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Inequalities and exponential decay in time varying delay differential equations

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ABSTRACT

We use Lyapunov functionals to obtain sufficient conditions that guarantee exponential decay of solutions to zero of the time varying delay differential equation

$$x'(t) = b(t)x(t) - a(t)x(t - h(t)).$$

The highlights of the paper are allowing $b(t)$ to change signs and the delay to vary with time. In addition, we obtain a criterion for the instability of the zero solution. Moreover, by comparison to existing literature we show effectiveness of our results.

1. Introduction

In this paper we consider the scalar linear differential equation with time varying delay

$$x'(t) = b(t)x(t) - a(t)x(t - h(t)) \tag{1.1}$$

where a, b and h are continuous with $0 < h(t) \leq r_0$ for positive constant r_0 and the function $t - h(t)$ is strictly increasing so that it has an inverse $r(t)$. We will use Lyapunov functionals and obtain some inequalities regarding the solutions of (1.1) from which we can deduce exponential asymptotic stability of the zero solution. Also, we will provide a criterion for the instability of the zero solution of (1.1) by means of a Lyapunov functional.

There are many results concerning equations similar to (1.1). This study is motivated by two recent papers [1,2] by Burton and Wang, respectively. In [1], Burton considered (1.1) when $b(t) = 0$ and when the delay is constant; $h(t) = h$ for all t, and used both Lyapunov functional and fixed point theory for the purpose of comparing both methods. Precisely, in Section 3 of [1], Burton displayed a Lyapunov type functional and showed that the zero solution of (1.1) is uniformly asymptotically stable. Our Lyapunov functional that we use here is a modified version of the one used in [1]. However, ours will lead to the derivative of the Lyapunov functionals V' along the solutions of (1.1) to satisfy $V' \leq -\beta V$, without requiring a sign condition on $b(t)$. Due to the choice of the Lyapunov functionals, we will deduce some inequalities on all solutions. As a consequence, the exponential decay of all solutions to zero is concluded. The main task in achieving this is to be able to relate the solutions back to V. That is, to find a lower bound on V in terms of x, where x is a solution of (1.1). For more on the stability of (1.1) when the delay is constant and the sign condition on $b(t)$ is required, we refer the reader to [3]. Also, for a general reading on stability, we refer the reader to [4–7], and [2]. For the stability results on impulsive delay differential equations one may consult with [8,9], and [10]. Our work generalizes some of the work in [2].

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Let $\psi : [-r_0, 0] \to (-\infty, \infty)$ be a given continuous initial function with

$$\|\psi\| = \max_{-r_0 \le s \le 0} |\psi(s)|.$$

It should cause no confusion to denote the norm of a continuous function $\varphi : [-r, \infty) \to (-\infty, \infty)$ with

$$\|\varphi\| = \sup_{-r \le s < \infty} |\varphi(s)|.$$

The notation x_t means that $x_t(\tau) = x(t + \tau),\ \tau \in [-r_0, 0]$ as long as $x(t + \tau)$ is defined. Thus, x_t is a function mapping an interval $[-r_0, 0]$ into $\mathbb{R}$. We say that $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \ge t_0$ and $x_{t_0}(s) = x(t_0 + s) = \psi(s),\ s \in [-r_0, 0]$.

In preparation for our main results, we rewrite (1.1) in the form

$$\begin{aligned} x'(t) &= \left(b(t) - \frac{a(r(t))}{1 - h'(r(t))}\right)x(t) + \frac{d}{dt}\int_{t-h(t)}^{t} \frac{a(r(s))}{1 - h'(r(s))}x(s)ds \\ &= \big(b(t) - c(t)\big)x(t) + \frac{d}{dt}\int_{t-h(t)}^{t} c(s)x(s)ds, \end{aligned} \tag{1.2}$$

where

$$c(t) = \frac{a(r(t))}{1 - h'(r(t))}.$$

2. Exponential decay

Now we turn our attention to the exponential decay of solutions of Eq. (1.1). For simplicity, we let $Q(t) := b(t) - c(t)$.

