J. Math. Anal. Appl. 435 (2016) 532–550

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Almost automorphic solutions of discrete delayed neutral system ☆

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ARTICLE INFO

Article history:
Received 13 January 2015
Available online 23 October 2015
Submitted by J. Shi

Keywords:
Almost automorphic
Almost periodic
Discrete exponential dichotomy
Discrete nonlinear neutral system
Krasnoselskii
Unique projection

ABSTRACT

We study almost automorphic solutions of the discrete delayed neutral dynamic system

$$x(t+1) = A(t)x(t) + \Delta Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t)))$$

by means of a fixed point theorem due to Krasnoselskii. Using discrete variant of exponential dichotomy and proving uniqueness of projector of discrete exponential dichotomy we invert the equation and obtain some limit results leading to sufficient conditions for the existence of almost automorphic solutions of the neutral system. Unlike the existing literature we prove our existence results without assuming boundedness of inverse matrix $A(t)^{-1}$. Hence, we significantly improve the results in the existing literature. We provide two examples to illustrate effectiveness of our results. Finally, we also provide an existence result for almost periodic solutions of the system.

1. Introduction

The theory of neutral type equations has a significant application potential in certain fields of applied mathematics, biology, and physics dealing with modelling and controlling the dynamics of real life processes (see [6,19,21,43], and references therein). In particular, investigation of periodic solutions of neutral dynamic systems has a special importance for researchers interested in biological models of certain type of populations having periodical structures (see [14,25,28]). There is a vast literature on stability analysis, oscillation theory, and periodic solutions of neutral differential and neutral functional equations (see e.g. [2,23,33,36,38,44]). We may refer to [3,24,34,35,37] for studies handling neutral difference and neutral dynamic equations on time scales.

☆ This study is supported by the Scientific and Technological Research Council of Turkey (Grant number 1649B031101152).
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http://dx.doi.org/10.1016/j.jmaa.2015.10.056
0022-247X/

Periodicity may be a strong restriction in some specific real life models including functions not strictly periodic but having values close enough to each other at every different period. Many mathematical models (see e.g. [22,31,32]) in signal processing and astrophysics require the use of almost periodic functions, informally, being a nearly periodic functions where any one period is virtually identical to its adjacent periods but not necessarily similar to periods much farther away in time. The theory of almost periodic functions was first introduced by H. Bohr [13] and generalized by A.S. Besicovitch, W. Stepanoff, S. Bochner, and J. von Neumann at the beginning of 20th century (see [7,10,11,40]). The idea of almost periodicity can roughly be regarded as a relaxation of strict the periodicity notion. A continuous function $f:\mathbb{R}\to\mathbb{R}$ is said to be almost periodic if the following characteristic property holds:

A. *For any* $\varepsilon>0$, *the set*

$$E(\varepsilon, f(x)) := \{\tau : |f(x+\tau)-f(x)|<\varepsilon \text{ for all } x\in\mathbb{R}\}$$

is relatively dense in the real line $\mathbb{R}$. *That is, for any* $\varepsilon>0$, there exists a number $l(\varepsilon)>0$ such that any interval of length $l(\varepsilon)$ contains a number in $E(\varepsilon, f(x))$.

Afterwards, S. Bochner showed that almost periodicity is equivalent to the following characteristic property which is also called *the normality condition*:

B. *From any sequence of the form* $\{f(x+h_n)\}$, *where* h_n *are real numbers, one can extract a subsequence converging uniformly on the real line (see [8,9,18]).*

Obviously, every periodic function is almost periodic. However, there exist almost periodic functions which are not periodic. For instance, the function

$$f(t)=e^{it}+e^{i\pi t}$$

is almost periodic but there is no any real number $\omega\neq 0$ such that $f(t+\omega)=f(t)$, since the functions e^{it} and $e^{i\pi t}$ are linearly independent.

Theory of almost automorphic functions was first studied by S. Bochner [10]. It is a property of a function which can be obtained by replacing convergence with uniform convergence in normality condition (B). More explicitly, a continuous function $f:\mathbb{R}\to\mathbb{R}$ is said to be almost automorphic if for every sequence $\{h'_n\}_{n\in\mathbb{Z}_+}$ of real numbers there exists a subsequence $\{h_n\}$ such that $\lim_{m\to\infty}\lim_{n\to\infty} f(t+h_n-h_m)=f(t)$ for each $t\in\mathbb{R}$. For more reading on almost automorphic functions, we refer to [20] and [30]. Obviously, almost periodicity implies almost automorphicity but not vice versa. It is shown in [42] that the function

$$f(t)=\frac{2+\exp(it)+\exp\left(i\sqrt{2}t\right)}{\left|2+\exp(it)+\exp\left(i\sqrt{2}t\right)\right|},\ t\in\mathbb{R}$$

is almost automorphic but not almost periodic.

Unlike the vast literature on almost periodicity, there is a poor research backlog on almost automorphic solutions of difference equations. To the best of our knowledge the study of almost automorphic solutions of difference equations was begun by Araya et al. in [4]. Afterwards, C. Lizama and J.G. Mesquita [27] studied almost automorphic solutions of non-autonomous difference equations

$$u(k+1)=A(k)u(k)+f(k,u(k)),\quad k\in\mathbb{Z}.$$

Employing exponential dichotomy and contraction mapping principle they proposed some existence results. Furthermore, in [26] C. Lizama and J.G. Mesquita perfectly generalized the notion of almost automorphy by studying of almost automorphic solutions of dynamic equations on time scales that are invariant under translation. In [15], S. Castillo and M. Pinto studied almost automorphic solutions of the system with constant coefficient matrix A

$$y(n+1) = Ay(n) + f(n)$$

using (μ_1, μ_2)-exponential dichotomy. In [29] I. Mishra et al. investigated almost automorphic solutions to functional differential equation

$$\frac{d}{dt}\left(x(t) - F_1(t, x(t-g(t)))\right) = A(t)x(t) + F_2(t, x(t), x(t-g(t))) \tag{1.1}$$

using the theory of evolution semigroup. Note that almost periodic solutions of Eq. (1.1) have also been studied in [1] by means of the theory of evolution semigroup.

In the present work, we propose some existence results for almost automorphic solutions of the discrete neutral delayed system

$$x(t+1) = A(t)x(t) + \Delta Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))) \tag{1.2}$$

by using fixed point theory. The highlights of the paper can be summarized as follows:

- In our analysis, we prefer using exponential dichotomy instead of theory of evolution semigroup since the conditions required by theory of evolution are strict and not easy to check (for regarding discussion see [16]). We prove uniqueness of projector of discrete exponential dichotomy. This result has a wide application potential in theory of difference equations.
- In [27, Relations (3.10) and (3.11)], the authors obtain the limiting properties of exponential dichotomy by using the product integral on discrete domain (see [39, Section 4] and [26, Section 4]). This method requires boundedness of inverse matrix $A(t)^{-1}$ as a compulsory condition. Different than [27], we obtain our limit results without assuming boundedness of the inverse matrix $A(t)^{-1}$ (see Theorem 4).
- Using a different approach we improve the existence results [27, Theorem 3.1 and Theorem 4.3] (see Example 2) and extend the results of [15] to the systems with nonconstant coefficient matrix $A(t)$. Two examples are given to illustrate the effectiveness of our results.

