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Progressive Type II censored order statistics for multivariate observations

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Abstract

For a sequence of independent and identically distributed random vectors $\mathbf{X}_i = (X_i^1, X_i^2, \dots, X_i^p)$, $i = 1, 2, \dots, n$, we consider the conditional ordering of these random vectors with respect to the magnitudes of $\mathbf{N}(\mathbf{X}_i)$, $i = 1, 2, \dots, n$, where \mathbf{N} is a p -variate continuous function defined on the support set of \mathbf{X}_1 and satisfying certain regularity conditions. We also consider the Progressive Type II right censoring for multivariate observations using conditional ordering. The need for the conditional ordering of random vectors exists for example, in reliability analysis when a system has n independent components each consisting of p arbitrarily dependent and parallel connected elements. Let the vector of life lengths for the i th component of the system be $\mathbf{X}_i = (X_i^1, X_i^2, \dots, X_i^p)$, $i = 1, 2, \dots, n$, where X_i^j denotes the life length of the j th element of the i th component. Then the first failure in the system occurs at time $\min \left\{ \max(X_1^1, X_1^2, \dots, X_1^p), \max(X_2^1, X_2^2, \dots, X_2^p), \dots, \max(X_n^1, X_n^2, \dots, X_n^p) \right\}$, and for this case $\mathbf{N}(\mathbf{X}_i) = \max(X_i^1, X_i^2, \dots, X_i^p)$. In this paper we introduce the conditionally ordered and Progressive Type II right-censored conditionally ordered statistics for multivariate observations and to study their distributional properties.

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1. Introduction

Ordered random variables are widely used in statistical theory and its applications. The details of the theory of univariate order statistics can be found in David [6] and Arnold et al. [1]. Many papers and several monographs have appeared on the theory of record values since Chandler [5] first introduced the concept. As noted by Kamps [8] there are several other models of ordered random variables with different interpretations and interesting applications in many fields, for example, in reliability theory, survival analysis, financial economics, etc. Kamps [8] described a natural modification of order statistics, the so-called sequential order statistics that appear when some component of the system fails and this has an influence on the life-length distributions of the remaining components. One of the interesting modifications of order statistics is the concept of Progressive Type II censored-order statistics, which is very useful in reliability and lifetime studies. Let X_1, X_2, \dots, X_n be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.s) representing failure times of n identical units placed on a life test. Under the Progressive Type II right-censoring scheme, at the time of the i th failure R_i ($i = 1, 2, \dots, m$ and $m \leq n$) surviving items are removed at random from the experiment, where $m + \sum_{i=1}^m R_i = n$. Let $\mathbf{R} = (R_1, R_2, \dots, R_m)$. Denote the m ordered observed failure times by $X_{1:m:n}^{(\mathbf{R})}, X_{2:m:n}^{(\mathbf{R})}, \dots, X_{m:m:n}^{(\mathbf{R})}$. These random variables are called Progressive Type II right-censored order statistics from a sample X_1, X_2, \dots, X_n with the progressive censoring scheme $(\mathbf{R} = (R_1, R_2, \dots, R_m))$. A good description of the theory, methods and applications of Progressive censoring can be found in Balakrishnan and Aggarwala [3]. If the failure times of the n items originally on test are from a continuous population with a cumulative distribution function (c.d.f.) F and a probability density function (p.d.f.) f , then the joint p.d.f. of all m progressively Type II censored-order statistics is

$$f_{1,2,\dots,m}(x_1, x_2, \dots, x_m) = c \prod_{i=1}^m f(x_i) \{1 - F(x_i)\}^{R_i}, \quad x_1 < x_2 < \dots < x_m, \quad (1)$$

where $c = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1)$. Note that the concept of generalized order statistics first proposed by Kamps [8] includes order statistics, records and Progressive Type II censored-order statistics as special cases.

In applications there are situations when we need to order multivariate random variables (or random vectors). For example, consider a model in reliability analysis when the system has n independent components each consisting of p arbitrarily dependent elements connected by using a parallel structure. Denote the life length of the i th component of the system by $\mathbf{X}_i = (X_i^1, X_i^2, \dots, X_i^p)$, $i = 1, 2, \dots, n$, where X_i^j denote the life length of the j th element of the i th component. Then the first failure in the system occurs at time

$$\min \left\{ \max(X_1^1, X_1^2, \dots, X_1^p), \max(X_2^1, X_2^2, \dots, X_2^p), \dots, \max(X_n^1, X_n^2, \dots, X_n^p) \right\}$$

and the corresponding vector of the length is denoted by $\mathbf{X}^{(1)}$. Similarly, the last failure occurs at time

$$\max \left\{ \max(X_1^1, X_1^2, \dots, X_1^p), \max(X_2^1, X_2^2, \dots, X_2^p), \dots, \max(X_n^1, X_n^2, \dots, X_n^p) \right\}.$$

and the corresponding vector of life length is denoted by $\mathbf{X}^{(n)}$. It is clear that in the case where the elements of a component are connected by using series structure, we have

$$\min \left\{ \min(X_1^1, X_1^2, \dots, X_1^p), \min(X_2^1, X_2^2, \dots, X_2^p), \dots, \min(X_n^1, X_n^2, \dots, X_n^p) \right\}$$

as the first failure time. Therefore to know the time of the first failure (and consequently, the time of the second failure, etc.) in such a system, we need to arrange random vectors $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ by the magnitude of the function $\mathbf{N}(\mathbf{X}_i) = \max(X_i^1, X_i^2, \dots, X_i^p)$, $i = 1, 2, \dots, n$ (or $\mathbf{N}(\mathbf{X}_i) = \min(X_i^1, X_i^2, \dots, X_i^p)$, etc.). The distributions of the norm-ordered statistics, i.e. the random vectors ordered with respect to the norm function $\mathbf{N}(\mathbf{x}) = \|\mathbf{x}\|$, $\mathbf{x} = (x_1, x_2, \dots, x_p)$ in a linear normed space, are studied in Bairamov and Gebizlioglu [2]. Recently, Eryilmaz [7] has studied the distributional properties of multivariate exceedances based on norm-ordered statistics.