Lemma 1. *Assume for $\delta > 0$,*

$$-\frac{\delta}{\delta r_0 + h(t)} \le Q(t) \le -r_0\delta c^2(t), \tag{2.1}$$

hold. If

$$V(t) = \left(x(t) - \int_{t-h(t)}^{t} c(s)x(s)ds\right)^2 + \delta\int_{-r_0}^{0}\int_{t+s}^{t} c^2(z)x^2(z)dz\, ds, \tag{2.2}$$

then, along the solutions of (1.1) we have

$$V'(t) \le Q(t)V(t).$$

Proof. First we note that due to condition (2.1), $Q(t) < 0$ for all $t \ge 0$. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (2.2). Then along with the solutions of (1.1) we have

$$\begin{aligned} V'(t) &= 2\left(x(t) - \int_{t-h(t)}^{t} c(s)x(s)ds\right)Q(t)x(t) + r_0\delta c^2(t)x^2(t) - \delta\int_{-r_0}^{0} c^2(t+s)x^2(t+s)ds \\ &\le Q(t)\left[x^2(t) - 2x(t)\int_{t-h(t)}^{t} c(s)x(s)ds\right] + r_0\delta c^2(t)x^2(t) - \delta\int_{-r_0}^{0} c^2(t+s)x^2(t+s)ds + Q(t)x^2(t) \\ &= Q(t)V(t) - Q(t)\left(\int_{t-h(t)}^{t} c(s)x(s)ds\right)^2 - \delta Q(t)\int_{-r_0}^{0}\int_{t+s}^{t} c^2(z)x^2(z)dz\, ds \\ &\quad + \big(r_0\delta c^2(t) + Q(t)\big)x^2(t) - \delta\int_{-r_0}^{0} c^2(t+s)x^2(t+s)ds. \end{aligned} \tag{2.3}$$

In what follows we perform some calculations to simplify (2.3). First, if we let $u = t + s$, then

$$\int_{-r_0}^{0} c^2(t+s)x^2(t+s)ds = \int_{t-r_0}^{t} c^2(s)x^2(s)ds. \tag{2.4}$$

Also, with the aid of Hölder's inequality, we have

$$\left(\int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right)^2 \le h(t)\int_{t-h(t)}^{t} c^2(s)x^2(s)\mathrm{d}s \le h(t)\int_{t-r_0}^{t} c^2(s)x^2(s)\mathrm{d}s. \tag{2.5}$$

Finally, we easily observe

$$\int_{-r_0}^{0}\int_{t+s}^{t} c^2(z)x^2(z)\mathrm{d}z\ \mathrm{d}s \le r_0\int_{t-r_0}^{t} c^2(s)x^2(s)\mathrm{d}s. \tag{2.6}$$

By invoking (2.2) and substituting expressions (2.4)–(2.6) into (2.3), yield

$$V'(t) \le Q(t)V(t) + \big(r_0\delta c^2(t) + Q(t)\big)x^2(t) + [-(\delta r_0 + h(t))Q(t) - \delta]\int_{t-r_0}^{t} c^2(s)x^2(s)\mathrm{d}s \le Q(t)V(t). \tag{2.7}$$

This completes the proof. □

In the next theorem we will furnish two inequalities; one for $t \ge t_0 + \gamma r_0$ and the other for $t \in [t_0, t_0 + \gamma r_0]$, for $\gamma > 0$.

Theorem 1. *Assume the hypothesis of Lemma 1 and let* $1 < \alpha \le 2$. *If*

$$\left(\frac{\alpha-1}{\alpha}\right) r_0 \le h(t) \le r_0, \quad \text{for all } t \ge 0, \tag{2.8}$$

then any solution $x(t) = x(t, t_0, \psi)$ *of (1.1) satisfies the exponential inequalities*

$$|x(t)| \le \sqrt{2\frac{\frac{(\alpha-1)\delta}{\alpha}}{1+\frac{(\alpha-1)\delta}{\alpha}}V(t_0)\mathrm{e}^{\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}[b(s)-c(s)]\mathrm{d}s}} \tag{2.9}$$

for $t \ge t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_0$, *and*

$$|x(t)| \le \|\psi\|\mathrm{e}^{\int_{t_0}^{t} b(s)\mathrm{d}s}\left[1 + \int_{t_0}^{t} |a(u)|\mathrm{e}^{-\int_{t_0}^{t} b(s)\mathrm{d}s}\mathrm{d}u\right]. \tag{2.10}$$

for $t \in \left[t_0, t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_0\right]$.

