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Numbers of near bivariate record-concomitant observations

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ABSTRACT

Let $\overline{Z}_1 = (X_1, Y_1), \overline{Z}_2 = (X_2, Y_2), \ldots$ be independent and identically distributed random vectors with continuous distribution. Let $L(n)$ and $X(n)$ denote the nth record time and the nth record value obtained from the sequence of Xs. Let $Y(n)$ denote the concomitant of the nth record value, which relates to the sequence of Ys. We call $\overline{Z}_i$ a near bivariate nth record-concomitant observation if $\overline{Z}_i$ belongs to the open rectangle $(X(n) - a, X(n)) \times (Y(n) - b_1, Y(n) + b_2)$, where $a, b_1, b_2 > 0$ and $L(n) < i < L(n+1)$. Asymptotic properties of the numbers of near bivariate record-concomitant observations are discussed in the present work. New techniques for generating bivariate record-concomitants, the numbers of near record observations and the numbers of near bivariate record-concomitant observations are also proposed.

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1. Introduction

Assume in the following, $\overline{Z} = (X, Y), \overline{Z}_1 = (X_1, Y_1), \overline{Z}_2 = (X_2, Y_2), \ldots$ are independent and identically distributed random vectors with continuous distribution $F(x, y)$ and marginal distributions $H(x) = P\{X \le x\}$ and $G(y) = P\{Y \le y\}$. In the case of existence, the corresponding densities will be denoted as $f(x, y)$, $h(x)$ and $g(y)$, respectively. For the sequence of X-s, let us define the sequences of record values $X(n)$ and record times $L(n)$:

$$L(1) = 1,$$
$$L(n+1) = \min\{j : j > L(n), X_j > X_{L(n)}\},$$
$$X(n) = X_{L(n)} \quad (n \ge 1).$$

The sequence of X-records $X(n)$ induces the sequence of their concomitants $Y(n)$, i.e. $Y(n) = Y_i$ if $X_i = X(n)$. Let us also denote bivariate record-concomitant observations as $\overline{Z}(n) = (X(n), Y(n))$.

The univariate theory of records can be found among others in the books of [1,3,18]. Concomitants of records are studied in [15,2,1,3,20].

Let in the following,

$$P(a_1, a_2, b_1, b_2) = P((X, Y) \in (a_1, a_2) \times (b_1, b_2)),$$

where $(a_1, a_2) \times (b_1, b_2)$ is an open rectangle and $-\infty \le a_1 < a_2 \le \infty, -\infty \le b_1 < b_2 \le \infty$. We further call $\overline{Z}_j = (X_j, Y_j)$ a near nth bivariate record-concomitant observation if $L(n) < j < L(n+1)$ and $\overline{Z}_j = (X_j, Y_j)$ belongs to the rectangle

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$(X(n) - a, X(n)) \times (Y(n) - b_1, Y(n) + b_2)$, where $a, b_1, b_2 > 0$. We also define the number of near nth bivariate record-concomitant observations by

$$\xi_n = \xi_n(a, b_1, b_2) = \#\{j : L(n) < j < L(n+1), \overline{Z}_j \in (X(n) - a, X(n)) \times (Y(n) - b_1, Y(n) + b_2)\} \quad (n \geq 1).$$

In the present work we study the properties of ξ_n in the case of continuous F.

Our study is the first investigation on the numbers of near bivariate record-concomitant observations. Two close topics such as the numbers of near bivariate maxima and the numbers of near univariate records are discussed in [11,12,9,4], and in [5–8], respectively.

The rest of this paper is organized as follows. In Section 2, we study the limiting behavior of the concomitants of records $Y(n)$ $(n \to \infty)$. This is important, because knowing this behavior we are able to put proper conditions on F and derive limit results for ξ_n. Distributions of the numbers of near bivariate record-concomitant observations are given in Section 3. In Section 4, we obtain limit theorems for ξ_n. Some limit results for the sums of near bivariate record-concomitant observations are derived in Section 5. Illustrative examples are given in Section 6. Section 7 of our work contains new simulation techniques for record values and the numbers of near records and near bivariate record-concomitant observations.

In the following $\stackrel{d}{\to}, \stackrel{p}{\to}, \stackrel{a.s.}{\to}$ stand for convergence in distribution, convergence in probability, and almost sure convergence, respectively.

2. Asymptotic behavior of $Y(n)$

The distributions of the bivariate record-concomitant observations $\overline{Z}(n) = (X(n), Y(n))$ and the concomitants of record values $Y(n)$ are given for $n \geq 2$ (see, for example, [20]) by

$$F_n(x, y) = P(X(n) \leq x, Y(n) \leq y) = \frac{1}{(n-2)!} \int_{-\infty}^{x} \frac{F(x, y) - F(u, y)}{1 - H(u)} [-\log(1 - H(u))]^{n-2} \, dH(u), \tag{2.1}$$

$$G_n(y) = P(Y(n) \leq y) = \frac{1}{(n-2)!} \int_{\mathbb{R}} \frac{G(y) - F(u, y)}{1 - H(u)} [-\log(1 - H(u))]^{n-2} \, dH(u), \tag{2.2}$$

and for $n = 1$ by

$$F_1(x, y) = F(x, y), \qquad G_1(y) = G(y).$$

It is known that $X(n) \stackrel{a.s.}{\to} r_H$ $(n \to \infty)$, where, in the following, we denote by $r_H = \sup\{x \in \mathbb{R} : H(x) < 1\}$ and $l_H = \inf\{x \in \mathbb{R} : H(x) > 0\}$ the right and left extremities of H, respectively.