2. Conditionally ordered random vectors

Let $(\Omega, \mathfrak{S}, P)$ be a probability space, where Ω is a non-empty set of points ω , \mathfrak{S} is a σ -field of subsets of Ω and P is a probability measure defined on $\{\Omega, \mathfrak{S}\}$. Let us consider the real Euclidean space R^p . Let \mathfrak{R}^p , $p \geq 1$ be the Borel σ -algebra of subsets of R^p . Let $\mathbf{X}(\omega)$, $\omega \in \Omega$, be the r.v. mapping Ω into R^p , so $\mathbf{X}^{-1}(B) \in \mathfrak{S}$ for any $B \in \mathfrak{R}^p$. If ω is fixed, then $\mathbf{X} = \mathbf{X}(\omega)$ is a point of R^p . If $\mathbf{N}(\mathbf{x})$ is a measurable function with respect to the Borel σ -algebra \mathfrak{R}^p , then $\mathbf{N}(\mathbf{X})(\omega)$ is a random variable. (In fact, $\{\omega : \mathbf{N}(\mathbf{X})(\omega) \leq x\} = \{\omega : \mathbf{N}(\mathbf{X})(\omega) \in S(0, x)\} \in \mathfrak{S}$, where $S(0, x) = \{\mathbf{y} \in R^m : \mathbf{N}(\mathbf{y}) \leq x\} \in \mathfrak{R}^p$.) Throughout this paper we assume that $\mathbf{N}(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \dots, x_p)$ is a continuous function of its arguments satisfying $\mathbf{N}(\mathbf{x}) \geq 0$, for all $\mathbf{x} \in R^p$ and $\mathbf{N}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$, where $\mathbf{0} = (0, 0, \dots, 0)$.

Suppose $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n \in \mathbf{S} \subseteq R^p$ are i.i.d. random variables ($p \geq 1$ random vectors) (r.v.'s) with p -variate c.d.f. $F(\mathbf{x})$, and p.d.f. $f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \dots, x_p)$ and \mathbf{S} is the support of \mathbf{X} . It is clear that $\mathbf{N}(\mathbf{X}_1), \mathbf{N}(\mathbf{X}_2), \dots, \mathbf{N}(\mathbf{X}_n)$ are i.i.d. random variables with c.d.f. $P\{\mathbf{N}(\mathbf{X}_i) \leq x\}$, $x \in R$. If F is assumed to be continuous, the probability of any two or more of these r.v.'s assuming equal magnitudes is zero. Therefore, there exists a unique ordered arrangement within the r.v.'s $\mathbf{N}(\mathbf{X}_i)$, $i = 1, 2, \dots, n$. We say that \mathbf{X}_1 precedes \mathbf{X}_2 (or that \mathbf{X}_1 is conditionally less than \mathbf{X}_2) if $\mathbf{N}(\mathbf{X}_1) \leq \mathbf{N}(\mathbf{X}_2)$ and write $\mathbf{X}_1 < \mathbf{X}_2$. Suppose $\mathbf{X}^{(1)}$ denotes the smallest of the set $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$; $\mathbf{X}^{(2)}$ denotes the second smallest, etc.; and $\mathbf{X}^{(n)}$ denotes the largest in the sense of conditional ordering with respect to a function $\mathbf{N}(\cdot)$. We call $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(n)}$ the conditionally \mathbf{N} -ordered statistics. Note that throughout this paper we also use $\mathbf{X}^{(1:n)}, \mathbf{X}^{(2:n)}, \dots, \mathbf{X}^{(n:n)}$ instead of $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(n)}$.

Denote $h(\mathbf{x}) = P\{\mathbf{N}(\mathbf{X}) \leq \mathbf{N}(\mathbf{x})\}$, and $\mathbf{x} = (x_1, x_2, \dots, x_p) \in \mathbf{S} \subseteq R^p$. The function $h(\mathbf{x})$ is called the structural function and plays an important role in our study.

Theorem 1. Let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n \in \mathbf{S} \subseteq R^p$ be i.i.d. continuous random vectors with p.d.f. $f(\mathbf{x})$, and $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(n)}$ be the conditionally \mathbf{N} -ordered statistics. Then for $1 \leq r \leq n$

the p.d.f. of $\mathbf{X}^{(r)}$ is

$$f_r(\mathbf{x}) = \frac{n!}{(r-1)!(n-r)!} [h(\mathbf{x})]^{r-1} [1-h(\mathbf{x})]^{n-r} f(\mathbf{x}). \tag{2}$$

In particular, the p.d.f.s of $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(n)}$ are, respectively,

$$f_1(\mathbf{x}) = n[1-h(\mathbf{x})]^{n-1} f(\mathbf{x}) \quad \text{and} \quad f_n(\mathbf{x}) = nh^{n-1}(\mathbf{x})f(\mathbf{x}), \quad \mathbf{x} \in R^p. \tag{3}$$

Proof. For simplicity, consider the case $p = 2$. Suppose F has the probability density function f , i.e. for $\mathbf{X}_i = (X_i^1, X_i^2) \in \mathbf{S} \subset R^2, i = 1, 2, \dots, n$

$$F(x, y) = P \{X_i^1 \leq x, X_i^2 \leq y\} = \int_{-\infty}^x \int_{-\infty}^y f(u, v) du dv.$$