The latter part of the paper is organized as follows: In the next section, we give basic definitions and properties of discrete almost automorphic functions and prove our limit results regarding discrete exponential dichotomy. In the final section, we propose some sufficient conditions for the existence of almost automorphic and almost periodic solutions of the system (1.2) by means of Krasnoselskii's fixed point theorem.

2. Almost automorphic functions and exponential dichotomy

Let $\mathcal{X}$ be a (real or complex) Banach space endowed with the norm $\|.\|_{\mathcal{X}}$ and $\mathcal{B}(\mathcal{X})$ is a Banach space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{X}$ with the norm $\|.\|_{\mathcal{B}(\mathcal{X})}$ given by

$$\|L\|_{\mathcal{B}(\mathcal{X})} := \sup\left\{\|Lx\|_{\mathcal{X}} : x \in \mathcal{X} \text{ and } \|x\|_{\mathcal{X}} \leq 1\right\}.$$

Following definitions and results can be found in [4] and [27].

Definition 1. A function $f : \mathbb{Z} \to \mathcal{X}$ is said to be discrete almost automorphic if for every integer sequence $\{k'_n\}_{n\in\mathbb{Z}_+}$ there exists a subsequence $\{k_n\}_{n\in\mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} f(t + k_n) =: \bar{f}(t) \tag{2.1}$$

is well defined for each $t \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{f}(t-k_n) = f(t) \tag{2.2}$$

for each $t \in \mathbb{Z}$.

Definition 2. A function $g : \mathbb{Z}\times\mathcal{X}\to\mathcal{X}$ of two variables is said to be discrete almost automorphic in $t \in \mathbb{Z}$ for each $x \in \mathcal{X}$, if for every integer sequence $\{k'_n\}_{n\in\mathbb{Z}_+}$, there exists a subsequence $\{k_n\}_{n\in\mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} g(t+k_n, x) =: \bar{g}(t,x)$$

is well defined for each $t \in \mathbb{Z}$, $x \in \mathcal{X}$ and

$$\lim_{n\to\infty} \bar{g}(t-k_n, x) =: g(t,x)$$

for each $t \in \mathbb{Z}$ and $x \in \mathcal{X}$.

Throughout the paper, $\mathcal{A}(\mathcal{X})$ represents the set of discrete almost automorphic functions taking values on $\mathcal{X}$. Notice that $\mathcal{A}(\mathcal{X})$ is a Banach space endowed by the norm

$$\|f\|_{\mathcal{A}(\mathcal{X})} := \sup_{t\in\mathbb{Z}} \|f(t)\|_{\mathcal{X}}\,.$$

Some properties of discrete almost automorphic functions are listed in the following theorems:

Theorem 1. *(See [4].) Let $f_1, f_2 : \mathbb{Z} \to\mathcal{X}$ and $g_1, g_2 : \mathbb{Z}\times\mathcal{X}\to\mathcal{X}$ be discrete almost automorphic functions in $t \in \mathbb{Z}$, then*

i. *$f_1 + f_2$ is discrete almost automorphic*
ii. *$g_1 + g_2$ is discrete almost automorphic in t for each $x \in \mathcal{X}$*
iii. *cf_1 is discrete almost automorphic for every scalar c*
iv. *For every scalar c, cg_1 is discrete almost automorphic in t for each $x \in \mathcal{X}$*
v. *For each fixed $k \in \mathbb{Z}$, the function $f_1(t+k)$ is discrete almost automorphic*
vi. *The function $\hat{f}_1 : \mathbb{Z} \to\mathcal{X}$ defined by $\hat{f}_1(t) := f_1(-t)$ is discrete almost automorphic*
vii. *$\sup_{t\in\mathbb{Z}} \|f_1(t)\|_{\mathcal{X}} < \infty$ for each $t \in \mathbb{Z}$*
viii. *$\sup_{t\in\mathbb{Z}} \|g_1(t,x)\|_{\mathcal{X}} < \infty$ for each $t \in \mathbb{Z}$ and $x \in \mathcal{X}$*
ix. *$\sup_{t\in\mathbb{Z}} \|\bar{f}_1(t)\|_{\mathcal{X}} \le \sup_{t\in\mathbb{Z}} \|f_1(t)\|_{\mathcal{X}}$ for all $t \in \mathbb{Z}$ where $\bar{f}_1$ is defined as in (2.1)*
x. *$\sup_{t\in\mathbb{Z}} \|\bar{g}_1(t,x)\|_{\mathcal{X}} < \infty$ for each $t \in \mathbb{Z}$ and $x \in \mathcal{X}$.*

Theorem 2. *(See [4].) Let $g : \mathbb{Z}\times\mathcal{X}\to\mathcal{X}$ be discrete almost automorphic in $t \in \mathbb{Z}$, for each $x \in \mathcal{X}$ satisfying Lipschitz condition in x uniformly in t, that is*

$$\|g(t,x) - g(t,y)\|_{\mathcal{A}(\mathcal{X})} \le L\,\|x-y\|_{\mathcal{X}}\,,\ \forall x,y \in \mathcal{X}.$$

Suppose $\varphi : \mathbb{Z} \to\mathcal{X}$ is discrete almost automorphic function, then the function $g(t,\varphi(t))$ is discrete almost automorphic.

Definition 3 *(Discrete exponential dichotomy).* Let $X(t)$ be the principal fundamental matrix solution of the linear homogeneous system

$$x(t+1) = A(t)x(t),\ x(t_0) = x_0. \tag{2.3}$$

Then (2.3) is said to admit an exponential dichotomy if there exist a projection P and positive constants $\alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$\left\|X(t)PX^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \leq \beta_1 (1+\alpha_1)^{s-t},\ t \geq s, \tag{2.4}$$

$$\left\|X(t)(I-P)X^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \leq \beta_2 (1+\alpha_2)^{t-s},\ s \geq t. \tag{2.5}$$

Remark 1. Notice that in [5] and [27], the discrete exponential dichotomy is defined by using the exponential function $\exp(\alpha(s-t))$ instead of the discrete exponential function $e_\alpha(t,s) = (1+\alpha)^{s-t}$ (satisfying $\Delta_t e_\alpha(t,s) = \alpha e_\alpha(t,s)$) in (2.4) and (2.5), respectively. For convenience we prefer using Definition 3 which is evidently equivalent to [27, Definition 2.11]. Notice that Definition 3 is also consistent with the unified version of exponential dichotomy (see [26, Definition 2.12]) which covers both the discrete and continuous cases. For more reading on exponential dichotomy we may refer to [17].