It is reasonable to expect that the limiting behavior of the concomitants of record values $Y(n)$ is influenced by the type of dependence between X and Y. The same remark is true for the limiting behavior of the concomitants of maxima. The latter is discussed in the recent paper of [4], where a method for studying such behavior is proposed. However, that method seems rather complicated. In our present work, we propose another method for investigating asymptotic properties of the concomitants of records. This method, which makes use of the limit in (2.3) below, is much simpler than the method proposed in [4]. This new method can also be applied for studying the limiting behavior of the concomitants of maxima.

Let us consider the following limit

$$\lim_{x \to r_H} \frac{G(y) - F(x, y)}{1 - H(x)} = \beta(y) \in [0, 1]. \tag{2.3}$$

Proposition 2.1 shows us that the function β in (2.3) is a limiting distribution for the concomitants of records.

Proposition 2.1. *Let the limit in (2.3) exist. Then*

$$G_n(y) \to \beta(y) \quad (n \to \infty). \tag{2.4}$$

Proof. Let us estimate the terms of $G_n(y) = F_n(x_0, y) + [G_n(y) - F_n(x_0, y)]$, where x_0 is some fixed positive number which will be defined more precisely later. Since $\frac{F(x_0,y)-F(u,y)}{1-H(u)} \leq 1$, and $[-\log(1 - H(u))]^{n-2}$ is an increasing in u function,

$$F_n(x_0, y) \leq Q_n(x_0) = \frac{[-\log(1 - H(x_0))]^{n-2}}{(n-2)!} \to 0 \quad (n \to \infty).$$

We have $G_n(y) - F_n(x_0, y) = I_1 + I_2$, where

$$I_1 = \frac{1}{(n-2)!} \int_{-\infty}^{x_0} \frac{G(y) - F(x_0, y)}{1 - H(u)} [-\log(1 - H(u))]^{n-2} \, dH(u)$$

and

$$I_2 = \frac{1}{(n-2)!} \int_{x_0}^{\infty} \frac{G(y) - F(u, y)}{1 - H(u)} [-\log(1 - H(u))]^{n-2} \, dH(u).$$

Observe that $I_1 = (G(y) - F(x_0, y))Q_{n+1}(x_0) \to 0$. Let us now choose x_0 such that

$$\frac{G(y) - F(u, y)}{1 - H(u)} < \beta(y) + \varepsilon(y) \quad (u > x_0).$$

Then $I_2 < (\beta(y) + \varepsilon(y))(1 - Q_n(x_0))$. Hence $G_n(y) < o(1) + \beta(y) + \varepsilon(y)$.

In the same way $G_n(y)$ can be estimated from below. The result readily follows. □

The function β defined by (2.3) and (2.4) is a limiting distribution for continuous G_n (even if in (2.3) the function β is obtained as left-continuous, it can be redefined as right-continuous). The distribution β can be continuous and can be degenerate. If the limit in (2.3) does not exist, the limiting distribution for the sequence of G_n does not exist either. When the limit in (2.3) exists we distinguish the following two cases.

(i) Suppose that for some $c \in [-\infty, \infty]$,

$$\beta(y) = 0 \quad (y < c) \quad \text{and} \quad \beta(y) = 1 \quad (y > c). \tag{2.5}$$

This means that $Y(n) \xrightarrow{p} c$. We then call F a c-stable record-concomitant distribution. Examples 6.1–6.5 illustrate this case.

(ii) If such c does not exist, then $Y(n)$ does not converge in probability. In this case, we call F an unstable record-concomitant distribution. For example, if $F(x, y) = H(x)G(y)$, i.e. X and Y are independent, then $G_n(y) = G(y)$ $(n \geq 1)$. Obviously, such c does not exist here. See Example 6.6 in this respect.

If X and Y are independent, or 'weakly' dependent, then the limiting distribution β is not degenerate, and F is an unstable record-concomitant distribution. If X and Y are 'strongly' dependent, i.e. the limiting distribution β is degenerate with atom at c, then F is a c-stable record-concomitant distribution.

We have just introduced a new concept — the stable record-concomitant distribution. In our work, we analyze the limit behavior of ξ_n in these terms.

3. Distributional results for the number of near bivariate record-concomitant observations

We start discussing the properties of the number of near bivariate record-concomitant observations by presenting the probability mass function of ξ_n. The proof of Theorem 3.1 is based on Nevzorov's [19] deletion argument.