The structural function of the sample $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ is $h(\mathbf{x}) = P \{\mathbf{N}(\mathbf{X}_1) \leq \mathbf{N}(\mathbf{x})\}$, where $\mathbf{x} = (x, y) \in R^2$. Consider $\mathbf{X}^{(1)} < \mathbf{X}^{(2)} < \dots < \mathbf{X}^{(n)}$. It is clear that the r.v.'s $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(n)}$ are not independent. Let us first derive the distribution of extreme vector $\mathbf{X}^{(n)}$. For any $B \in \mathfrak{N}^2$ one can write

$$\begin{aligned} P \{ \mathbf{X}^{(n)} \in B \} &= nP \{ \mathbf{X}_1 \in B, \mathbf{N}(\mathbf{X}_1) \geq \mathbf{N}(\mathbf{X}_i), i = 1, 2, \dots, n; i \neq 1 \} \\ &= n \int \int_B P \{ \mathbf{X}_1 \in B, \mathbf{N}(\mathbf{X}_1) \geq \mathbf{N}(\mathbf{X}_i), i = 1, 2, \dots, n; i \neq 1 | \\ &\quad X_1^1 = x, X_1^2 = y \} dF(x, y) \\ &= n \int \int_B P \{ \mathbf{X}_1 \in B, \mathbf{N}(\mathbf{X}_i) \leq \mathbf{N}(\mathbf{x}), i = 1, 2, \dots, n; i \neq 1 \} dF(x, y) \\ &= n \int \int_B [P \{ \mathbf{N}(\mathbf{X}_1) \leq \mathbf{N}(\mathbf{x}) \}]^{n-1} dF(x, y) \\ &= n \int \int_B [h(x, y)]^{n-1} dF(x, y). \end{aligned}$$

Hence, the probability density function of $\mathbf{X}^{(n)}$ is

$$f_n(x, y) = n [h(x, y)]^{n-1} f(x, y),$$

where by $f_r(x, y)$ we denote the probability density function of r.v. $\mathbf{X}^{(r)}, r = 1, 2, \dots, n$. Similarly, one can write

$$f_1(x, y) = n [1-h(x, y)]^{n-1} f(x, y).$$

In general, one has for $B \in \mathfrak{R}^2$

$$\begin{aligned}
 P \left\{ \mathbf{X}^{(r)} \in B \right\} &= \sum_{k=1}^n P \left\{ X_k \in B, \mathbf{N}(\mathbf{X}_k) \text{ is the } r\text{th smallest among} \right. \\
 &\quad \left. \mathbf{N}(\mathbf{X}_1), \mathbf{N}(\mathbf{X}_2), \dots, \mathbf{N}(\mathbf{X}_n) \right\} \\
 &= \sum_{k=1}^n \binom{n-1}{r-1} P \left\{ X_k \in B, \mathbf{N}(\mathbf{X}_1) \leq \mathbf{N}(\mathbf{X}_k), \mathbf{N}(\mathbf{X}_2) \leq \mathbf{N}(\mathbf{X}_k), \dots, \right. \\
 &\quad \left. \mathbf{N}(\mathbf{X}_{r-1}) \leq \mathbf{N}(\mathbf{X}_k), \mathbf{N}(\mathbf{X}_{r+1}) > \mathbf{N}(\mathbf{X}_k), \dots, \mathbf{N}(\mathbf{X}_n) > \mathbf{N}(\mathbf{X}_k) \right\} \\
 &= \sum_{k=1}^n \binom{n-1}{r-1} \int \int_B [P \{ \mathbf{N}(\mathbf{X}_1) \leq \mathbf{N}(\mathbf{x}) \}]^{r-1} \\
 &\quad \times [1 - P \{ \mathbf{N}(\mathbf{X}_1) \leq \mathbf{N}(\mathbf{x}) \}]^{n-r} dF(x, y) \\
 &= \frac{n!}{(r-1)!(n-r)!} \int \int_B [h(x, y)]^{r-1} [1 - h(x, y)]^{n-r} dF(x, y).
 \end{aligned}$$

Therefore the probability density function of $\mathbf{X}^{(r)}$, $1 \leq r \leq n$, is

$$f_r(x, y) = \frac{n!}{(r-1)!(n-r)!} [h(x, y)]^{r-1} [1 - h(x, y)]^{n-r} f(x, y). \quad \square \quad (4)$$

2.1. The joint distributions of two or more conditionally ordered statistics

Let $1 \leq r < s \leq n$, $\mathbf{X}^{(r)} < \mathbf{X}^{(s)}$. Consider $B_1, B_2 \in \mathfrak{R}^p$, such that $\mathbf{N}(\mathbf{x}_1) \leq \mathbf{N}(\mathbf{x}_2)$ for any $\mathbf{x}_1 \in B_1, \mathbf{x}_2 \in B_2, \mathbf{x}_k = (x_k^1, x_k^2, \dots, x_k^p), k = 1, 2$. Suppose F has the probability density function f .

Theorem 2. Let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n \in \mathbf{S} \subseteq R^p$ be an i.i.d. continuous random vectors and $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(n)}$ be the conditionally \mathbf{N} -ordered statistics. Then, the joint probability density function of $\mathbf{X}^{(r)}$ and $\mathbf{X}^{(s)}$ is

$$\begin{aligned}
 &f_{rs}(\mathbf{x}_1, \mathbf{x}_2) \\
 &= \begin{cases} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [h(\mathbf{x}_1)]^{r-1} \\ \quad \times [h(\mathbf{x}_2) - h(\mathbf{x}_1)]^{s-r-1} [1 - h(\mathbf{x}_2)]^{n-s} f(\mathbf{x}_1) f(\mathbf{x}_2) & \text{if } \mathbf{N}(\mathbf{x}_1) \leq \mathbf{N}(\mathbf{x}_2), \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Proof. For simplicity, consider the case $p = 2$. For $1 \leq r < s \leq n$ we have