Proof. By changing the order of integration we have

$$\int_{-r_0}^{0}\int_{t+s}^{t} c^2(z)x^2(z)\mathrm{d}z\ \mathrm{d}s = \int_{t-r_0}^{t}\int_{-r_0}^{z-t} c^2(z)x^2(z)\mathrm{d}s\ \mathrm{d}z = \int_{t-r_0}^{t} c^2(z)x^2(z)(z-t+r_0)\mathrm{d}z. \tag{2.11}$$

For $1 < \alpha \le 2$, and if $t - \frac{r_0}{\alpha} \le z \le t$, then $\left(\frac{\alpha-1}{\alpha}\right) r_0 \le z - t + r_0 \le r_0$. Expression (2.11) yields,

$$\begin{aligned}\int_{-r_0}^{0}\int_{t+s}^{t} c^2(z)x^2(z)\mathrm{d}z\ \mathrm{d}s &= \int_{t-r_0}^{t} c^2(z)x^2(z)(z-t+r_0)\mathrm{d}z \\ &= \int_{t-r_0}^{t-\frac{r_0}{\alpha}} c^2(z)x^2(z)(z-t+r_0)\mathrm{d}z + \int_{t-\frac{r_0}{\alpha}}^{t} c^2(z)x^2(z)(z-t+r_0)\mathrm{d}z \\ &\ge \int_{t-\frac{r_0}{\alpha}}^{t} c^2(z)x^2(z)(z-t+r_0)\mathrm{d}z \\ &\ge \left(\frac{\alpha-1}{\alpha}\right) r_0\int_{t-\frac{r_0}{\alpha}}^{t} c^2(z)x^2(z)\mathrm{d}z.\end{aligned} \tag{2.12}$$

Let $V(t)$ be given by (2.2). Then,

$$V(t) \geq \delta \int_{-r_0}^{0} \int_{t+s}^{t} c^2(z)x^2(z)\mathrm{d}z\ \mathrm{d}s \geq \delta \left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-\frac{r_0}{\alpha}}^{t} c^2(z)x^2(z)\mathrm{d}z.$$

This implies that, for $1 < \alpha \leq 2$, we have $-r_0 + \frac{r_0}{\alpha} \geq -\frac{r_0}{\alpha}$ and hence

$$\begin{aligned} V\left(t - \left(\frac{\alpha-1}{\alpha}\right) r_0\right) &\geq \delta \left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-r_0}^{t-r_0+\frac{r_0}{\alpha}} c^2(z)x^2(z)\mathrm{d}z \\ &\geq \delta \left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-r_0}^{t-\frac{r_0}{\alpha}} c^2(z)x^2(z)\mathrm{d}z. \end{aligned} \tag{2.13}$$

Note that since $V'(t) \leq 0$ we have for $t \geq t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_0$ that

$$0 \leq V(t) + V\left(t - \left(\frac{\alpha-1}{\alpha}\right) r_0\right) \leq 2V\left(t - \left(\frac{\alpha-1}{\alpha}\right) r_0\right).$$

We note that

$$\begin{aligned} r_0 \int_{t-r_0}^{t} c^2(z)x^2(z)\mathrm{d}z &\geq h(t) \int_{t-h(t)}^{t} c^2(z)x^2(z)\mathrm{d}z \\ &\geq \left(\int_{t-h(t)}^{t} c(z)x(z)\mathrm{d}z\right)^2. \end{aligned} \tag{2.14}$$