Theorem 3.1. *The probability mass function of* $\xi_n(a, b_1, b_2)$ $(a, b_1, b_2 > 0,\ n \geq 1)$ *is given by*

$$P(\xi_n(a, b_1, b_2) = k) = \int_{\mathbb{R}^2} [1 - \gamma(x, y, a, b_1, b_2)]\gamma(x, y, a, b_1, b_2)^k\, F_n(\mathrm{d}x, \mathrm{d}y) \quad (k \geq 0), \tag{3.1}$$

where $F_n(x, y)$ *is the distribution function of* $\overline{Z}(n)$ *determined by* (2.1), *and* $\gamma(x, y, a, b_1, b_2) \in [0, 1]$ *is defined by*

$$\gamma(x, y, a, b_1, b_2) = \frac{P(x - a, x, y - b_1, y + b_2)}{P(x - a, x, y - b_1, y + b_2) + P(x, \infty, -\infty, \infty)}.$$

Proof. First, we have

$$P(\xi_n(a, b_1, b_2) = k) = \int_{\mathbb{R}^2} P(\xi_n(a, b_1, b_2) = k \mid X(n) = x, Y(n) = y)\, F_n(\mathrm{d}x, \mathrm{d}y).$$

Now, in order to estimate the conditional probability

$$P(\xi_n(a, b_1, b_2) = k \mid X(n) = x, Y(n) = y), \tag{3.2}$$

we employ Nevzorov's [19] deletion argument proposed for the univariate case. For more accessible reference on this method one can refer to [25,5,24].

From the sequence of independent $\overline{Z}_i$s $(i > L(n))$ we delete all those $\overline{Z}_i$s which do not belong to the union of the rectangles

$$(x - a, x) \times (y - b_1, y + b_2) \bigcup (x, \infty) \times (-\infty, \infty).$$

The remaining sequence of independent $\overline{Z}_{i_1}, \overline{Z}_{i_2}, \ldots$ is such that $L(n) < i_1 < i_2 < \cdots$. This deletion procedure does not alter the number of near bivariate record-concomitant observations in the open rectangle $(x - a, x) \times (y - b_1, y + b_2)$. Let $\widetilde{Z}_j = (\widetilde{X}_j, \widetilde{Y}_j)$ $(j \geq 1)$ denote random vectors which are conditionally independent given $\{X(n) = x, Y(n) = y\}$. For

$$(\widetilde{x}, \widetilde{y}) \in (x - a, x) \times (y - b_1, y + b_2) \bigcup (x, \infty) \times (-\infty, \infty),$$

let

$$P(\widetilde{X}_j \leq \widetilde{x}, \widetilde{Y}_j \leq \widetilde{y}) = P\left(X_{i_j} \leq \widetilde{x}, Y_{i_j} \leq \widetilde{y} \mid \overline{Z}_{i_j} \in (x - a, x) \times (y - b_1, y + b_2) \bigcup (x, \infty) \times (-\infty, \infty)\right).$$

Then,

$$P(\xi_n(a,b_1,b_2)=k \mid X(n)=x, Y(n)=y) = P(\widetilde{Z}_1 \in (x-a,x)\times(y-b_1,y+b_2),\ldots, \widetilde{Z}_{k-1} \in (x-a,x)\times(y-b_1,y+b_2), \widetilde{Z}_k \in (x,\infty)\times(-\infty,\infty)),$$

which readily yields (3.1). □

Remark 3.1. It should be mentioned that the analysis of the conditional probability in (3.2) could be done differently by means of the method proposed in the proof of Theorem 2.1 of [21].

4. Limit results

For the rest of the paper we assume that $r_H = \infty$. The following limit

$$\lim_{x\to\infty} \frac{1-H(x+a)}{1-H(x)} = \tau(a) \in [0,1], \tag{4.1}$$

proposed in [23], is used for distribution tail classification in the univariate theories of near-maxima and near-records. Let the limit in (4.1) exist. The distribution tail $1-H(x)$ is classified as 'thin' if $\tau(a)=0$, 'medium' if $0<\tau(a)<1$ and 'thick' if $\tau(a)=1$. Based on this classification different limit laws are obtained for the numbers of univariate near-maxima and near-records. In particular, it is shown in [5] that if the tail $1-H(x)$ is 'medium', then the limiting distribution of the number of near-records is geometric.

In the bivariate case a similar result holds true.

Theorem 4.1. *Let F be a c-stable record-concomitant distribution, where $-\infty < l_G \le c \le r_G < \infty$, and suppose that the limit in (4.1) exists with $\tau(a) \in (0,1)$. Then*

$$\xi_n(a,b_1,b_2) \xrightarrow{d} Geo(\tau(a)),$$

where $Geo(p)$ is a geometrically distributed random variable with parameter p.

Proof. It follows from (2.5) that

$$P(\xi_n(a,b_1,b_2)=k) = \int_{-\infty}^{\infty}\int_{c-b_2}^{c+b_1} (1-\gamma(x,y,a,b_1,b_2))\gamma(x,y,a,b_1,b_2)^k\, F_n(\mathrm{d}x,\mathrm{d}y) + o(1).$$

It should be noted that for any $y \in (c-b_2, c+b_1)$,

$$\gamma(x,y,a,b_1,b_2) \to 1-\tau(a) \quad (x\to\infty).$$

The result readily follows. □

Comment 4.1. *Observe that the limiting distribution of $\xi_n(a,b_1,b_2)$ is free of b_1 and b_2. Indeed, when F is a c-stable record-concomitant distribution and x is large, the probability mass is concentrated near the line $y=c$. If n is large enough, $Y(n)$ is close to c with a probability arbitrarily near unity. The height of the rectangle $(X(n)-a, X(n))\times(c-b_1, c+b_2)$ is then unimportant for counting near bivariate record-concomitant observations registered in this rectangle, because these observations are located near the line $y=c$.*

Using the argument proposed in the proof of Theorem 4.1, one can obtain the following.