We prove the following lemmas for further use in our analysis:

Lemma 1. *Let $\varphi : \mathbb{Z} \to (0,\infty)$ and $\psi : \mathbb{Z} \to (0,\infty)$ be two functions satisfying*

$$\varphi(t)\sum_{j=-\infty}^{t-1}\varphi(j)^{-1} \leq \mu,\ t \in \mathbb{Z}, \tag{2.6}$$

$$\psi(t)\sum_{j=t}^{\infty}\psi(j)^{-1} \leq \gamma,\ t \in \mathbb{Z}, \tag{2.7}$$

for some constants $\mu > 0$ and $\gamma > 0$. Then for any $t_0 \in \mathbb{Z}$, there exist positive constants c and $\widetilde{c}$ such that

$$\varphi(t) \leq c\left(1+\mu^{-1}\right)^{t_0-t} \text{ for } t \geq t_0$$

and

$$\psi(t) \leq \widetilde{c}\left(1+\gamma^{-1}\right)^{t-t_0} \text{ for } t \leq t_0.$$

Proof. Define the function

$$u(t) := \sum_{j=-\infty}^{t-1}(\varphi(j))^{-1}$$

with $\Delta u(t) = (\varphi(t))^{-1}$, where Δ is the forward difference operator. By (2.6), we have

$$u(t) \leq \mu \Delta u(t),$$

and hence,

$$u(t) \geq \left(1+\mu^{-1}\right)^{t-t_0} u(t_0),\ \text{for } t \geq t_0.$$

This implies

$$\begin{aligned}\varphi(t) &\leq \mu u(t)^{-1} \\ &\leq \mu u(t_0)^{-1}\left(1+\mu^{-1}\right)^{t_0-t} \\ &\leq c\left(1+\mu^{-1}\right)^{t_0-t}\end{aligned}$$

for $c = \mu u(t_0)^{-1}$ and $t \geq t_0$. Similarly, the function

$$v(t) := \sum_{j=t}^{\infty} \psi(j)^{-1},$$

with $\Delta v(t) = -\psi(t)^{-1}$, satisfies

$$\gamma \Delta v(t) \leq -v(t+1).$$

Solving the last inequality for $t_0 \geq t$, we get

$$v(t_0) - v(t)\left(1+\gamma^{-1}\right)^{t-t_0} \leq 0.$$

By using (2.7), we obtain

$$\begin{aligned} \psi(t) &\leq \gamma v(t)^{-1} \\ &\leq \gamma v(t_0)^{-1}\left(1+\gamma^{-1}\right)^{t-t_0} \\ &\leq \widetilde{c}\left(1+\gamma^{-1}\right)^{t-t_0} \end{aligned}$$

for $\widetilde{c} = \gamma v(t_0)^{-1}$ and $t_0 \geq t$. The proof is complete. □

Lemma 2. *If the system (2.3) admits an exponential dichotomy, then $x = 0$ is the unique bounded solution of the system (2.3).*

Proof. Let B_0 be set of initial conditions ξ belonging to bounded solutions of (2.3). Assume $(I-P)\xi \neq 0$ and define $\phi(t)^{-1} := \|X(t)(I-P)\xi\|_{\mathcal{B}(\mathcal{X})}$. Using the equality $(I-P)^2 = I-P$ we get

$$\sum_{j=t}^{\infty} X(t)(I-P)\xi\phi(j) = \sum_{j=t}^{\infty} X(t)(I-P)X^{-1}(j)X(j)(I-P)\xi\phi(j).$$

Taking the norm of both sides, we obtain

$$\begin{aligned} \phi(t)^{-1}\sum_{j=t}^{\infty}\phi(j) &\leq \sum_{j=t}^{\infty}\left\|X(t)(I-P)X^{-1}(j)\right\|_{\mathcal{B}(\mathcal{X})}\phi^{-1}(j)\phi(j) \\ &\leq \sum_{j=t}^{\infty}\beta_2(1+\alpha_2)^{t-j} \\ &= \sum_{j=0}^{\infty}\beta_2(1+\alpha_2)^{-j} \\ &= \beta_2\frac{1+\alpha_2}{\alpha_2}. \end{aligned}$$

This yields

$$\sum_{j=t}^{\infty}\phi(j) \leq \phi(t)\beta_2\frac{1+\alpha_2}{\alpha_2},$$

uniformly in t. Hence, we get

$$\lim \inf_{j\in[t,\infty)} \phi(j) = 0,$$

which means that $\|X(t)(I-P)\xi\|_{\mathcal{B}(\mathcal{X})}$ have to be unbounded.

Similarly, if we assume that $P\xi \neq 0$, define $(\theta(t))^{-1} = \|X(t)P\xi\|_{\mathcal{B}(\mathcal{X})}$, and repeat the above procedure, we get

$$\sum_{j=-\infty}^{t-1} \theta(j)X(t)P\xi = \sum_{j=-\infty}^{t-1} X(t)PX^{-1}(j)\,X(j)P\xi\theta(j)$$

since $P^2 = P$. Taking the norm of both sides we obtain

$$\begin{aligned}(\theta(t))^{-1} \sum_{j=-\infty}^{t-1} \theta(j) &= \sum_{j=-\infty}^{t-1} \left\|X(t)PX^{-1}(j)\right\|_{\mathcal{B}(\mathcal{X})} \theta^{-1}(j)\theta(j)\\ &\leq \sum_{j=-\infty}^{t-1} \beta_1 (1+\alpha_1)^{j-t}\\ &\leq \frac{\beta_1}{\alpha_1}\end{aligned}$$

and

$$\sum_{j=-\infty}^{t-1} \theta(j) \leq \theta(t)\frac{\beta_1}{\alpha_1}.$$

This shows that

$$\lim \inf_{j\in[-\infty,t-1)} \theta(j) = 0$$

and hence $\|X(t)P\xi\|_{\mathcal{B}(\mathcal{X})}$ must be unbounded. Consequently, boundedness of a solution of the system (2.3) is possible only if $B_0 = \{0\}$, which means, if $x(t)$ is a bounded solution of (2.3), then $x(t) = 0$. The proof is complete. □

Theorem 3. *If the homogeneous system (2.3) admits an exponential dichotomy, then the projection P of the exponential dichotomy is unique.*

Proof. Suppose that the system (2.3) admits an exponential dichotomy. At first, we need to show that $\|X(t)P\|_{\mathcal{B}(\mathcal{X})}$ is bounded for $t \geq t_0$ and $\|X(t)(I-P)\|_{\mathcal{B}(\mathcal{X})}$ is bounded for $t \leq t_0$. Define the function $\varphi(t) := \|X(t)P\|_{\mathcal{B}(\mathcal{X})}$ and consider the following equality

$$\sum_{j=-\infty}^{t-1} X(t)P\varphi(j)^{-1} = \sum_{j=-\infty}^{t-1} X(t)PX^{-1}(j)X(j)P\varphi(j)^{-1}.$$

Taking the norm of both sides, we get

$$\varphi(t)\sum_{j=-\infty}^{t-1} \varphi(j)^{-1} \leq \frac{\beta_1}{\alpha_1} := K.$$

Employing Lemma 1, there exists a positive constant c such that

$$\varphi(t) \le c\left(1+K^{-1}\right)^{t_0-t} \text{ for } t \ge t_0,$$

which means $\|X(t)P\|_{\mathcal{B}(\mathcal{X})}$ is bounded for $t \ge t_0$.