$$\begin{aligned}
 & P \left\{ \mathbf{X}^{(r)} \in B_1, \mathbf{X}^{(s)} \in B_2 \right\} \\
 &= \sum_{k \neq j} P \left\{ \mathbf{X}^{(r)} \in B_1, \mathbf{X}^{(s)} \in B_2, \mathbf{X}_k \text{ is} \right. \\
 &\quad \left. \text{the } r\text{th conditionally smallest (with respect to } \mathbf{N}(\mathbf{x}), \right. \\
 &\quad \left. \mathbf{X}_j \text{ is the } s\text{th conditionally smallest} \right\} \\
 &= \sum_{k \neq j} \binom{n-2}{r-1} \binom{n-2-(r-1)}{s-r-1} \\
 &\quad \times \sum_{k \neq j} P \left\{ \mathbf{X}_k \in B_1, \mathbf{X}_j \in B_2, \mathbf{N}(\mathbf{X}_1) < \mathbf{N}(\mathbf{X}_k), \dots, \right. \\
 &\quad \left. \mathbf{N}(\mathbf{X}_{r-1}) < \mathbf{N}(\mathbf{X}_k), \mathbf{N}(\mathbf{X}_k) < \mathbf{N}(\mathbf{X}_{r+1}) < \mathbf{N}(\mathbf{X}_j), \dots, \right. \\
 &\quad \left. \mathbf{N}(\mathbf{X}_k) < \mathbf{N}(\mathbf{X}_{s-1}) < \mathbf{N}(\mathbf{X}_j), \mathbf{N}(\mathbf{X}_j) < \mathbf{N}(\mathbf{X}_{s+1}), \dots, \mathbf{N}(\mathbf{X}_j) < \mathbf{N}(\mathbf{X}_n) \right\} \\
 &= \binom{n-2}{r-1} \binom{n-r-1}{s-r-1} \sum_{k \neq j} \int \int \int \int_{B_1 \times B_2} P \left\{ \mathbf{X}_k \in B_1, \mathbf{X}_j \in B_2, \mathbf{N}(\mathbf{X}_1) \right. \\
 &\quad \left. < \mathbf{N}(\mathbf{X}_k), \dots, \mathbf{N}(\mathbf{X}_{r-1}) < \mathbf{N}(\mathbf{X}_k), \mathbf{N}(\mathbf{X}_k) < \mathbf{N}(\mathbf{X}_{r+1}) < \mathbf{N}(\mathbf{X}_j), \dots, \right. \\
 &\quad \left. \mathbf{N}(\mathbf{X}_k) < \mathbf{N}(\mathbf{X}_{s-1}) < \mathbf{N}(\mathbf{X}_j), \mathbf{N}(\mathbf{X}_j) < \mathbf{N}(\mathbf{X}_{s+1}), \dots, \mathbf{N}(\mathbf{X}_j) < \mathbf{N}(\mathbf{X}_n) \mid \right. \\
 &\quad \left. X_k^1 = x_1, X_k^2 = y_1, X_j^1 = x_2, X_j^2 = y_2 \right\} dF(x_1, y_1) dF(x_2, y_2) \\
 &= n(n-1) \binom{n-2}{r-1} \binom{n-r-1}{s-r-1} \int \int \int \int_{B_1 \times B_2} [P \{ \mathbf{N}(\mathbf{X}_1) < \mathbf{N}(\mathbf{x}_1) \}]^{r-1} \\
 &\quad \times [P \{ \mathbf{N}(\mathbf{X}_1) < \mathbf{N}(\mathbf{x}_2) \} - P \{ \mathbf{N}(\mathbf{X}_1) < \mathbf{N}(\mathbf{x}_1) \}]^{s-r-1} \\
 &\quad \times [1 - P \{ \mathbf{N}(\mathbf{X}_1) < \mathbf{N}(\mathbf{x}_2) \}]^{s-r} dF(x_1, y_1) dF(x_2, y_2) \\
 &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \\
 &\quad \times \int \int \int \int_{B_1 \times B_2} [h(x_1, y_1)]^{r-1} [h(x_2, y_2) - h(x_1, y_1)]^{s-r-1} \\
 &\quad \times [1 - h(x_2, y_2)]^{n-s} dF(x_1, y_1) dF(x_2, y_2). \\
 &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \\
 &\quad \times \int \int \int \int_{B_1 \times B_2} [h(x_1, y_1)]^{r-1} [h(x_2, y_2) - h(x_1, y_1)]^{s-r-1} \\
 &\quad \times [1 - h(x_2, y_2)]^{n-s} dF(x_1, y_1) dF(x_2, y_2).
 \end{aligned}$$

Hence the proof. \square

The following Theorem is an obvious extension of Theorem 2.

Theorem 3. The joint p.d.f. of random vectors $\mathbf{X}^{(r_1)}, \mathbf{X}^{(r_2)}, \dots, \mathbf{X}^{(r_k)}, 1 \leq r_1 < r_2 < \dots < r_k \leq n$ is

$$\begin{aligned}
 & f_{r_1, r_2, \dots, r_k}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) \\
 &= \frac{n!}{(r_1 - 1)!(r_2 - r_1 - 1)! \dots (n - r_k)!} [h(\mathbf{x}_1)]^{r_1 - 1} \\
 &\quad \times [h(\mathbf{x}_2) - h(\mathbf{x}_1)]^{r_2 - r_1 - 1} \dots [h(\mathbf{x}_k) - h(\mathbf{x}_{k-1})]^{r_k - r_{k-1} - 1} [1 - h(\mathbf{x}_k)]^{n - r_k} \\
 &\quad \times f(\mathbf{x}_1) f(\mathbf{x}_2) \dots f(\mathbf{x}_k), \quad \text{if } \mathbf{N}(\mathbf{x}_1) \leq \mathbf{N}(\mathbf{x}_2) \leq \dots \leq \mathbf{N}(\mathbf{x}_k)
 \end{aligned} \tag{5}$$

and $f_{r_1, r_2, \dots, r_k}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) = 0$; otherwise, where $\mathbf{x}_i = (x_i^1, x_i^2, \dots, x_i^p)$, $i = 1, 2, \dots, n$.