Remark 4.1. Let F be a c-stable record-concomitant distribution with $-\infty < l_G \le c \le r_G < \infty$. Let the limit in (4.1) exist. If $\tau(a)=0$, then $\xi_n(a,b_1,b_2) \xrightarrow{p} \infty$; and if $\tau(a)=1$, then $\xi_n(a,b_1,b_2) \xrightarrow{p} 0$.

The following asymptotic result is valid for $\pm\infty$-stable record-concomitant distributions.

Theorem 4.2. *Let F be a $\pm\infty$-stable record-concomitant distribution. Then*

$$\xi_n(a,b_1,b_2) \xrightarrow{p} 0. \tag{4.2}$$

Proof. We present the proof only for the case when F is an ∞-stable record-concomitant distribution. It follows from the definition of the ∞-stable record-concomitant distribution that for any y_0

$$P(\xi_n(a,b_1,b_2)=0) = \int_{-\infty}^{\infty}\int_{y_0}^{\infty} \frac{P(x,\infty,-\infty,\infty)}{P(x-a,x,y-b_1,y+b_2)+P(x,\infty,-\infty,\infty)} F_n(\mathrm{d}x,\mathrm{d}y) + o(1).$$

When y_0 increases to infinity, $P(x-a,x,y-b_1,y+b_2)$ tends to zero and, correspondingly, $P(\xi_n(a,b_1,b_2)=0)$ tends to one. The result follows. □

In Theorems 4.3 and 4.4, conditions for the strong convergence of ξ_n are found.

Theorem 4.3. *Let*

$$\int_{\mathbb{R}^2} \frac{P(x-a, x, y-b_1, y+b_2)}{(1-H(x))^2} F(\mathrm{d}x, \mathrm{d}y) < \infty. \tag{4.3}$$

Then $\xi_n(a, b_1, b_2) \overset{a.s.}{\to} 0$.

Proof. Indeed,

$$\begin{aligned} \sum_{n=1}^{\infty} P\{\xi_n(a, b_1, b_2) > 0\} &= \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} \gamma(x, y, a, b_1, b_2) F_n(\mathrm{d}x, \mathrm{d}y) \\ &= \int_{\mathbb{R}^2} \frac{P(x-a, x, y-b_1, y+b_2)}{(1-H(x))(1-H(x)+P(x-a, x, y-b_1, y+b_2))} F(\mathrm{d}x, \mathrm{d}y) \\ &< \int_{\mathbb{R}^2} \frac{P(x-a, x, y-b_1, y+b_2)}{(1-H(x))^2} F(\mathrm{d}x, \mathrm{d}y). \end{aligned}$$

The result follows from the Borel–Cantelli lemma. □

Remark 4.2. Let $\widetilde{\xi}_n(a)$ be the number of univariate near-records, defined by

$$\widetilde{\xi}_n(a) = \#\{j : L(n) < j < L(n+1), X_j \in (X(n)-a, X(n))\} \quad (n \geq 1).$$

It should be noted that $\widetilde{\xi}_n(a) = \xi_n(a, \infty, \infty)$. The following limit result is obtained for $\widetilde{\xi}_n(a)$ in [5]. If

$$\int_{\mathbb{R}} \frac{H(x+a)-H(x)}{(1-H(x))^2} \mathrm{d}H(x) < \infty, \tag{4.4}$$

then $\widetilde{\xi}_n(a) \overset{a.s.}{\to} 0$. Observe that if condition (4.4) holds, then (4.3) also holds.

Theorem 4.4. *Let*

$$\int_{\mathbb{R}^2} \frac{F(\mathrm{d}x, \mathrm{d}y)}{1-H(x)+P(x-a, x, y-b_1, y+b_2)} < \infty. \tag{4.5}$$

Then $\xi_n(a, b_1, b_2) \overset{a.s.}{\to} \infty$.

Proof. Indeed, for any positive integer k

$$\begin{aligned} \sum_{n=1}^{\infty} P\{\xi_n(a, b_1, b_2) \leq k\} &= \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} [1-\gamma(x, y, a, b_1, b_2)^k] F_n(\mathrm{d}x, \mathrm{d}y) \\ &\leq k \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} (1-\gamma(x, y, a, b_1, b_2)) F_n(\mathrm{d}x, \mathrm{d}y) \\ &< k \int_{\mathbb{R}^2} \frac{F(\mathrm{d}x, \mathrm{d}y)}{1-H(x)+P(x-a, x, y-b_1, y+b_2)}. \end{aligned}$$

The result follows from the Borel–Cantelli lemma. □

5. Sums of near bivariate record-concomitant observations

Assume in this section that X_i, Y_i $(i \geq 1)$ can take only positive values, and as was supposed before, $r_H = \infty$. Let

$$\overline{S}_n(a, b_1, b_2) = (S_n^X(a, b_1, b_2), S_n^Y(a, b_1, b_2))$$

be the vector of sums of near nth bivariate record-concomitant observations, i.e.

$$S_n^X(a, b_1, b_2) = \sum_{i=L(n)+1}^{L(n+1)-1} I_{i,n} X_i \quad \text{and} \quad S_n^Y(a, b_1, b_2) = \sum_{i=L(n)+1}^{L(n+1)-1} I_{i,n} Y_i$$

where $I_{i,n}$ is the indicator function of the event $(X_i, Y_i) \in (X(n)-a, X(n)) \times (Y(n)-b_1, Y(n)+b_2)$.