Performing the similar procedure for the function $\psi(t) := \|X(t)(I-P)\|_{\mathcal{B}(\mathcal{X})}$ we get

$$\sum_{j=t}^{\infty} X(t)(I-P)(\psi(j))^{-1} = \sum_{j=t}^{\infty} X(t)(I-P)X^{-1}(j)X(j)(I-P)(\psi(j))^{-1},$$

which implies that

$$\psi(t)\sum_{j=t}^{\infty}\psi(j)^{-1} \le \beta_2\frac{1+\alpha_2}{\alpha_2} := \hat{K}.$$

By Lemma 1, we can find a constant $\widehat{c} > 0$ such that

$$\psi(t) \le \widehat{c}\left(1+\hat{K}^{-1}\right)^{t-t_0} \text{ for } t_0 \ge t.$$

This shows that $\|X(t)(I-P)\|_{\mathcal{B}(\mathcal{X})}$ is bounded for $t_0 \ge t$.

Suppose that there exists another projection $\tilde{P} \ne P$ of exponential dichotomy of (2.3). Using the similar arguments we may find constants N and $\widetilde{N}$ such that

$$\|X(t)\tilde{P}\|_{\mathcal{B}(\mathcal{X})} \le N \text{ for } t \ge t_0,$$

and

$$\|X(t)(I-\tilde{P})\|_{\mathcal{B}(\mathcal{X})} \le \widetilde{N} \text{ for } t_0 \ge t.$$

Using (2.4)–(2.5), for any arbitrary nonzero vector ξ, we get

$$\begin{aligned}\|X(t)P(I-\tilde{P})\xi\|_{\mathcal{B}(\mathcal{X})} &= \|X(t)PX^{-1}(t_0)X(t_0)(I-\tilde{P})\xi\|_{\mathcal{B}(\mathcal{X})}\\ &\le \|X(t)PX^{-1}(t_0)\|_{\mathcal{B}(\mathcal{X})}\|X(t_0)(I-\tilde{P})\xi\|_{\mathcal{B}(\mathcal{X})}\\ &\le \beta_1\|(I-\tilde{P})\xi\|_{\mathcal{X}} \text{ for } t \ge t_0\end{aligned}$$

and

$$\begin{aligned}\|X(t)P(I-\tilde{P})\xi\|_{\mathcal{B}(\mathcal{X})} &= \|X(t)PX^{-1}(t)X(t)(I-\tilde{P})X^{-1}(t_0)X(t_0)(I-\tilde{P})\xi\|_{\mathcal{B}(\mathcal{X})}\\ &\le \|X(t)PX^{-1}(t)\|_{\mathcal{B}(\mathcal{X})}\|X(t)(I-\tilde{P})X^{-1}(t_0)\|_{\mathcal{B}(\mathcal{X})}\|X(t_0)(I-\tilde{P})\xi\|_{\mathcal{B}(\mathcal{X})}\\ &\le \beta_1\beta_2\|(I-\tilde{P})\xi\|_{\mathcal{X}} \text{ for } t_0 \ge t\end{aligned}$$

since $X(t_0) = I$. Then $x(t) = X(t)P(I-\tilde{P})\xi$ is bounded solution of (2.3). Observe that $x(t) = X(t)(I-P)\tilde{P}\xi$ is also a bounded solution of (2.3). Employing Lemma 2, we get $x = 0$, and hence, $P = P\tilde{P} = \tilde{P}$. The proof is complete. □

Theorem 4. *Suppose that the system (2.3) admits an exponential dichotomy with the projection P and the positive constants α_1, α_2, β_1, and β_2. Let the matrix valued function $A(t)$ in (2.3) be almost automorphic. That is, for any sequence $\{\widetilde{\theta}_k\}_{k\in\mathbb{Z}_+}$ of integers there exists a subsequence $\{\theta_k\}_{k\in\mathbb{Z}_+}$ such that*

$$\lim_{k\to\infty} A(t+\theta_k) := \bar{A}(t)$$

is well defined and

$$\lim_{k\to\infty} \bar{A}(t-\theta_k) = A(t)$$

for each $t \in \mathbb{Z}$. Then

$$\lim_{k\to\infty} X(t+\theta_k)PX^{-1}(s+\theta_k) := \bar{X}(t)\bar{P}\bar{X}^{-1}(s) \text{ for } s \in (-\infty, t] \cap \mathbb{Z} \tag{2.8}$$

and

$$\lim_{k\to\infty} X(t+\theta_k)\left(I-P\right)X^{-1}(s+\theta_k) := \bar{X}(t)\left(I-\bar{P}\right)\bar{X}^{-1}(s) \text{ for } s \in [t,\infty) \cap \mathbb{Z} \tag{2.9}$$

are well defined for each $t \in \mathbb{Z}$ and the limiting system

$$x(t+1) = \bar{A}(t)x(t),\ x(t_0) = x_0 \tag{2.10}$$

admits an exponential dichotomy with the projection $\bar{P}$ and the same constants. Furthermore, for each $t \in \mathbb{Z}$ we have

$$\lim_{k\to\infty} \bar{X}(t-\theta_k)\bar{P}\bar{X}^{-1}(s-\theta_k) = X(t)PX^{-1}(s),\quad s \in (-\infty, t] \cap \mathbb{Z} \tag{2.11}$$

and

$$\lim_{k\to\infty} \bar{X}(t-\theta_k)\left(I-\bar{P}\right)\bar{X}^{-1}(s-\theta_k) = X(t)\left(I-P\right)X^{-1}(s),\ s \in [t,\infty) \cap \mathbb{Z}. \tag{2.12}$$

Proof. We first show that the sequence $\left\{X(t_0+\theta_k)PX^{-1}(t_0+\theta_k)\right\}$ is convergent. Suppose the contrary, then there exist two subsequences

$$\left\{X(t_0+\theta_{k_m})PX^{-1}(t_0+\theta_{k_m})\right\} \quad \text{and} \quad \left\{X(t_0+\theta_{k'_m})PX^{-1}(t_0+\theta_{k'_m})\right\}$$

converging two different numbers $\overline{P}$ and $\underline{P}$, respectively. From (2.4) we have

$$\left\|X(t+\theta_{k_m})PX^{-1}(s+\theta_{k_m}))\right\|_{\mathcal{B}(\mathcal{X})} \le \beta_1\left(1+\alpha_1\right)^{s-t},\ t \ge s, \tag{2.13}$$

and

$$\left\|X(t+\theta_{k'_m})PX^{-1}(s+\theta_{k'_m}))\right\|_{\mathcal{B}(\mathcal{X})} \le \beta_1\left(1+\alpha_1\right)^{s-t},\ t \ge s. \tag{2.14}$$

