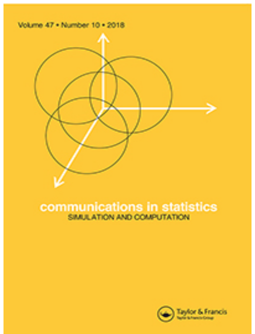

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Reliability Analysis

Consecutive k-Out-of-n : G System in Stress-Strength Setup

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A consecutive k-out-of-n: G system consists of n linearly ordered components functions if and only if at least k consecutive components function. In this article we investigate the consecutive k-out-of-n : G system in a setup of multicomponent stress-strength model. Under this setup, a system consists of n components functions if and only if there are at least k consecutive components survive a common random stress. We consider reliability and its estimation of such a system whenever there is a change and no change in strength. We provide minimum variance unbiased estimation of system reliability when the stress and strength distributions are exponential with unknown scale parameters. A nonparametric minimum variance unbiased estimator is also provided.

Keywords Consecutive k-out-of-n : G system; Longest run; Minimum variance unbiased estimator; Stress-strength reliability.

Mathematics Subject Classification 90B25; 62F10.

1. Introduction

A consecutive k-out-of-n : G system consists of n linearly ordered components functions if and only if at least k consecutive components function. We denote this system by $(C, k, n : G)$. Clearly, $(C, n, n : G)$ and $(C, 1, n : G)$ represent series and parallel systems, respectively. If ξ_i denotes the state of component i ($\xi_i = 1(0)$ if component i is operating (has failed)), then the structure function of the $(C, k, n : G)$ system is given by

$$\phi(\xi_1, \xi_2, \ldots, \xi_n) = 1 - \prod_{j=1}^{n-k+1} \left\{1 - \prod_{i=j}^{j+k-1} \xi_i\right\}.$$

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The reliability of $(C, k, n : G)$ system is defined by

$$R_{n,k} = P\{L_n \geq k\},$$

where L_n denotes the longest run of successes (1s) in $\xi_1, \xi_2, \ldots, \xi_n$. Exact expressions, bounds, and approximations for $R_{n,k}$ are well studied in the literature whenever the states of components are independent or dependent in a Markovian fashion. We refer the reader to Balakrishnan and Koutras (2002), Fu and Lou (2003), and Kuo and Zuo (2003). The recent articles of Fu et al. (2003), Boland and Samaniego (2004), Jalali et al. (2005), Yun et al. (2007), Eryılmaz (2007), and Navarro and Eryılmaz (2007) also contribute to the state of the art in consecutive k-out-of-n systems developments. In this work, we consider $(C, k, n : G)$ system in a setup of multicomponent stress-strength model which makes the states of components dependent. $(C, k, n : G)$ system in this setup may appear in various real life reliability applications. For example, a bridge consists of n cables survives if minimum k consecutive cables survive a common random stress (as a function of load on the bridge, corrosion, etc.) available in an environment. Qualification procedures for the systems whose components subject to a random stress may also involve the requirement of passing k consecutive tests.

Stress-strength model is of special importance in reliability analysis. Since it was first considered by Birnbaum (1956), there have been many contributions. For an extensive and lucid review of the topic we refer to Kotz et al. (2003). Suppose that a system consists of n components and Y_i $(i = 1, 2, \ldots, n)$ denotes the strength of the ith component subject to a stress X. The component fails if the applied stress exceeds its strength at any moment, i.e., if $Y_i > X$ then the ith component operates otherwise fails. Thus, $P\{Y_i > X\}$ gives the reliability of the ith component. Define the indicators

$$\xi_i = \begin{cases} 1 & \text{if } Y_i > X \\ 0 & \text{if } Y_i \leq X \end{cases}, \quad i = 1, 2, \ldots, n, \tag{1}$$

where we assume $Y_1, Y_2, \ldots, Y_n$ are independent random strengths having cumulative distributions $F_1, F_2, \ldots, F_n$, respectively, and independent of the random stress X having cumulative distribution F_X.

In most of the statistical process control problems, first n_1 observations follow a certain statistical model and after these observations the model may change and the remaining $n - n_1$ observations follow another model. This kind of modeling is closely related to the change point problem which has been extensively studied in the literature. One of the main interests in change point problem is to test

$$H_0 : F_1 = \cdots = F_n$$

against

$$\begin{gathered} H_1 : \text{There exists } n_1 < n \text{ such that} \\ F_1 = \cdots = F_{n_1} \neq F_{n_1+1} = \cdots = F_n. \end{gathered} \tag{2}$$

Suppose in (2) F_i, $i = 1, 2, \ldots, n$ denotes the cumulative distribution of an ith component's strength. It is clear that if H_0 is true then the random indicators defined by (1) are exchangeable, i.e., the joint distribution of $\xi_1, \xi_2, \ldots, \xi_n$ is invariant under the permutation of its arguments. In this study we consider the reliability of $(C, k, n : G)$ system in a multicomponent stress-strength setup under the hypotheses H_0 and H_1.