Corollary 1. The joint p.d.f. of all conditionally \mathbf{N} -ordered statistics $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(n)}$ is

$$f_{1, 2, \dots, n}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \begin{cases} n! f(\mathbf{x}_1) f(\mathbf{x}_2) \dots f(\mathbf{x}_n) & \text{if } \mathbf{N}(\mathbf{x}_1) \leq \mathbf{N}(\mathbf{x}_2) \leq \dots \leq \mathbf{N}(\mathbf{x}_n), \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Let us consider some examples:

Example 2.1. Let X_1, X_2, \dots, X_n be i.i.d. r.v.'s and $\mathbf{X}_1 = (X_1^1, X_1^2, \dots, X_1^k), k \geq 1$, where $X_1^1, X_1^2, \dots, X_1^k$ are i.i.d. normally distributed random variables with $EX_1^1 = 0, \text{var}(X_1^1) = 1$. The probability density function of \mathbf{X}_1 is

$$f(x_1, x_2, \dots, x_k) = \frac{1}{(2\pi)^{\frac{k}{2}}} \exp \left\{ -\frac{x_1^2 + x_2^2 + \dots + x_k^2}{2} \right\}.$$

Suppose for any $\mathbf{x} = (x_1, x_2, \dots, x_k) \in R^k$ the function $\mathbf{N}(\mathbf{x})$ is defined as $\mathbf{N}(\mathbf{x}) = x_1^2 + x_2^2 + \dots + x_k^2$. Then $(X_1^1)^2 + (X_1^2)^2 + \dots + (X_1^k)^2$ is distributed as a random variable with χ^2 distribution with k degrees of freedom

$$P \{ \mathbf{N}(\mathbf{X}_1) \leq x \} = G_{\frac{1}{2}, \frac{k}{2}}(x),$$

where $G_{\alpha, \beta}(x)$ denotes the c.d.f. of gamma distribution with the parameters (α, β) . Then

$$h(x_1, x_2, \dots, x_k) = G_{\frac{1}{2}, \frac{k}{2}}(x_1^2 + x_2^2 + \dots + x_k^2).$$

Example 2.2. Let $(X_1^1, X_1^2, \dots, X_1^k)$ have a probability density function

$$f(x_1, x_2, \dots, x_k) = \begin{cases} \lambda^k \exp \{-\lambda(x_1 + x_2 + \dots + x_k)\} & \text{if } x_1 \geq 0, x_2 \geq 0, \dots, x_k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose for any $\mathbf{x} = (x_1, x_2, \dots, x_k) \in R^k$ $\mathbf{N}(\mathbf{x}) = \sum_{i=1}^k |x_i|$. One can write for $\mathbf{X}_1 = (X_1^1, X_1^2, \dots, X_1^k)$

$$\begin{aligned} h(x_1, x_2, \dots, x_k) &= P \left\{ \left| X_1^1 \right| + \left| X_1^2 \right| + \dots + \left| X_1^k \right| \leq |x_1| + |x_2| + \dots + |x_k| \right\} \\ &= P \left\{ X_1^1 + X_1^2 + \dots + X_1^k \leq |x_1| + |x_2| + \dots + |x_k| \right\} \\ &= G_{k,\lambda}(|x_1| + |x_2| + \dots + |x_k|), \end{aligned}$$

since $X_1^1, X_1^2, \dots, X_1^k$ are i.i.d. r.v.'s with c.d.f. $F(u) = 1 - e^{-\lambda u}, u \geq 0$.

Example 2.3. Let $(X_1^1, X_1^2, \dots, X_1^k)$ have a probability density function

$$f(x_1, x_2, \dots, x_k) = \begin{cases} \lambda_1 \lambda_2 \dots \lambda_k \exp \\ \times \{-(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k)\} & \text{if } x_1 \geq 0, x_2 \geq 0, \dots, x_k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consider $\mathbf{N}(\mathbf{x}) = \max(x_1, x_2, \dots, x_k), \mathbf{x} = (x_1, x_2, \dots, x_k)$. Then

$$\begin{aligned} h(x_1, x_2, \dots, x_k) &= P \left\{ \max(X_1^1, X_1^2, \dots, X_1^k) \leq \max(x_1, x_2, \dots, x_k) \right\} \\ &= \prod_{i=1}^k [1 - \exp(-\lambda_i \max(x_1, x_2, \dots, x_k))]. \end{aligned}$$

In the next section we investigate the distributional properties of the conditionally ordered random vectors under a Progressive Type II censoring scheme. A motivation for this study is the need for progressive censoring in many applications, for example, in the model of reliability analysis when the system consists of n independent units each having p arbitrarily dependent elements and if the progressive censoring of the life lengths of the elements is required.

3. Progressive Type II censored conditionally ordered statistics

Let a system have n independent components (units) each consisting of p arbitrarily dependent elements. Consider $\mathbf{X}_i = (X_i^1, X_i^2, \dots, X_i^p), i = 1, 2, \dots, n$, where X_i^j denotes the life length of the j th element of the i th unit. Let $\mathbf{R} = (R_1, R_2, \dots, R_m)$, where $m + \sum_{i=1}^m R_i = n$. Assume that $\mathbf{X}_i, i = 1, 2, \dots, n$ are continuous random vectors with the p -variate c.d.f. $F(\mathbf{x})$ and p.d.f. $f(\mathbf{x}), \mathbf{x} = (x_1, x_2, \dots, x_p) \in R^p$. Underly Progressive Type II censoring scheme the n units are placed on the test at time zero. The life length of the i th item is $\mathbf{N}(\mathbf{X}_i), i = 1, 2, \dots, n$. Then, the first failure occurs at time $\mathbf{N}(\mathbf{X}^{(1)}) = \min \{\mathbf{N}(\mathbf{X}_1), \mathbf{N}(\mathbf{X}_2), \dots, \mathbf{N}(\mathbf{X}_n)\}$. Let us denote the vector of the life length corresponding to this time as $\mathbf{X}_R^{(1:m:n)}$. Immediately following the first failure, R_1 surviving units are removed from the test at random. The second failure in the system (or first failure among the remaining $n - R_1 - 1$ items) occurs at time $\mathbf{N}(\mathbf{X}_R^{(2:m:n)})$ and immediately R_2 items are randomly removed from the test. This process continues until, at the time of the m th observed failure, the remaining $R_m = n - R_1 - R_2 - \dots - R_{m-1} - m$ units are all removed from