Let $X_{k_m}(t)$ and $X_{k'_m}(t)$ denote the principal fundamental matrix solutions of the systems:

$$x(t+1) = A(t+\theta_{k_m})x(t),\ x(t_0) = x_0, \tag{2.15}$$

and

$$x(t+1) = A(t+\theta_{k'_m})x(t),\ x(t_0) = x_0, \tag{2.16}$$

respectively. Then we must have

$$X(t+\theta_{k_m}) = X_{k_m}(t)X(t_0+\theta_{k_m}), \tag{2.17}$$

since

$$\Delta\left[X_{k_m}(t)^{-1}X(t+\theta_{k_m})\right] = 0.$$

Similarly, we get

$$X(t+\theta_{k'_m}) = X_{k'_m}(t)X(t_0+\theta_{k'_m}). \tag{2.18}$$

Since $A(t+\theta_{k_m}) \to \bar{A}(t)$, $A(t+\theta_{k'_m})x(t) \to \bar{A}(t)x(t)$ as $m \to \infty$ for each $t \in \mathbb{Z}$, we have

$$A(t+\theta_{k_m})x(t) \to \bar{A}(t)x(t),$$

and

$$A(t+\theta_{k'_m})x(t) \to \bar{A}(t)x(t).$$

Thus, the sequences $\{X_{k_m}(t)\}$ and $\left\{X_{k'_m}(t)\right\}$ converge to $\bar{X}(t)$ as $m \to \infty$ for each $t \in \mathbb{Z}$. Now, the exponential dichotomy of the linear homogeneous system (2.3) plays a crucial role. Using (2.17) along with (2.13) and (2.14), we get

$$\left\|X_{k_m}(t)X(t_0+\theta_{k_m})PX^{-1}(t_0+\theta_{k_m})X_{k_m}^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \leq \beta_1 (1+\alpha_1)^{s-t},\ t \geq s$$

and

$$\left\|X_{k'_m}(t)X(t_0+\theta_{k'_m})PX^{-1}(t_0+\theta_{k'_m})X_{k'_m}^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \leq \beta_1 (1+\alpha_1)^{s-t},\ t \geq s.$$

Taking the limit as $m \to \infty$, we obtain

$$\left\|\bar{X}(t)\overline{P}\bar{X}^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \leq \beta_1 (1+\alpha_1)^{s-t},\ t \geq s, \tag{2.19}$$

$$\left\|\bar{X}(t)\underline{P}\bar{X}^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \leq \beta_1 (1+\alpha_1)^{s-t},\ t \geq s. \tag{2.20}$$

Applying the similar procedure we arrive at the following inequalities

$$\left\|\bar{X}(t)\left(I-\overline{P}\right)\bar{X}^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \leq \beta_2 (1+\alpha_2)^{t-s},\ s \geq t, \tag{2.21}$$

$$\left\|\bar{X}(t)\left(I-\underline{P}\right)\bar{X}^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \leq \beta_2 (1+\alpha_2)^{t-s},\ s \geq t. \tag{2.22}$$

Inequalities (2.19)–(2.22) show that the limiting system (2.10) admits exponential dichotomy and both $\overline{P}$ and $\underline{P}$ are projections. By Theorem 4 we conclude that $\overline{P} = \underline{P}$. This means the sequence $\left\{X(t_0+\theta_k)PX^{-1}(t_0+\theta_k)\right\}$ is convergent as desired. Assume that $X(t_0+\theta_k)PX^{-1}(t_0+\theta_k) \to \overline{P}$ and that $X_k(t)$ is the principal fundamental matrix solution of the system

$$x(t+1) = A(t+\theta_k)x(t),\ x(t_0) = x_0.$$

Then $X_k(t) \to \bar{X}(t)$ and $X_k^{-1}(s) \to \bar{X}^{-1}(s)$ as $k \to \infty$ for each $t, s \in \mathbb{Z}$. This means for each $t \in \mathbb{Z}$

$$X(t+\theta_k)PX^{-1}(s+\theta_k) \to \bar{X}(t)\overline{P}\bar{X}^{-1}(s) \text{ for } s \in (-\infty, t] \cap \mathbb{Z}$$

and

$$X(t+\theta_k)\,(I-P)\,X^{-1}(s+\theta_k) \to \bar{X}(t)\left(I-\overline{P}\right)\bar{X}^{-1}(s) \text{ for } s \in [t, \infty) \cap \mathbb{Z}.$$

Hence, we prove (2.8) and (2.9). From (2.13) and (2.14) we also have

$$\left\|X(t+\theta_k)PX^{-1}(s+\theta_k))\right\|_{\mathcal{B}(\mathcal{X})} \le \beta_1\,(1+\alpha_1)^{s-t},\ t \ge s$$

and similarly,

$$\left\|X(t+\theta_k)\,(I-P)\,X^{-1}(s+\theta_k)\right\|_{\mathcal{B}(\mathcal{X})} \le \beta_2\,(1+\alpha_2)^{s-t},\ s \ge t.$$

Taking limit as $k \to \infty$ we get

$$\left\|\bar{X}(t)\overline{P}\bar{X}^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \le \beta_1\,(1+\alpha_1)^{s-t},\ t \ge s,$$
$$\left\|\bar{X}(t)\left(I-\overline{P}\right)\bar{X}^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \le \beta_2\,(1+\alpha_2)^{s-t},\ s \ge t.$$

This shows that the limiting system (2.10) admits exponential dichotomy with the projection $\overline{P}$ and the positive constants α_1, α_2, β_1, and β_2. To prove (2.11) and (2.12), we can follow the similar procedure that we used to get (2.8) and (2.9). This completes the proof. □

3. Existence results

In this section, we propose some sufficient conditions for existence of almost automorphic solutions of the nonlinear neutral delay difference system

$$x(t+1) = A(t)x(t) + \Delta Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))), \tag{3.1}$$

where $A(t)$ is an $n \times n$ matrix function, $g : \mathbb{Z} \to \mathbb{Z}_+$ is scalar, and the functions $Q : \mathbb{Z}\times\mathcal{X}\to\mathcal{X}$ and $G : \mathbb{Z} \times \mathcal{X} \times \mathcal{X} \to\mathcal{X}$ are continuous in x.

In our analysis, we use the following fixed point theorem:

Theorem 5 *(Krasnoselskii). Let $\mathbb{M}$ be a closed, convex and nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that H_1 and H_2 map $\mathbb{M}$ into $\mathbb{B}$ such that*

i. *$x, y \in \mathbb{M}$ implies $H_1x + H_2y \in \mathbb{M}$,*
ii. *H_2 is continuous and $H_2\mathbb{M}$ contained in a compact set,*
iii. *H_1 is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = H_1z + H_2z$.