In summary, (2.13)–(2.14) imply that

$$\begin{aligned} V(t) + V\left(t - \left(\frac{\alpha-1}{\alpha}\right) r_0\right) &\geq \left(x(t) - \int_{t-ht}^{t} c(s)x(s)\mathrm{d}s\right)^2 \\ &\quad + \delta \int_{-r_0}^{0} \int_{t+s}^{t} c^2(z)x^2(z)\mathrm{d}z\ \mathrm{d}s + \delta \left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-r_0}^{t-\frac{r_0}{\alpha}} c^2(z)x^2(z)\mathrm{d}z \\ &\geq \left(x(t) - \int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right)^2 + \delta \left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-\frac{r_0}{\alpha}}^{t} c^2(z)x^2(z)\mathrm{d}z \\ &\quad + \delta \left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-r_0}^{t-\frac{r_0}{\alpha}} c^2(z)x^2(z)\mathrm{d}z \\ &= \left(x(t) - \int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right)^2 + \delta \left(\frac{\alpha-1}{\alpha}\right) r_0 \int_{t-r_0}^{t} c^2(z)x^2(z)\mathrm{d}z \\ &\geq \left(x(t) - \int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right)^2 + \delta \left(\frac{\alpha-1}{\alpha}\right) \left(\int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right)^2 \quad (\text{by } (2.14)) \\ &= \frac{\frac{(\alpha-1)\delta}{\alpha}}{1+\frac{(\alpha-1)\delta}{\alpha}} x^2(t) + \left[\frac{1}{\sqrt{1+\delta\left(\frac{\alpha-1}{\alpha}\right)}} x(t) - \sqrt{1+\delta\left(\frac{\alpha-1}{\alpha}\right)} \int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right]^2 \\ &\geq \frac{\frac{(\alpha-1)\delta}{\alpha}}{1+\frac{(\alpha-1)\delta}{\alpha}} x^2(t). \end{aligned} \tag{2.15}$$

Thus, (2.15) shows that

$$\begin{aligned} \frac{\frac{(\alpha-1)\delta}{\alpha}}{1+\frac{(\alpha-1)\delta}{\alpha}} x^2(t) &\leq V(t) + V\left(t - \left(\frac{\alpha-1}{\alpha}\right) r_0\right) \\ &\leq 2V\left(t - \left(\frac{\alpha-1}{\alpha}\right) r_0\right). \end{aligned} \tag{2.16}$$

An integration of (2.7) from t_0 to t yields the inequality

$$V(t) \le V(t_0)e^{\int_{t_0}^{t}[b(s)-c(s)]ds}.$$

As a consequence of (2.16),

$$V\left(t-\left(\frac{\alpha-1}{\alpha}\right)r_0\right) \le V(t_0)e^{\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}[b(s)-c(s)]ds},$$

and

$$|x(t)| \le \sqrt{2\frac{\frac{(\alpha-1)\delta}{\alpha}}{1+\frac{(\alpha-1)\delta}{\alpha}}V(t_0)e^{\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}[b(s)-c(s)]ds}}$$

for $t \ge t_0 + \left(\frac{\alpha-1}{\alpha}\right)r_0$. For $t \in \left[t_0, t_0 + \left(\frac{\alpha-1}{\alpha}\right)r_0\right]$, we observe from (2.8) that

$$\left(\frac{\alpha-1}{\alpha}\right)r_0 - h\left(t_0+\left(\frac{\alpha-1}{\alpha}\right)r_0\right) \le 0. \tag{2.17}$$

Thus for $t \in \left[t_0, t_0 + \left(\frac{\alpha-1}{\alpha}\right)r_0\right]$ and by (2.17), we have $x'(t) = b(t)x(t) - a(t)x(t-h(t)) = b(t)x(t) - a(t)\psi(t)$. Since $\psi(t)$ is the known initial function, we can easily solve for $x(t)$ using the variations of the parameters formula. That is

$$x(t) = e^{\int_{t_0}^{t} b(s)ds}\left[\psi(t_0) - \int_{t_0}^{t} a(u)\psi(u)e^{-\int_{t_0}^{t} b(s)ds}du\right].$$

