sigma_2 \circ \delta(t^*)$. The proof is completed by using (3.2). $\square$

Hereafter, we shall use the above given terminology to give the proof of [1, Corollary 1] which is omitted in the original paper.

Theorem 1 (*Chain Rule [3, Theorem 1.93]*). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $\omega^{\widetilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(\omega \circ \nu)^{\Delta} = (\omega^{\widetilde{\Delta}} \circ \nu)\nu^{\Delta}.$$

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be the time scales defined as in (3.1). Hence, if for any differentiable function $\omega : \mathbb{T}_2 \to \mathbb{R}$ and for $t \in \mathbb{T}_1$ the derivative $(\omega \circ \delta)^{\Delta_1}(t)$ exists, then from Theorem 1 we have

$$(\omega \circ \delta)^{\Delta}(t) = (\omega \circ \delta)^{\Delta_1}(t) = \omega^{\Delta_2}(\delta(t))\delta^{\Delta_1}(t) = \omega^{\Delta}(\delta(t))\delta^{\Delta}(t). \tag{3.3}$$

Let ω be defined by

$$\omega(u) = \int_u^{t_0} f(s)\Delta_2 s,$$

where f is an *rd*-continuous function on $\mathbb{T}_2$. From (3.1),

$$\omega(u) = \int_u^{t_0} f(s)\Delta_2 s = \int_u^{t_0} f(s)\Delta s, \quad \text{for all } u \in \mathbb{T}_2.$$

By [3, Theorem 1.117] we know that $\omega^{\Delta_2}(u) = -f(u)$. Since $\delta(t) \in \mathbb{T}_2$ for all $t \in \mathbb{T}_1$, we get by (3.3) that

$$(\omega \circ \delta)^{\Delta}(t) = (\omega \circ \delta)^{\Delta_1}(t) = \omega^{\Delta_2}(\delta(t))\delta^{\Delta_1}(t) = -f(\delta(t))\delta^{\Delta}(t) \tag{3.4}$$

for all $t \in \mathbb{T}_1^{\kappa} = \mathbb{T}_1$. This verifies [1, Corollary 1]. Hence, the formula

$$\left[\int_{\delta(t)}^{t} f(s)\Delta s\right]^{\Delta} = f(t) - f(\delta(t))\delta^{\Delta}(t) \tag{3.5}$$

in [1, Lemma 3] follows from (3.4) and the equality

$$\int_{\delta(t)}^{t} f(s)\Delta s = \int_{\delta(t)}^{t_0} f(s)\Delta s + \int_{t_0}^{t} f(s)\Delta s, \quad (t_0 \in \mathbb{T} \text{ is fixed})$$

(see [3, Theorem 1.77 (iv)]).

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References

[1] Murat Adıvar, Youssef N. Raffoul, Stability and periodicity in dynamic delay equations, Computers and Mathematics with Applications 58 (2009) 264–272.
[2] Douglas R. Anderson, Robert J. Krueger, Allan C. Peterson, Delay dynamic equations with stability, Advances in Difference Equations 2006 (2006) 19. doi:10.1155/ADE/2006/94051. Article ID 94051.
[3] Martin Bohner, Allan C. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser Boston Inc., Boston, MA, 2001.