Hereafter, we suppose that the following conditions hold:

A1 Functions $A(t)$, $g(t)$, $Q(t, u)$ and $G(t, u, v)$ are almost automorphic in t.

A2 For $\zeta, \psi \in \mathcal{A}(\mathcal{X})$, there exists a constant $E_1 > 0$ such that

$$\|Q(t,\zeta) - Q(t,\psi)\|_{\mathcal{X}} \leq E_1 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} \text{ for all } t \in \mathbb{Z}.$$

A3 For $\zeta, \psi \in \mathcal{A}(\mathcal{X})$, there exists a constant $E_2 > 0$ such that

$$\|G(t,u,\zeta) - G(t,u,\psi)\|_{\mathcal{X}} \leq E_2 \left(\|u - v\|_{\mathcal{X}} + \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})}\right) \text{ for all } t \in \mathbb{Z}$$

and for any vector valued functions u and v defined on $\mathcal{X}$.

A4 The homogeneous system (2.3) admits an exponential dichotomy.

The following result can be proven similar to [16, Lemma 2.4], hence we omit it.

Lemma 3. *If u, $v : \mathbb{Z} \to \mathcal{X}$ are almost automorphic functions, then $u(t - v(t))$ is also discrete almost automorphic.*

We say that $x : \mathbb{Z} \to \mathcal{X}$ is a solution of

$$x(t+1) = A(t)x(t) + f(t, x(t))$$

if it satisfies

$$x(t) = \sum_{j=-\infty}^{t-1} X(t)PX^{-1}(j+1)f(j, x(j)) - \sum_{j=t}^{\infty} X(t)(I-P)X^{-1}(j+1)f(j, x(j)),$$

where $X(t)$ is principal fundamental matrix solution of system (2.3) (see [27, Definition 4.1]).

Now, define the mapping H by

$$(Hx)(t) := (H_1x)(t) + (H_2x)(t),$$

where

$$(H_1x)(t) := Q(t, x(t - g(t))), \tag{3.2}$$

and

$$(H_2x)(t) := \sum_{j=-\infty}^{t-1} X(t)PX^{-1}(j+1)\Lambda(j,x) - \sum_{j=t}^{\infty} X(t)(I-P)X^{-1}(j+1)\Lambda(j,x),$$

where $\Lambda(j,x)$ is given by

$$\Lambda(j,x) := (A(j) - I)\,Q(j, x(j - g(j))) + G(j, x(j), x(j - g(j))). \tag{3.3}$$

Lemma 4. *The mapping H maps $\mathcal{A}(\mathcal{X})$ into $\mathcal{A}(\mathcal{X})$.*

Proof. Suppose that $x \in \mathcal{A}(\mathcal{X})$. First, we deduce by using (A1–A3) along with Theorem 2 that the functions Q and G are discrete almost automorphic. That is,

$$\lim_{n\to\infty} Q(t + k_n, x(t + k_n - g(t + k_n))) := \overline{Q}(t, \overline{x}(t - \overline{g}(t)))$$

and

$$\lim_{n\to\infty} G(t+k_n, x\left(t+k_n\right), x(t+k_n-g(t+k_n))) := \overline{G}(t, \overline{x}(t), \overline{x}(t-\overline{g}(t)))$$

are well defined for each $t \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \overline{Q}(t-k_n, \overline{x}(t-k_n-\overline{g}(t-k_n))) = Q(t, x(t-g(t)))$$

and

$$\lim_{n\to\infty} \overline{G}(t-k_n, \overline{x}\left(t-k_n\right), \overline{x}(t-k_n-\overline{g}(t-k_n))) = G(t, x(t), x(t-g(t)))$$

for each $t \in \mathbb{Z}$. Second, we have

$$(Hx)(t+k_n) = Q(t+k_n, x(t+k_n-g(t+k_n))) + \sum_{j=-\infty}^{t-1} X(t+k_n)PX^{-1}(j+k_n+1)\Lambda(j+k_n, x)$$
$$-\sum_{j=t}^{\infty} X(t+k_n)\left(I-P\right)X^{-1}(j+k_n+1)\Lambda(j+k_n, x).$$

Taking the limit $n \to \infty$ and employing Lebesgue convergence theorem, we conclude that

$$\overline{(Hx)}(t) := \lim_{n\to\infty} (Hx)(t+k_n) = \overline{Q}(t, \overline{x}(t-\overline{g}(t))) + \sum_{j=-\infty}^{t-1} \overline{X}(t)\overline{PX}^{-1}(j+1)\overline{\Lambda}(j, x)$$
$$-\sum_{j=t}^{\infty} \overline{X}(t)\left(I-\overline{P}\right)\overline{X}^{-1}(j+1)\overline{\Lambda}(j, x),$$

is well defined for each $t \in \mathbb{Z}$, where

$$\overline{\Lambda}(j, x) := \left(\overline{A}(j) - I\right)\overline{Q}(j, \overline{x}(j-\overline{g}(j))) + \overline{G}(j, \overline{x}(j), \overline{x}(j-\overline{g}(j))).$$

Applying similar procedure to the following

$$\overline{(Hx)}(t-k_n) = \overline{Q}(t-k_n, \overline{x}(t-k_n-\overline{g}(t-k_n))) + \sum_{j=-\infty}^{t-1} \overline{X}(t-k_n)\overline{PX}^{-1}(j-k_n+1)\overline{\Lambda}(j-k_n)$$
$$-\sum_{j=t}^{\infty} \overline{X}(t-k_n)\left(I-\overline{P}\right)\overline{X}^{-1}(j-k_n+1)\overline{\Lambda}(j-k_n, x),$$

we get

$$\lim_{n\to\infty} \overline{(Hx)}(t-k_n) = (Hx)(t),$$

for each $t \in \mathbb{Z}$. This means $Hx \in \mathcal{A}(\mathcal{X})$. This completes the proof. □

The following result is a direct consequence of (A2) and (3.2).

Lemma 5. *Assume (A2). If $E_1 < 1$, then the operator H_1 is a contraction.*

Lemma 6. *Assume (A1–A4). Define the set*

$$\Pi_M := \left\{ x \in \mathcal{A}(\mathcal{X}), \|x\|_{\mathcal{A}(\mathcal{X})} \leq M \right\}$$

where M is a fixed constant. The operator H_2 is continuous and the image $H_2(\Pi_M)$ is contained in a compact set.