Thus for $t \in \left[t_0, t_0 + \left(\frac{\alpha-1}{\alpha}\right)r_0\right]$, the above expression implies

$$|x(t)| \le \|\psi\|e^{\int_{t_0}^{t} b(s)ds}\left[1 + \int_{t_0}^{t} |a(u)|e^{-\int_{t_0}^{t} b(s)ds}du\right].$$

This completes the proof. □

Remark 1. Since the delay $h(t)$ is time varying, condition (2.17) is the price we paid to obtain two different inequalities on two different intervals. In the case $h(t) = r_0$, where r_0 is constant, then condition (2.17) is automatically satisfied.

Remark 2. It follows from (2.1) and inequality (2.9) that

$$\begin{aligned}|x(t)| &\le \sqrt{2\frac{\frac{(\alpha-1)\delta}{\alpha}}{1+\frac{(\alpha-1)\delta}{\alpha}}V(t_0)e^{\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}[b(s)-c(s)]ds}} \\ &\le \sqrt{2\frac{\frac{(\alpha-1)\delta}{\alpha}}{1+\frac{(\alpha-1)\delta}{\alpha}}V(t_0)e^{-r_0\delta\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_0}c^2(s)ds}}.\end{aligned}$$

Thus, if $\int_{t_0}^{\infty} c^2(s)\, ds = \infty$, then the zero solution of (1.1) is exponentially stable.

Now we are in a position to deduce some results regarding the totally time varying delay differential equation

$$x'(t) = -a(t)x(t-h(t)). \tag{2.18}$$

Theorem 2. *Assume for $\delta > 0$,*

$$-\frac{\delta}{\delta r_0 + h(t)} \le -c(t) \le -r_0\delta c^2(t), \tag{2.19}$$

and (2.17) hold. Then any solutions $x(t)$ of (2.18) satisfy inequalities (2.9) and (2.10) with $b(t) = 0$.

The proof is a direct consequence of Lemma 1 and Theorem 1, and hence we omit it.

In [2], Wang used a similar method and showed the constant delay equation

$$x'(t) = b(t)x(t) - a(t)x(t-r) \tag{2.20}$$

and derived similar inequalities to (2.9) and (2.10) provided that

$$-\frac{1}{2r} \le b(t) - a(t+r) \le -ra^2(t+r) \tag{2.21}$$

hold. According to equation (2.20), our condition (2.1) becomes

$$-\frac{\delta}{r(1+\delta)} \le b(t) - a(t) \le -r\delta a^2(t). \tag{2.22}$$

Thus for the sake of comparison, if we take $b(t) = -2 + \sqrt{11}$, $a(t) = \sqrt{11}$, $r = \frac{1}{5}$, $\delta = \frac{2}{3}$, then we easily see that our condition (2.22) is satisfied, while condition (2.21) of [2] is not. Next we compare our results to [1]. In [1], Burton used the Lyapunov functional

$$V(t) = \left(x(t) + \int_{t-h}^{t} a(s)x(s)\mathrm{d}s\right)^2 + \int_{-h}^{0}\int_{t+s}^{t} |a(z)|x^2(z)\mathrm{d}z\ \mathrm{d}s,$$

and showed that any solution of

$$x'(t) = -a(t)x(t-r) \tag{2.23}$$

tends to zero as $t \to \infty$. We summarize his results in the following theorem.

Theorem 3 ([1]). *If there is a $\gamma > 0$ with*

$$a(t+r) \ge \gamma, \quad \text{for all } t \ge 0,$$

and $\epsilon > 0$ with

$$a(t+r)\int_{t-r}^{t} a(s+r)\mathrm{d}s - 2 \le -\epsilon, \quad \text{for all } t \ge 0, \tag{2.24}$$

and if there is a $\eta > 0$ with

$$\eta[a(t) + a(t+r)] \le \frac{\epsilon}{2}a(t+r), \quad \text{for all } t \ge 0$$

then the zero solution of (2.23) is uniformly asymptotically stable.