the experiment. The times of failures are $\mathbf{N}(\mathbf{X}_R^{(1:m:n)})$, $\mathbf{N}(\mathbf{X}_R^{(2:m:n)})$, \dots , $\mathbf{N}(\mathbf{X}_R^{(m:m:n)})$ and the random variables $\mathbf{X}_R^{(1:m:n)}$, $\mathbf{X}_R^{(2:m:n)}$, \dots , $\mathbf{X}_R^{(m:m:n)}$ are called Multivariate Progressive Type II censored conditionally N -ordered statistics.

Theorem 4. Let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ be i.i.d. random vectors with c.d.f. $F(\mathbf{x})$ and p.d.f. $f(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \dots, x_p) \in R^p$. Then the joint p.d.f. of the Progressive Type II conditionally N -ordered statistics $\mathbf{X}_R^{(1:m:n)}$, $\mathbf{X}_R^{(2:m:n)}$, \dots , $\mathbf{X}_R^{(m:m:n)}$ is

$$f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) = \begin{cases} c \prod_{i=1}^m f(\mathbf{x}_i) [1 - h(\mathbf{x}_i)]^{R_i} & \text{if } \mathbf{N}(\mathbf{x}_1) \leq \mathbf{N}(\mathbf{x}_2) \leq \dots \leq \mathbf{N}(\mathbf{x}_m), \\ 0 & \text{otherwise,} \end{cases}$$

where, $c = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1)$ and $h(\mathbf{x}) = P\{\mathbf{N}(\mathbf{X}) \leq \mathbf{N}(\mathbf{x})\}$.

The proof is straightforward and follows from the definition of multivariate Progressive Type II censored conditionally N -ordered statistics.

The following lemma will be useful for further developments.

Lemma 1. For any $\mathbf{y} \in R^p$

$$\int_{\mathbf{N}(\mathbf{x}) \geq \mathbf{N}(\mathbf{y})} f(\mathbf{x}) [1 - h(\mathbf{x})]^k d\mathbf{x} = \frac{1}{k + 1} [1 - h(\mathbf{y})]^{k+1} \tag{7}$$

and

$$\int_{\mathbf{N}(\mathbf{x}) \leq \mathbf{N}(\mathbf{y})} f(\mathbf{x}) [1 - h(\mathbf{x})]^k d\mathbf{x} = \frac{1}{k + 1} \left\{ 1 - [1 - h(\mathbf{y})]^{k+1} \right\}. \tag{8}$$

Proof. Consider (7). The proof of (8) is similar. It is clear that

$$\begin{aligned} \int_{\mathbf{N}(\mathbf{x}) \geq \mathbf{N}(\mathbf{y})} f(\mathbf{x}) [1 - h(\mathbf{x})]^k d\mathbf{x} &= \frac{1}{k + 1} P\left\{ \mathbf{N}(\mathbf{X}^{(1:k+1)}) \geq \mathbf{N}(\mathbf{y}) \right\} \\ &= \frac{1}{k + 1} P\left\{ \mathbf{N}(\mathbf{X}_1) \geq \mathbf{N}(\mathbf{y}), \dots, \mathbf{N}(\mathbf{X}_{k+1}) \geq \mathbf{N}(\mathbf{y}) \right\} \\ &= \frac{1}{k + 1} [1 - h(\mathbf{y})]^{k+1}, \end{aligned}$$

where $(k + 1) f(\mathbf{x}) [1 - h(\mathbf{x})]^k$ is the p.d.f. of $\mathbf{X}^{(1:k+1)}$, i.e. the p.d.f. of the smallest of $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{k+1}$, in the sense of conditional ordering with respect to a function \mathbf{N} . \square

The following theorem shows that analogous to the usual Progressive Type II censored-order statistics the joint distribution of the first r ($1 \leq r \leq m \leq n$) conditionally N -ordered Progressive Type II censored statistics does not depend on $R_{r+1}, R_{r+2}, \dots, R_m$.

Theorem 5. *The joint p.d.f of $\mathbf{X}_R^{(1:m:n)}, \mathbf{X}_R^{(2:m:n)}, \dots, \mathbf{X}_R^{(r:m:n)}$, $(1 \leq r \leq m \leq n)$ is*

$$\begin{aligned}
 & f_{\mathbf{X}_R^{(1:m:n)}, \mathbf{X}_R^{(2:m:n)}, \dots, \mathbf{X}_R^{(r:m:n)}}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r) \\
 &= c_r \prod_{i=1}^r f(\mathbf{x}_i) \prod_{i=1}^{r-1} [1 - h(\mathbf{x}_i)]^{R_i} [1 - h(\mathbf{x}_r)]^{n - R_1 - R_2 - \dots - R_{r-1} - r}, \tag{9}
 \end{aligned}$$

where $c_r = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{r-1} - r + 1)$.