These two sums, when X_i, Y_i are positive, are quantities of interest, since they can be interpreted as the sums of insurance claims falling in the random intervals $(X(n)-a, X(n))$ and $(Y(n)-b_1, Y(n)-b_2)$. These claims are to be registered only on receiving an outstanding X-claim. The registration of these claims is stopped at the next record time $L(n+1)$. When the first sum consists of the claims registered near the record values, the second sum is the sum of near induced record values.

Such a situation may happen when clients appeal to the insurance company to cover the damages obtained from serious car incidents (claims in respect to the X-records). At the same time the clients wish the insurance company to cover their medical expenses (claims in respect to the Y-concomitants).

Pakes [22], Li and Pakes [16], Hashorva [10], Hashorva and Hüsler [13,14], and Balakrishnan et al. [5] discussed sums of near maxima and near records as well as applications to insurance in the univariate case.

The limiting behavior of $S_n^X(a, b_1, b_2)$ can be easily obtained from the results of the previous section and the inequality

$$(X(n) - a)\xi_n(a, b_1, b_2) < S_n^X(a, b_1, b_2) \le X(n)\xi_n(a, b_1, b_2),$$

which holds on almost all sample paths for sufficiently large n. Since $X(n) \overset{a.s.}{\to} \infty$, we get

$$\frac{S_n^X(a, b_1, b_2)}{X(n)} \sim \xi_n(a, b_1, b_2) \quad \text{a.s.}.$$

Let us analyze the limit behavior of S_n^Y. Let F be an ∞-stable record-concomitant distribution. It follows from (2.2) and the Borel–Cantelli lemma that if for any y we have

$$\int_{\mathbb{R}} \frac{G(y) - F(x, y)}{(1 - H(x))^2} \mathrm{d}H(x) < \infty, \tag{5.1}$$

then $Y(n) \overset{a.s.}{\to} \infty$. Hence

$$\frac{S_n^Y(a, b_1, b_2)}{Y(n)} \sim \xi_n(a, b_1, b_2) \quad \text{a.s.}.$$

The following theorem is a simple consequence of Theorem 4.2.

Theorem 5.1. *Let F be an ∞-stable record-concomitant distribution and (5.1) hold. Then*

$$\left(\frac{S_n^X(a, b_1, b_2)}{X(n)}, \frac{S_n^Y(a, b_1, b_2)}{Y(n)}\right) \overset{p}{\to} (0, 0).$$

Suppose now F is a c-stable record-concomitant distribution, where c is finite. For any $\varepsilon > 0$, let

$$\int_{\mathbb{R}} \frac{G(c - \varepsilon) - F(x, c - \varepsilon)}{(1 - H(x))^2} \mathrm{d}H(x) < \infty \tag{5.2}$$

and

$$\int_{\mathbb{R}} \frac{1 - H(x) - G(c + \varepsilon) - F(x, c + \varepsilon)}{(1 - H(x))^2} \mathrm{d}H(x) < \infty. \tag{5.3}$$

It follows from the Borel–Cantelli lemma that $P(Y(n) < c - \varepsilon \ i.o.) = 0,\ P(Y(n) > c + \varepsilon \ i.o.) = 0$, and, consequently, $Y(n) \overset{a.s.}{\to} c$. The next theorem results from Remark 4.1.

Theorem 5.2. *Let (5.2) and (5.3) hold and $\tau(a) = 1$. Then*

$$\left(\frac{S_n^X(a, b_1, b_2)}{X(n)}, \frac{S_n^Y(a, b_1, b_2)}{Y(n)}\right) \overset{p}{\to} (0, 0).$$

6. Examples

Example 6.1. Let

$$F(x, y) = \int_0^x \int_0^y \frac{1}{u} \mathrm{e}^{-u - v/u} \mathrm{d}v \mathrm{d}u \quad (x > 0, y > 0)$$

with marginal distribution $H(x) = 1 - \mathrm{e}^{-x}$ and $F(\mathrm{d}x, y) = \mathrm{e}^{-x}(1 - \mathrm{e}^{-y/x})\mathrm{d}x$. It is clear that

$$\lim_{x \to \infty} \frac{F(\mathrm{d}x, y)}{h(x)\mathrm{d}x} = \lim_{x \to \infty} (1 - \mathrm{e}^{-y/x}) = 0 \quad (y \ge 0),$$

i.e. $\beta(y) = 0$. So, we have an ∞-stable record-concomitant distribution and

$$(X(n), Y(n)) \overset{p}{\to} (\infty, \infty). \tag{6.1}$$

One of the referees showed us how to obtain (6.1) differently. We are grateful to him/her for that. This different approach is not associated with the concept of stable record-concomitant distributions. We apply this approach here and in Examples 6.2, 6.3 and 6.5.