According to (2.23), our condition (2.19) becomes

$$-\frac{\delta}{\delta(1+r)} \le -a(t) \le -r\delta a^2(t), \tag{2.25}$$

If we choose $r = \frac{32}{17}$, $a = \frac{1}{4}$, and $\delta = 1$, then condition (2.24) cannot be satisfied, while our condition (2.25) is satisfied. Moreover, from our results we deduced exponential stability for the zero solution.

The next theorem gives sufficient conditions for the exponential stability of the zero solution of (1.1).

Theorem 4. *Define a continuous function $\xi(t) \ge 0$ such that for some $\tau > 0$*

$$\xi(t) := \frac{\mathrm{e}^{\int_0^t b(s)\mathrm{d}s}}{1 + h_0\int_t^{t+\tau} \mathrm{e}^{\int_0^u b(s)\mathrm{d}s}\mathrm{d}u} \tag{2.26}$$

and

$$|a(t)| \le h_0\xi(t)(1 - h'(t)). \tag{2.27}$$

Then, every solution of (1.1) with $x(t_0) = \varphi$ satisfies the inequality

$$|x(t)| \le V(t_0, \varphi)\int_{t_0}^{t} \mathrm{e}^{\int_{t_0}^{t}[b(s)+h_0\xi(s)]\mathrm{d}s}, \tag{2.28}$$

where

$$\begin{aligned} V(t_0, \varphi) &= |\varphi| + h_0\xi(t)\int_{t_0-h(t_0)}^{t_0} |\varphi(s)|\mathrm{d}s \\ &\le |\varphi| + h_0\xi(t)\int_{t_0-h_0}^{t_0} |\varphi(s)|\mathrm{d}s. \end{aligned}$$

Proof. First we note that a simple calculation proves that $\xi'(t) \le b(t)\xi(t) + h_0\xi^2(t)$. For $t \ge 0$, let

$$V(t, x_t) = |x(t)| + h_0\xi(t)\int_{t-h(t)}^{t} |x(s)|ds.$$

Then along with solutions of (1.1) we have

$$\begin{aligned} V'(t, x_t) &= \frac{x(t)}{|x(t)|}(b(t)x(t) - a(t)x(t-h(t))) + h_0\xi(t)|x(t)| \\ &\quad - h_0\xi(t)(1-h'(t))|x(t-h(t))| + h_0\xi'(t)\int_{t-h(t)}^{t} |x(s)|\, ds \\ &\le (b(t) + h_0\xi(t))|x(t)| + (|a(t)| - h_0\xi(t)(1-h'(t)))|x(t-h(t))| \\ &\quad + h_0(b(t) + h_0\xi(t))\xi(t)\int_{t-h(t)}^{t} |x(s)|\, ds \\ &\le [b(t) + h_0\xi(t)]\left[|x(t)| + h_0\xi(t)\int_{t-h(t)}^{t} |x(s)|ds\right] \\ &= (b(t) + h_0\xi(t))V(t). \end{aligned} \tag{2.29}$$

An integration of (2.29) from t_0 to t and then applying the fact that $|x(t)| \le V(t)$ yields inequality (2.28). □

3. A criterion for instability

In this section, we use a non-negative definite Lyapunov functional and obtain a criterion that can be easily applied to test for instability of the zero solution of (1.1).

Theorem 5. *Suppose there exists a positive constant $D > r_0$ such that*

$$Q(t) - Dc^2(t) \ge 0. \tag{3.1}$$

If

$$V(t) = \left(x(t) - \int_{t-h(t)}^{t} c(s)x(s)ds\right)^2 - D\int_{t-r_0}^{t} c^2(z)x^2(z)dz, \tag{3.2}$$

then, along the solutions of (1.1) *we have*

$$V'(t) \ge Q(t)V(t).$$

Proof. First we observe that condition (3.1) implies that $Q(t) > 0$ for all $t \ge 0$. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (3.2). Then along solutions of (1.1) we have