Proof. It is clear that

$$\begin{aligned}
 & f_{\mathbf{X}_R^{(1:m:n)}, \mathbf{X}_R^{(2:m:n)}, \dots, \mathbf{X}_R^{(r:m:n)}}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r) \\
 &= c f(\mathbf{x}_1) f(\mathbf{x}_2) \dots f(\mathbf{x}_r) [1 - h(\mathbf{x}_1)]^{R_1} \dots [1 - h(\mathbf{x}_r)]^{R_r} \\
 &\quad \times \int \dots \int_{\mathbf{N}(\mathbf{x}_r) \leq \mathbf{N}(\mathbf{x}_{r+1}) \leq \dots \leq \mathbf{N}(\mathbf{x}_m)} f(\mathbf{x}_{r+1}) \dots f(\mathbf{x}_m) [1 - h(\mathbf{x}_{r+1})]^{R_{r+1}} \times \dots \\
 &\quad \times [1 - h(\mathbf{x}_m)]^{R_m} d\mathbf{x}_{r+1} \dots d\mathbf{x}_m, \tag{10}
 \end{aligned}$$

where $c = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1)$. From Lemma 1, using (7), we have

$$\int_{\mathbf{N}(\mathbf{x}_{m-1}) \leq \mathbf{N}(\mathbf{x}_m)} f(\mathbf{x}_m) [1 - h(\mathbf{x}_m)]^{R_m} d\mathbf{x}_m = \frac{1}{R_m + 1} [1 - h(\mathbf{x}_{m-1})]^{R_m + 1};$$

then,

$$\begin{aligned}
 & \frac{1}{R_m + 1} \int_{\mathbf{N}(\mathbf{x}_{m-2}) \leq \mathbf{N}(\mathbf{x}_{m-1})} f(\mathbf{x}_{m-1}) [1 - h(\mathbf{x}_{m-1})]^{R_{m-1}} \\
 &\quad \times [1 - h(\mathbf{x}_{m-1})]^{R_m + 1} d\mathbf{x}_{m-1} \\
 &= \frac{1}{(R_m + 1)(R_m + R_{m-1} + 2)} [1 - h(\mathbf{x}_{m-2})]^{R_m + R_{m-1} + 2},
 \end{aligned}$$

continuing this procedure and using (7) again we obtain

$$\begin{aligned}
 & \int \dots \int_{\mathbf{N}(\mathbf{x}_r) \leq \mathbf{N}(\mathbf{x}_{r+1}) \leq \dots \leq \mathbf{N}(\mathbf{x}_m)} f(\mathbf{x}_{r+1}) \dots f(\mathbf{x}_m) [1 - h(\mathbf{x}_1)]^{R_{r+1}} \times \dots \\
 &\quad \times [1 - h(\mathbf{x}_m)]^{R_m} d\mathbf{x}_{r+1} \dots d\mathbf{x}_m \\
 &= \frac{1}{(R_m + 1)(R_m + R_{m-1} + 2) \dots (R_m + R_{m-1} + \dots + R_{r+1} + m - r)} \\
 &\quad \times [1 - h(\mathbf{x}_r)]^{R_m + R_{m-1} + \dots + R_{r+1} + m - r}. \tag{11}
 \end{aligned}$$

From (11) and (10) we have

$$\begin{aligned}
 & f_{\mathbf{X}_R^{(1:m:n)}, \mathbf{X}_R^{(2:m:n)}, \dots, \mathbf{X}_R^{(r:m:n)}}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r) \\
 &= \frac{n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1)}{(R_m + 1)(R_m + R_{m-1} + 2) \dots (R_m + R_{m-1} + \dots + R_{r+1} + m - r)} \\
 &\quad \times f(\mathbf{x}_1) f(\mathbf{x}_2) \dots f(\mathbf{x}_r) [1 - h(\mathbf{x}_1)]^{R_1} \dots [1 - h(\mathbf{x}_{r-1})]^{R_{r-1}} \\
 &\quad \times [1 - h(\mathbf{x}_r)]^{R_m + R_{m-1} + \dots + R_r + m - r}. \tag{12}
 \end{aligned}$$

Since $R_1 + R_2 + \dots + R_m + m = n$, it is not difficult to observe that (12) is the same as (9). \square

To derive the explicit expression of the p.d.f. of the r th multivariate Progressive Type II censored conditionally ordered statistic we need the following lemma which is a modification of Lemma 1 by Balakrishnan et al. [4].

Lemma 2. Let $f(\mathbf{x}), \mathbf{x} \in R^p$ be the p.d.f. of an absolutely continuous random vector $\mathbf{X} = (X_1^1, X_1^2, \dots, X_1^p)$, and $h(\mathbf{x}), \mathbf{x} \in R^p$ be the structural function with respect to a function $\mathbf{N}(\mathbf{x}), \mathbf{x} \in R^p$. Then for $r \geq 1$

$$\int \dots \int_{\mathbf{N}(\mathbf{x}_1) \leq \mathbf{N}(\mathbf{x}_2) \leq \dots \leq \mathbf{N}(\mathbf{x}_{r-1}) \leq \mathbf{N}(\mathbf{x}_r)} \prod_{i=1}^{r-1} f(\mathbf{x}_i) [1 - h(\mathbf{x}_i)]^{a_i-1} d\mathbf{x}_1 \dots d\mathbf{x}_{r-1} = \sum_{i=0}^{r-1} c_{i,r-1}(\mathbf{a}_{r-1}) \{1 - h(\mathbf{x}_r)\}^{b_{i,r-1}(\mathbf{a}_{r-1})},$$

where $\mathbf{a}_{r-1} = (a_1, a_2, \dots, a_{r-1})$,

$$c_{i,r-1}(\mathbf{a}_{r-1}) = \frac{(-1)^i}{\left\{ \prod_{j=1}^i \sum_{k=r-i}^{r-1-i+j} a_k \right\} \left\{ \prod_{j=1}^{r-1-i} \sum_{k=j}^{r-1-i} a_k \right\}}, \quad b_{i,r-1}(\mathbf{a}_{r-1}) = \sum_{i=r-i}^{r-1} a_i$$

and by convention $\prod_{j=1}^0 d_j \equiv 1$ and $\sum_{j=i}^{i-1} d_j \equiv 0$.