The random vector (X, Y) given in this example can be presented as

$$(X, Y) \stackrel{d}{=} (\epsilon, \epsilon'\epsilon),$$

where ϵ, ϵ' are independent standard exponential random variables. Since $X(n) \stackrel{d}{=} \epsilon(n)$, where the $\epsilon(n)$s are the records in the sequence of $\epsilon_1, \epsilon_2, \ldots$, it follows that

$$(X(n), Y(n)) \stackrel{d}{=} (\epsilon(n), \epsilon'_n\epsilon(n)),$$

where the ϵ'_ns are standard exponential, mutually independent, and independent of the $\epsilon(n)$ variables. The convergence in (6.1) follows since $\epsilon(n) \stackrel{a.s.}{\to} \infty$.

By Theorem 4.2, for any fixed $a, b_1, b_2 > 0$

$$\xi_n(a, b_1, b_2) \stackrel{p}{\to} 0 \quad (n \to \infty).$$

However, condition (4.3) does not hold and Theorem 4.3 can not be applied here. Indeed, for large positive x_0, y_0 the integral

$$\int_{x_0}^{\infty}\int_{y_0}^{\infty} \frac{P(x-a, x, y-b_1, y+b_2)}{(1-H(x))^2} F(\mathrm{d}x, \mathrm{d}y)$$

behaves like the divergent integral

$$\int_{x_0}^{\infty}\int_{y_0}^{\infty} \frac{\mathrm{e}^{-\frac{2y}{x}}}{x^2} \mathrm{d}x\mathrm{d}y.$$

Example 6.2. Let

$$F(x, y) = 1 - \mathrm{e}^{-x} - \frac{1 - \mathrm{e}^{-x(y+1)}}{y+1} \quad (x > 0, y > 0)$$

with marginal distributions $H(x) = 1 - \mathrm{e}^{-x}$ $(x > 0)$ and $G(y) = 1 - \frac{1}{y+1}$ $(y > 0)$. In this case, for any $y > 0$

$$\beta(y) = \lim_{x\to\infty}\left(1 - \frac{\mathrm{e}^{-xy}}{y+1}\right) = 1.$$

We have a 0-stable record-concomitant distribution and

$$(X(n), Y(n)) \stackrel{p}{\to} (\infty, 0). \tag{6.2}$$

The convergence in (6.2) can be obtained as proposed by the referee.

As in Example 6.1, we have $(X, Y) \stackrel{d}{=} (\epsilon, \epsilon'/\epsilon)$. The convergence in (6.2) follows since $(\epsilon(n), Y(n)) \stackrel{d}{=} (\epsilon(n), \epsilon'_n/\epsilon(n))$ and $\epsilon(n) \stackrel{a.s.}{\to} \infty$.

Observe that $\tau(a) = \mathrm{e}^{-a}$. By Theorem 4.1, for any fixed $a, b_2 > 0$

$$\xi_n(a, 0, b_2) \stackrel{d}{\to} Geo(\mathrm{e}^{-a}).$$

Example 6.3. Let

$$F(x, y) = 1 - \mathrm{e}^{-(y-1)} - \frac{1 - \mathrm{e}^{-(y-1)(x+1)}}{x+1} \quad (x > 0,\ y > 1)$$

with marginal distributions

$$H(x) = 1 - \frac{1}{x+1} \quad (x > 0) \quad \text{and} \quad G(y) = 1 - \mathrm{e}^{-(y-1)} \quad (y > 1).$$

In this case, for $y > 1$

$$\beta(y) = \lim_{x\to\infty}(1 - \mathrm{e}^{-(y-1)(x+1)}) = 1. \tag{6.3}$$

We have 1-stable record-concomitant distribution and $(X(n), Y(n)) \stackrel{p}{\to} (\infty, 1)$.

The referee observes that (6.3) also follows from the [17] characterization of the gamma law $(X, Y) \stackrel{d}{=} (\widetilde{X}, 1 + \frac{\epsilon+\epsilon'}{1+\widetilde{X}})$, where independent ϵ, ϵ' are introduced as above and $\widetilde{X}$ has the distribution H. It follows that

$$(X(n), Y(n)) \stackrel{d}{=} \left(\widetilde{X}(n), 1 + \frac{\epsilon + \epsilon'}{1 + \widetilde{X}(n)}\right) \xrightarrow{p} (\infty, 1).$$

It should be noted that $\tau(a) = 1$. By Remark 4.1, $\xi_n(a, 0, b_2) \xrightarrow{p} 0$ for any fixed $a, b_2 > 0$. Condition (4.4) holds true, and by Remark 4.2, $\xi_n(a, 0, b_2) \xrightarrow{a.s.} 0$.

Now, we would like to show that $Y(n) \xrightarrow{a.s.} 1$. Here, it is enough to show that (5.3) holds. We have

$$\int_{\mathbb{R}} \frac{1 - H(x) - G(c + \varepsilon) - F(x, c + \varepsilon)}{(1 - H(x))^2} \mathrm{d}H(x) = \int_0^{\infty} \frac{\mathrm{e}^{-\varepsilon(x+1)}}{x + 1} \mathrm{d}x < \infty.$$

Then $Y(n)$ converges to one with probability one and, by Theorem 5.2,

$$\left(\frac{S_n^X(a, 0, b_2)}{X(n)}, S_n^Y(a, 0, b_2)\right) \xrightarrow{p} (0, 0).$$

Example 6.4. Let F be a bivariate normal distribution with $\sigma_x = \sigma_y = 1,\ \mu_x = \mu_y = 0$, the joint density

$$f(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left[-\frac{1}{2(1 - \rho^2)}(x^2 + y^2 - 2\rho xy)\right],$$

and marginal standard normal distributions.