Proof. By (A4), we have the following:

$$\|(H_2 x)\|_{\mathcal{A}(\mathcal{X})} \leq \|f(., x(.)\|_{\mathcal{A}(\mathcal{X})} \left[\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right], \tag{3.4}$$

where f is defined by (3.3). To see that H_2 is continuous, suppose ζ, $\psi \in \mathcal{A}(\mathcal{X})$ and define the number $\delta(\varepsilon) > 0$ by

$$\delta := \frac{\varepsilon}{\left[(\|A\| + 1) E_1 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} + 2E_2 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} \right] \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right)},$$

for any given $\varepsilon > 0$ where

$$\|A\| = \sup_{t \in \mathbb{Z}} |A(t)|$$

and

$$|A(t)| := \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}(t)|.$$

If $\|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} < \delta$, then we have

$$\begin{aligned}
\|H_2(\zeta)(t) - H_2(\psi)(t)\|_{\mathcal{X}} \leq & \sum_{j=-\infty}^{t-1} \Big(\left\| X(t) P X^{-1}(j+1) \right\|_{\mathcal{B}(\mathcal{X})} \\
& \times \left[(\|A\| + 1) \|Q(j, \zeta(j - g(j))) - Q(s, \psi(j - g(j)))\|_{\mathcal{X}} \right. \\
& \left. + \|G(j, \zeta(j), \zeta(j - g(j))) - G(j, \psi(j), \psi(j - g(j)))\|_{\mathcal{X}} \right] \Big) \\
& + \sum_{j=t}^{\infty} \Big(\left\| X(t)(I - P) X^{-1}(j+1) \right\|_{\mathcal{B}(\mathcal{X})} \\
& \times \left[(\|A\| + 1) \|Q(j, \zeta(j - g(j))) - Q(s, \psi(j - g(j)))\|_{\mathcal{X}} \right. \\
& \left. + \|G(j, \zeta(j), \zeta(j - g(j))) - G(j, \psi(j), \psi(j - g(j)))\|_{\mathcal{X}} \right] \Big).
\end{aligned}$$

By (vii) of Theorem 1 and (A2–A4), we get

$$\begin{aligned}
\|H_2(\zeta)(t) - H_2(\psi)(t)\|_{\mathcal{X}} \leq & \sum_{j=-\infty}^{t-1} \beta_1 (1 + \alpha_1)^{j+1-t} \left[(\|A\| + 1) E_1 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} + 2E_2 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} \right] \\
& + \sum_{j=t}^{\infty} \beta_2 (1 + \alpha_2)^{t-j-1} \left[(\|A\| + 1) E_1 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} + 2E_2 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} \right]
\end{aligned}$$

$$\leq \left[(\|A\|+1)\, E_1 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} + 2E_2 \|\zeta - \psi\|_{\mathcal{A}(\mathcal{X})} \right] \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right)$$
$$< \varepsilon,$$

which shows that H_2 is continuous.

Now, we show that $H_2(\Pi_M)$ is contained in a compact set. For any $\zeta, \psi \in \Pi_M$ we have

$$\begin{aligned} \|G(t,\zeta(t),\psi(t-g(t)))\|_{\mathcal{X}} &\leq \|G(t,\zeta(t),\psi(t-g(t))) - G(t,0,0)\|_{\mathcal{X}} + \|G(t,0,0)\|_{\mathcal{X}} \\ &\leq E_2 \left(\|\zeta\|_{\mathcal{A}(\mathcal{X})} + \|\psi\|_{\mathcal{A}(\mathcal{X})} \right) + a \\ &\leq 2ME_2 + a, \end{aligned}$$

and

$$\begin{aligned} \|Q(t,\zeta(t-g(t)))\|_{\mathcal{X}} &\leq \|Q(t,\zeta(t-g(t))) - Q(t,0)\|_{\mathcal{X}} + \|Q(t,0)\|_{\mathcal{X}} \\ &\leq E_1 \|\zeta\|_{\mathcal{A}(\mathcal{X})} + b \\ &\leq E_1 M + b \end{aligned}$$

where $a := \|G(t,0,0)\|_{\mathcal{X}}$ and $b := \|Q(t,0)\|_{\mathcal{X}}$. This implies

$$\|H_2(\zeta_n)(t)\|_{\mathcal{A}(\mathcal{X})} \leq [(\|A\|+1)(E_1 M + b) + 2E_2 M + a] \left[\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right]$$

for any sequence $\{\zeta_n\}$ in Π_M. Moreover, from (A1), (A4) and (3.4), we deduce that $\Delta(H_2(\zeta_n(t)))$ is bounded. That means, $H_2(\zeta_n)$ is uniformly bounded an equicontinuous. The proof follows from Arzela–Ascoli theorem. □

Theorem 6. *Assume (A1–A4). Let M_0 be a constant satisfying the following inequality*

$$E_1 M_0 + b + [(\|A\|+1)(E_1 M_0 + b) + 2E_2 M_0 + a] \left[\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right] \leq M_0,$$

where $E_1 \in (0,1)$ and

$$a := \|G(t,0,0)\|_{\mathcal{X}}, b := \|Q(t,0)\|_{\mathcal{X}}.$$

Then the equation (3.1) has an almost automorphic solution in Π_{M_0}.

Proof. For $\psi \in \Pi_{M_0}$, we have

$$\begin{aligned} \|H_1(\psi(t)) + H_2(\psi(t))\|_{\mathcal{X}} \leq{} & \|Q(t,\psi(t-g(t))) - Q(t,0)\|_{\mathcal{X}} + \|Q(t,0)\|_{\mathcal{X}} \\ & + \sum_{j=-\infty}^{t-1} \Big\{ \left\| X(t) P X^{-1}(j+1) \right\|_{\mathcal{B}(\mathcal{X})} \\ & \times \|(A(j)-I)\, Q(j, x(j-g(j))) + G(j,\psi(j),\psi(j-g(j)))\|_{\mathcal{X}} \Big\} \\ & + \sum_{j=t}^{\infty} \Big\{ \left\| X(t)(I-P) X^{-1}(j+1) \right\|_{\mathcal{B}(\mathcal{X})} \\ & \times \|(A(j)-I)\, Q(j, x(j-g(j))) + G(j,\psi(j),\psi(j-g(j)))\|_{\mathcal{X}} \Big\} \end{aligned}$$

$$\leq E_1M_0 + b + [(\|A\| + 1)(E_1M_0 + b) + 2E_2M_0 + a]\left[\beta_1\frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right]$$
$$\leq M_0$$

which means $H_1(\psi)+H_2(\psi) \in \Pi_{M_0}$. Then all conditions of fixed point theorem are satisfied and there exists a $x \in \Pi_{M_0}$ such that $x(t) = H_1(x(t)) + H_2(x(t))$. The proof is complete. □