The proof is a modification of the proof of Lemma 1 in Balakrishnan et al. [4] and we omit it.

The result given in the following lemma is obtained from straightforward integration by repeated application of Lemma 1.

Lemma 3. Let $f(\mathbf{x}), \mathbf{x} \in R^p$ be the p.d.f. of an absolutely continuous random vector $\mathbf{X} = (X_1^1, X_1^2, \dots, X_1^p)$, and $h(\mathbf{x}), \mathbf{x} \in R^p$ be the structural function with respect to a function $\mathbf{N}(\mathbf{x}), \mathbf{x} \in R^p$. Then for $r \geq 1$

$$\int \dots \int_{\mathbf{N}(\mathbf{x}_m) \geq \mathbf{N}(\mathbf{x}_{m-1}) \geq \dots \geq \mathbf{N}(\mathbf{x}_{r+1}) \geq \mathbf{N}(\mathbf{x}_r)} \prod_{i=r+1}^m f(\mathbf{x}_i) [1 - h(\mathbf{x}_i)]^{R_i+1} d\mathbf{x}_{r+1} \dots d\mathbf{x}_m = \mathbf{Q}_{m,r} \{1 - h(\mathbf{x}_r)\}^{\sum_{i=m}^{r+1} R_i+m-r},$$

where

$$\mathbf{Q}_{m,r} \equiv \frac{1}{R_m + 1} \frac{1}{R_m + R_{m-1} + 2} \dots \frac{1}{R_m + R_{m-1} + \dots + R_{r+1} + m - r - 1} \times \frac{1}{R_m + R_{m-1} + \dots + R_{r+1} + m - r}.$$

Lemmas 2 and 3 allow us to write the explicit distribution of the r th conditionally ordered Progressive Type II order statistic.

Theorem 6. *The p.d.f of p -variate random vector $\mathbf{X}_R^{(r:m:n)}$ is*

$$\begin{aligned}
 f_r(\mathbf{x}_r) &= c \mathbf{Q}_{m,r} f(\mathbf{x}_r) \sum_{i=0}^{r-1} c_{i,r-1}(R_1 + 1, R_2 + 1, \dots, R_{r-1} + 1) \\
 &\quad \times \{1 - h(\mathbf{x}_r)\}^{b_{i,r-1}(R_1+1, R_2+1, \dots, R_{r-1}+1) + t_{m,r}}, \\
 \mathbf{x}_r &= (x_r^1, x_r^2, \dots, x_r^p) \in R^p,
 \end{aligned} \tag{13}$$

where

$$\begin{aligned}
 c &= n(n - R_1 - 1) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1), \\
 t_{m,r} &= R_m + R_{m-1} + \cdots + R_r + m - r, \\
 \mathbf{Q}_{m,r} &= \frac{1}{R_m + 1} \frac{1}{R_m + R_{m-1} + 2} \cdots \frac{1}{R_m + R_{m-1} + \cdots + R_{r+1} + m - r - 1} \\
 &\quad \times \frac{1}{R_m + R_{m-1} + \cdots + R_{r+1} + m - r}.
 \end{aligned}$$

Proof. Using Lemmas 1 and 2 integrating the joint p.d.f. of $\mathbf{X}_R^{(1:m:n)}, \mathbf{X}_R^{(2:m:n)}, \dots, \mathbf{X}_R^{(m:m:n)}$ with respect to $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{r-1}, \mathbf{x}_{r+1}, \dots, \mathbf{x}_m$ in

$$\{(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m) : \mathbf{N}(\mathbf{x}_1) \leq \mathbf{N}(\mathbf{x}_2) \leq \cdots \leq \mathbf{N}(\mathbf{x}_m)\}$$

we complete the proof.

Note. If $R_1 = R_2 = \cdots = R_m = 0$ then we have the usual conditionally N -ordered statistics. It would be interesting to obtain from (13) the p.d.f. of r th conditionally ordered statistics $\mathbf{X}^{(r:n)}$. For this let us write the expression for $c_{i,r-1}(R_1 + 1, \dots, R_{r-1} + 1)$ in the form

$$\begin{aligned}
 &c_{i,r-1}(R_1 + 1, \dots, R_{r-1} + 1) \\
 &= (-1)^i \frac{1}{a_{r-i}(a_{r-i} + a_{r-i+1}) \cdots (a_{r-i} + a_{r-i+1} + \cdots + a_{r-1})} \\
 &\quad \times \frac{1}{a_{r-1-i}(a_{r-1-i} + a_{r-2-i}) \cdots (a_{r-1-i} + a_{r-2-i} + \cdots + a_1)}.
 \end{aligned}$$

Then we have

$$\begin{aligned}
 c_{i,r-1}(1, 1, \dots, 1) &= \frac{(-1)^i}{i!(r - 1 - i)!}, \\
 b_{i,r-1}(1, 1, \dots, 1) &= i, \\
 c &= n!, \\
 \mathbf{Q}_{m,r} &= \frac{1}{(n - r)!}, \\
 t_{m,r} &= n - r
 \end{aligned}$$

and

$$\begin{aligned} f_r(\mathbf{x}_r) &= \frac{n!}{(n-r)!} f(\mathbf{x}_r) \sum_{i=0}^{r-1} \frac{(-1)^i}{i!(r-1-i)!} \{1-h(\mathbf{x}_r)\}^{i+n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} (1-h(\mathbf{x}_r))^{n-r} h^{r-1}(\mathbf{x}_r) f(\mathbf{x}_r), \end{aligned}$$

which coincides with (2). \square

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