If $\rho = 0$, then X and Y are independent, and F is an unstable record-concomitant distribution.

Let $\rho \neq 0$. We have

$$F(\mathrm{d}x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{y} \exp\left[-\frac{1}{2(1 - \rho^2)}(x^2 + v^2 - 2\rho xv)\right] \mathrm{d}v\mathrm{d}x.$$

Then

$$\begin{aligned}\frac{G(y) - F(x, y)}{1 - H(x)} &\sim \frac{F(\mathrm{d}x, y)}{h(x)\mathrm{d}x} \\ &= \Phi\left(\frac{y - \rho x}{\sqrt{1 - \rho^2}}\right) \to \begin{cases} 0 & \text{if } \rho > 0, x \to \infty, \\ 1 & \text{if } \rho < 0, x \to \infty, \end{cases}\end{aligned}$$

where Φ is the standard normal distribution. Hence, if $\rho > 0$ the distribution F is an ∞-stable record-concomitant distribution and $(X(n), Y(n)) \xrightarrow{p} (\infty, \infty)$. By Theorem 4.2, $\xi_n(a, b_1, b_2) \xrightarrow{p} 0$. When $\rho < 0$, we have $(X(n), Y(n)) \xrightarrow{p} (\infty, -\infty)$. Obviously, in this case F is a $-\infty$-stable record-concomitant distribution, and by Theorem 4.2, $\xi_n(a, b_1, b_2) \xrightarrow{p} 0$.

We now show that when $\rho > 0$ the sequence $Y(n)$ tends with probability one to infinity. It is known that when H is a standard normal distribution and h is its density, then

$$1 - H(x) \sim h(x)/x \quad (x \to \infty).$$

The integrand in (5.1) can be estimated as

$$\frac{G(y) - F(x, y)}{(1 - H(x))^2} h(x) \sim \frac{xF(\mathrm{d}x, y)}{h(x)} = x\Phi\left(\frac{y - \rho x}{\sqrt{1 - \rho^2}}\right). \tag{6.4}$$

Observe that for any large positive k,

$$x^k \Phi(c_1 - c_2 x) \to 0 \quad (x \to \infty),$$

where c_1, c_2 are constants and c_2 is positive. Since the expression in (6.4) contains the estimate of the integrand in (5.1), it follows that the integrand goes to zero quicker then $1/x^k$, where k is a large positive number. This implies the convergence in (5.1). Hence $(X(n), Y(n)) \xrightarrow{a.s.} (\infty, \infty)$. It follows from Theorem 5.1 that

$$\left(\frac{S_n^X(a, b_1, b_2)}{X(n)}, \frac{S_n^Y(a, b_1, b_2)}{Y(n)}\right) \xrightarrow{p} (0, 0).$$

The same asymptotic result is valid when $\rho < 0$.

Example 6.5. Let

$$F(x,y) = 1 - e^{-x^2} - \frac{1 - e^{-x^2(y+1)}}{y+1} \quad (x > 0, y > 0)$$

with marginal distributions $H(x) = 1 - e^{-x^2}$ $(x > 0)$ and $G(y) = 1 - \frac{1}{y+1}$ $(y > 0)$. In this case,

$$\beta(y) = \lim_{x\to\infty} \left(1 - \frac{e^{-x^2 y}}{y+1}\right) = 1.$$

We have a 0-stable record-concomitant distribution and

$$(X(n), Y(n)) \xrightarrow{p} (\infty, 0). \tag{6.5}$$

We again apply the method proposed by the referee to obtain (6.5). If $\widetilde{X} \stackrel{d}{=} \sqrt{\epsilon}$, then $(X, Y) \stackrel{d}{=} (\widetilde{X}, \epsilon'/\widetilde{X}^2)$, where ϵ, ϵ' are the variables introduced above. Obviously, $(\widetilde{X}(n), \epsilon'_n/\widetilde{X}^2(n)) \xrightarrow{p} (\infty, 0)$.

We have $\tau(a) = 0$, i.e. $\xi_n(a, 0, b_2) \xrightarrow{p} \infty$. Furthermore, since (4.5) holds, we have $\xi_n(a, 0, b_2) \xrightarrow{a.s.} \infty$.

Example 6.6. Let

$$F(x,y) = xy[1 + \alpha(1-x)(1-y)] \quad (0 < x, y < 1,\ \alpha \in [0, 1])$$

with marginal distributions $H(x) = x$ $(0 < x < 1)$ and $G(y) = y$ $(0 < y < 1)$. In this case,

$$\beta(y) = \alpha y^2 + (1-\alpha)y \qquad (0 < y < 1).$$

We have an unstable record-concomitant distribution.

7. Simulation results

In this section, we present new simulation techniques for generating bivariate record-concomitant observations, numbers of near records and near bivariate record-concomitant observations. We also supplement Example 6.2 of the previous section with simulation results.

(i) First, let us consider the univariate case, where independent identically distributed $X_1, X_2, \ldots$ are taken with continuous H. The known method for generating the sequence of record values $X(1), X(2), \ldots$ makes use of Nevzorov's deletion argument (1986).