$$\begin{aligned} V'_{(1.1)}(t) &= 2\left(x(t) - \int_{t-h(t)}^{t} c(s)x(s)ds\right)[Q(t)x(t)] - Dc^2(t)x^2(t) + Dc^2(t-r_0)x^2(t-r_0) \\ &\ge 2\left(x(t) - \int_{t-h(t)}^{t} c(s)x(s)ds\right)[Q(t)x(t)] - Dc^2(t)x^2(t) \\ &= Q(t)V(t) + Q(t)\left[-\left(\int_{t-h(t)}^{t} c(s)x(s)ds\right)^2 + D\int_{t-r_0}^{t} c^2(z)x^2(z)dz\right] + \big(Q(t) - Dc^2(t)\big)x^2(t) \\ &\ge Q(t)V(t) + \big(Q(t) - Dc^2(t)\big)x^2(t) + Q(t)\big(-r_0 + D\big)\int_{t-r_0}^{t} c^2(s)x^2(s)ds \\ &\ge Q(t)V(t), \end{aligned} \tag{3.3}$$

where we have used

$$\left(\int_{t-h(t)}^{t} c(s)x(s)ds\right)^2 \le h(t)\int_{t-h(t)}^{t} c^2(s)x^2(s)ds \le r_0\int_{t-r_0}^{t} c^2(s)x^2(s)ds.$$

This completes the proof. □

Theorem 6. *Suppose the hypothesis of Theorem 5 holds. Then the zero solution of (1.1) is unstable, provided that*

$$\int_{t_0}^{\infty} c^2(s)\,\mathrm{d}s = \infty.$$

Proof. An integration of (3.3) from t_0 to t yields

$$V(t) \geq V(t_0)\mathrm{e}^{\int_{t_0}^{t}\left(b(s)-c(s)\right)\,\mathrm{d}s}. \tag{3.4}$$

Let $V(t)$ be given by (3.2). Then

$$V(t) = x^2(t) - 2x(t)\int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s + \left[\int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right]^2 - D\int_{t-r_0}^{t} c^2(z)x^2(z)\mathrm{d}z. \tag{3.5}$$

Let $\beta = D - r_0$. Then from

$$\left(\frac{\sqrt{r_0}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{r_0}}b\right)^2 \geq 0,$$

we have

$$2ab \leq \frac{r_0}{\beta}a^2 + \frac{\beta}{r_0}b^2.$$

With this in mind we arrive at,

$$\begin{aligned} -2x(t)\int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s &\leq 2|x(t)|\left|\int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right| \\ &\leq \frac{r_0}{\beta}x^2(t) + \frac{\beta}{r_0}\left[\int_{t-h(t)}^{t} c(s)x(s)\mathrm{d}s\right]^2 \\ &\leq \frac{r_0}{\beta}x^2(t) + \frac{\beta}{r_0}r_0\int_{t-r_0}^{t} c^2(s)x^2(s)\mathrm{d}s. \end{aligned}$$

A substitution of the above inequality into (3.5) yields,

$$\begin{aligned} V(t) &\leq x^2(t) + \frac{r_0}{\beta}x^2(t) + (\beta + r_0 - D)\int_{t-r_0}^{t} c^2(s)x^2(s)\mathrm{d}s \\ &= \frac{\beta + r_0}{\beta}x^2(t) \\ &= \frac{D}{D - r_0}x^2(t). \end{aligned}$$

Using inequality (3.4), we get

$$\begin{aligned} |x(t)| &\geq \sqrt{\frac{D - r_0}{D}}V^{1/2}(t) \\ &= \sqrt{\frac{D - r_0}{D}}V^{1/2}(t_0)\mathrm{e}^{\frac{1}{2}\int_{t_0}^{t}\left(b(s)-c(s)\right)\,\mathrm{d}s} \\ &\geq \sqrt{\frac{D - r_0}{D}}V^{1/2}(t_0)\mathrm{e}^{\frac{D}{2}\int_{t_0}^{t} c^2(s)\,\mathrm{d}s}. \end{aligned}$$

This completes the proof. □

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