Example 1. Let the neutral delay discrete system be given by

$$x(t+1) = \frac{1}{3}sgn\,(\cos 2\pi t\theta)\,x(t)I + \frac{1}{10}\Delta x\,(t-\tau) + \begin{bmatrix}\sin\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\sqrt{2}\right) \\ \cos\pi t + \cos\pi t\sqrt{2}\end{bmatrix} + \frac{1}{20}x\,(t-\tau)\,, \tag{3.5}$$

where θ is an irrational number, τ is a positive integer with $t > \tau$ and Banach space $\mathcal{X} = \mathbb{R}$. In [41], it is shown that $sgn\,(\cos 2\pi t\theta)$ is an almost automorphic function for $t \in \mathbb{Z}$ and θ is irrational. Therefore, the matrix function

$$A(t) = \begin{bmatrix}\frac{1}{3}sgn\,(\cos 2\pi t\theta) & 0 \\ 0 & \frac{1}{3}sgn\,(\cos 2\pi t\theta)\end{bmatrix}$$

is discrete almost automorphic. Comparing (3.5) with (3.1), we have vector functions

$$Q(t, x(t-g(t)))) = \begin{bmatrix}\frac{1}{10}x_1\,(t-\tau) \\ \frac{1}{10}x_2\,(t-\tau)\end{bmatrix}$$

and

$$G(t, x(t), x(t-g(t))) = \begin{bmatrix}\sin\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\sqrt{2}\right) + \frac{1}{20}x_1\,(t-\tau) \\ \cos\pi t + \cos\pi t\sqrt{2} + \frac{1}{20}x_2\,(t-\tau)\end{bmatrix},$$

which are discrete almost automorphic. Then assumption (A1) is satisfied. For any ς, $\psi \in \Pi_{M_0}$, we have

$$|Q(t, \varsigma(t-g(t))) - Q(t, \psi(t-g(t)))| \leq \frac{1}{10}\|\varsigma - \psi\|_{A(\mathbb{R})}$$

and

$$|G(t, \varsigma(t), \varsigma(t-g(t))) - G(t, \psi(t), \psi(t-g(t)))| \leq \frac{1}{20}\|\varsigma - \psi\|_{A(\mathbb{R})}\,.$$

Then (A2–A3) hold with $E_1 = \frac{1}{10}$, $E_2 = \frac{1}{20}$, $a = 2$ and $b = 0$.

By using Putzer algorithm (see [12, Theorem 5.35]) with P-matrices under the special case $\mathbb{T} = \mathbb{Z}$, we get $P_0 = I_{2\times 2}$ and $P_1 = 0_{2\times 2}$. Then the principal fundamental matrix solution of the homogeneous system

$$x(t+1) = \frac{1}{3}sgn\,(\cos 2\pi t\theta)\,x(t)I$$

can be written as

$$X\,(t) = \begin{bmatrix}3^{-t}\left(\prod_{j=0}^{t-1} sgn\,(\cos 2\pi j\theta)\right) & 0 \\ 0 & 3^{-t}\left(\prod_{j=0}^{t-1} sgn\,(\cos 2\pi j\theta)\right)\end{bmatrix}.$$

Since

$$\left| 3^{s-t} \left(\prod_{j=s}^{t-1} sgn \left(\cos 2\pi j\theta \right) \right) \right| = 3^{s-t} \leq \beta_1 \left(1 + \alpha_1 \right)^{s-t} \text{ for } t \geq s$$

is satisfied for $\beta_1 = 1$ and $\alpha_1 = 1$, the homogeneous system admits exponential dichotomy, as desired. Moreover, we may assume $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ since $P_1 = 0_{2\times 2}$. That is, all assumptions of Theorem 6 hold. Hence, we conclude that the system (3.5) has an almost automorphic solution in Π_{M_0} whenever M_0 satisfies the inequality

$$\frac{1}{10} M_0 + \frac{4}{10} M_0 + \frac{3}{10} M_0 + 6 \leq M_0$$

or equivalently

$$30 \leq M_0.$$

The following existence result is given in [27]:

Theorem 7. *(See [27, Theorem 4.3].) Suppose $A(k)$ is discrete almost automorphic and a non-singular matrix and the set $\left\{ A^{-1}(k) \right\}_{k\in\mathbb{Z}}$ is bounded. Also, assume the homogeneous system $U(k+1) = A(k) U(k)$, $k \in \mathbb{Z}$, admits an exponential dichotomy on $\mathbb{Z}$ with positive constants η, ν, β, α and the function $f : \mathbb{Z} \times E^n \to E^n$ is discrete almost automorphic in k for each u in E^n, satisfying the following condition:*

1. *There exists a constant $0 < L < \frac{(1-e^{-\alpha})(e^{\beta}-1)}{\eta(e^{\beta}-1)+\nu(1-e^{-\alpha})}$ such that*

$$\| f(k,u) - f(k,v) \| \leq L \| u - v \|$$

for every $u, v \in E^n$ and $k \in \mathbb{Z}$. Then the system

$$U(k+1) = A(k) U(k) + f(k, u(k)), k \in \mathbb{Z}$$

has a unique almost automorphic solution.

Example 2. The conditions of our existence result are weaker than the conditions of [27, Theorem 4.3]. In [27, Theorem 4.3], the authors require boundedness of the inverse matrix $A^{-1}(t)$ to deduce existence of almost automorphic solutions of the system

$$x(t+1) = A(t) x(t) + f(t,x).$$

In particular, [27, Theorem 4.3] is invalid for the system

$$x(t+1) = \begin{bmatrix} \frac{1}{2}\sin\left(\frac{\pi}{2}t\right) & 0 \\ 0 & \frac{1}{2}\sin\left(\frac{\pi}{2}t\right) \end{bmatrix} x(t) + f(t,x), \quad t \in \mathbb{Z} \tag{3.6}$$

since the matrix

$$A(t) = \begin{bmatrix} \frac{1}{2}\sin\left(\frac{\pi}{2}t\right) & 0 \\ 0 & \frac{1}{2}\sin\left(\frac{\pi}{2}t\right) \end{bmatrix}$$

is singular for some integers. However, Theorem 6 implies the existence of discrete almost automorphic solution of the system (3.6) for an almost automorphic function $f(t,x)$ satisfying (A1) and (A3).

One may repeat the same procedure in the last section by replacing $\mathcal{A}(\mathcal{X})$ with $\mathcal{AP}(\mathcal{X})$, the space of all almost periodic functions on $\mathcal{X}$, and the assumption (A1) with the following

A1′ Functions $A(t)$, $g(t)$, $Q(t,u)$ and $G(t,u,v)$ are almost periodic in t

to arrive at the following result:

Theorem 8 *(Almost periodic solutions of the system (3.1)). Assume (A1′) and (A2–A4). Let M_0 be a constant satisfying the following inequality*

$$E_1M_0 + b + [(\|A\|+1)(E_1M_0+b) + 2E_2M_0 + a]\left[\beta_1\frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right] \leq M_0,$$

where $E_1 \in (0,1)$ and

$$a := \|G(t,0,0)\|_{\mathcal{X}}, b := \|Q(t,0)\|_{\mathcal{X}}.$$

Then the equation (3.1) has an almost periodic solution in $\widetilde{\Pi}_{M_0} := \left\{x \in \mathcal{AP}(\mathcal{X}), \|x\|_{\mathcal{AP}(\mathcal{X})} \leq M_0\right\}$.

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