First, we generate $X(1) = X_1$ with $H(x)$. If $n \geq 2$ and $X(n-1) = x(n-1)$, then $X(n)$ is obtained as a single observation from the distribution function

$$\frac{H(x) - H(x(n-1))}{1 - H(x(n-1))} \quad (x > x(n-1)).$$

With little modification this known method can be applied for generating $\widetilde{\xi}_n(a)$ — the numbers of near records in the univariate case. On obtaining $X(n) = x(n)$, we generate $\widetilde{X}_1, \widetilde{X}_2, \ldots$ with distribution

$$\frac{H(x) - H(x(n) - a)}{1 - H(x(n) - a)} \quad (x > x(n) - a).$$

Then $\widetilde{\xi}_n(a) = t - 1$, where t is the least k such that $\widetilde{X}_k > x(n)$ $(k \geq 1)$.

(ii) In the bivariate case, as before $\overline{Z}_1 = (X_1, Y_1), \overline{Z}_2 = (X_2, Y_2), \ldots$ are independent and identically distributed random vectors with continuous distribution function $F(x, y)$ and marginal distribution functions $H(x) = P\{X \leq x\}$ and $G(y) = P\{Y \leq y\}$. For obtaining the bivariate record-concomitants $\overline{Z}(1), \overline{Z}(2), \ldots$, we first generate $\overline{Z}(1) = \overline{Z}_1$ with $F(x, y)$. When $n \geq 2$ we obtain $\overline{Z}(n)$ as a single observation from the distribution function

$$\frac{F(x,y) - F(x(n-1), y)}{1 - H(x(n-1))} \quad (x > x(n-1)).$$

For generating the numbers of near bivariate record-concomitants ξ_n, we make use of the arguments proposed above in this section and the proof of Theorem 3.1. On obtaining $\overline{Z}(n) = (X(n), Y(n)) = (x(n), y(n))$, we generate vectors $\widetilde{Z}_1 = (\widetilde{X}_1, \widetilde{Y}_1), \widetilde{Z}_2 = (\widetilde{X}_2, \widetilde{Y}_2), \ldots$ with distribution

$$\widetilde{F}(\widetilde{x}, \widetilde{y}) = P\left(\widetilde{X}_i \leq \widetilde{x}, \widetilde{Y}_i \leq \widetilde{y} \mid \widetilde{Z}_i \in (x(n) - a, x(n)) \times (y(n) - b_1, y(n) + b_2) \bigcup (x(n), \infty) \times (-\infty, \infty)\right),$$

where

$$(\widetilde{x}, \widetilde{y}) \in (x(n) - a, x(n)) \times (y(n) - b_1, y(n) + b_2) \bigcup (x(n), \infty) \times (-\infty, \infty).$$

The distribution $\widetilde{F}$ can be also written as

$$\widetilde{F}(\widetilde{x},\widetilde{y}) = \begin{cases} \dfrac{F(\widetilde{x},\widetilde{y}) - F(x(n),\widetilde{y}) + P(x(n)-a, x(n), y(n)-b_1, y(n)+b_2)}{1-H(x(n)) + P(x(n)-a, x(n), y(n)-b_1, y(n)+b_2)} & \text{if } \widetilde{x} > x(n),\ y \in \mathbb{R} \\ \dfrac{P(x(n)-a, \widetilde{x}, y(n)-b_1, \widetilde{y})}{1-H(x(n)) + P(x(n)-a, x(n), y(n)-b_1, y(n)+b_2)} & \text{if } x(n)-a \le \widetilde{x} \le x(n), \\ & y(n)-b_1 \le \widetilde{y} \le y(n)+b_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\xi_n(a, b_1, b_2) = t - 1$, where t is the least k such that $\widetilde{Z}_k \in (x(n), \infty) \times (-\infty, \infty)$ $(k \ge 1)$.

Let us apply the above generating techniques to Example 6.2. We put $a = 1$, $b_1 = 0$, $b_2 = 1$. On obtaining $(x(n), y(n))$, we generate the sequence of random numbers $U_1, U_2, \ldots$ while

$$U_i \le \frac{P(x(n)-1, x(n), y(n), y(n)+1)}{\mathrm{e}^{-x(n)} + P(x(n)-1, x(n), y(n), y(n)+1)}.$$

Then $\xi_n(a, b_1, b_2) = t - 1$, where t is the least i such that

$$U_i > \frac{P(x(n)-1, x(n), y(n), y(n)+1)}{\mathrm{e}^{-x(n)} + P(x(n)-1, x(n), y(n), y(n)+1)}.$$

Applying MATLAB and the above argument, we generated 500 sequences of 500 bivariate record-concomitants. On obtaining each sequence of record-concomitants we found 500 times $\xi_{500}(1, 0, 1)$. The average value of $(X(500), Y(500))$ was about the point (500, 0.0002). The average value of $\xi_{500}(1, 0, 1)$ was located in the interval (1.2, 2.5). This agrees with our theoretical results, where $(X(n), Y(n)) \xrightarrow{p} (\infty, 0)$ and $\xi_n(1, 0, 1) \xrightarrow{d} Geo(\mathrm{e}^{-1})$ with $EGeo(\mathrm{e}^{-1}) = 1.7183$.

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