2. Reliability Formulas

Eryılmaz and Demir (2007) obtained the reliability of $(C, k, n : G)$ system in a stress-strength setup under the hypothesis H_0:

$$R^0_{n,k} = P\{L_n \geq k \mid H_0\}$$
$$= 1 - \sum_{l=0}^{n} \sum_{j=0}^{\min([\frac{l}{k}], n-l+1)} \sum_{i=0}^{n-l} (-1)^j (-1)^i \binom{n-l+1}{j} \binom{n-kj}{n-l} \binom{n-l}{i} \lambda_{l+i}, \quad (3)$$

where $\lambda_t = P\{\xi_1 = \cdots = \xi_t = 1\} = P\{Y_1 > X, \ldots, Y_t > X\}$. They also obtained the following easier formula when $2k \geq n$:

$$R^0_{n,k} = (n-k+1)\lambda_k - (n-k)\lambda_{k+1}. \quad (4)$$

Now, suppose that out of the n components, n_1 are of one category and their random strengths are assumed to have a common distribution $F^{(1)}$, i.e., $F_1 = \cdots = F_{n_1} = F^{(1)}$ and the remaining $n_2 = n - n_1$ components are of another category and their common strength distribution is $F^{(2)}$, i.e., $F_{n_1+1} = \cdots = F_n = F^{(2)}$. This corresponds to the hypothesis H_1 if $F^{(1)} \neq F^{(2)}$ and here $Y_1, \ldots, Y_{n_1}$ are i.i.d. $F^{(1)}$, $Y_{n_1+1}, \ldots, Y_n$ are i.i.d. $F^{(2)}$ and the random stress X is distributed as F_X. Under the hypothesis H_1, the exchangeability property of $\xi_1, \xi_2, \ldots, \xi_n$ is violated. Therefore the formulas (3) and (4) are no longer valid under H_1.

The reliability of $(C, k, n : G)$ under the hypothesis H_1 can be computed by the function defined by

$$g(a, b, c, d) := E_{F_X}[(1 - F^{(1)}(X))^a (1 - F^{(2)}(X))^b (F^{(1)}(X))^c (F^{(2)}(X))^d]$$
$$= \int (1 - F^{(1)}(x))^a (1 - F^{(2)}(x))^b (F^{(1)}(x))^c (F^{(2)}(x))^d dF_X(x). \quad (5)$$

It is clear that if $F^{(1)} = F^{(2)}$, then $g(a, 0, 0, 0) = \lambda_a$. In the following theorem we provide the reliability of $(C, k, n : G)$ under the hypothesis H_1 when $2k \geq n = n_1 + n_2$ and n_1 is known.

Theorem 2.1. *For* $2k \geq n = n_1 + n_2$

$$R^1_{n,k} = P\{L_n \geq k \mid H_1\} = \sum_{j=k}^{n} p(j, k),$$

where

$$p(j,k) = \begin{cases} g(0,0,1,0) - \sum_{m=0}^{k-1} g(m,0,2,0) & \text{if } k \le j \le n_1 - 1 \\ g(0,0,0,1) - \sum_{m=j-n_1}^{k-1} g(m+n_1-j, j-n_1, 1, 1) & \\ \quad - \sum_{m=0}^{j-n_1-1} g(0,m,0,2) & \text{if } n_1 \le j \le n_1+n_2-1 \, . \\ 1 - \sum_{m=n-n_1}^{k-1} g(m+n_1-n, n-n_1, 1, 0) & \\ \quad - \sum_{m=0}^{n-n_1-1} g(0,m,0,1) & \text{if } j = n_1 + n_2, \end{cases}$$

and $g(a, b, c, d)$ *is given by* (5).

Proof. Denote by η_j the length of consecutive 1's in $\xi_1, \xi_2, \ldots, \xi_n$ at jth trial, i.e., the event $\{\eta_j = m\}$ is equivalent to $\{\xi_j = 1, \ldots, \xi_{j-m+1} = 1, \xi_{j-m} = 0\}$ and $\{\eta_j = 0\}$ iff $\{\xi_j = 0\}$, where by convention, $\xi_0 = 0$. If $2k \ge n = n_1 + n_2$ then

$$R^1_{n,k} = P\left\{ \bigcup_{j=k}^{n} E_{j,k} \right\},$$

where the events $E_{j,k}$ are defined by:

$$E_{j,k} \equiv \{\eta_j \ge k, \xi_{j+1} = 0\}, \quad k \le j \le n-1$$
$$E_{n,k} \equiv \{\eta_n \ge k\}.$$

Since $2k \ge n$, there can be at most one success run of length at least k consecutive operating components

$$R^1_{n,k} = \sum_{j=k}^{n-1} P\{\eta_j \ge k, \xi_{j+1} = 0\} + P\{\eta_n \ge k\}.$$

For $k \le j \le n_1 - 1$,

$$\begin{aligned} p(j,k) = P\{\eta_j \ge k, \xi_{j+1} = 0\} &= P\{\xi_{j+1} = 0\} - \sum_{m=0}^{k-1} P\{\eta_j = m, \xi_{j+1} = 0\} \\ &= P\{\xi_{j+1} = 0\} - P\{\xi_{j+1} = 0, \xi_j = 0\} \\ &\quad - \sum_{m=1}^{k-1} P\{\xi_{j+1} = 0, \xi_j = 1, \ldots, \xi_{j-m+1} = 1, \xi_{j-m} = 0\} \\ &= P\{Y_{j+1} \le X\} - P\{Y_{j+1} \le X, Y_j \le X\} \\ &\quad - \sum_{m=1}^{k-1} P\{Y_{j+1} \le X, Y_j > X, \ldots, Y_{j-m+1} > X, Y_{j-m} \le X\}, \end{aligned}$$

by conditioning on X we have:

$$p(j,k) = \int F^{(1)}(x)dF_X(x) - \sum_{m=0}^{k-1} \int (1 - F^{(1)}(x))^m (F^{(1)}(x))^2 dF_X(x)$$
$$= g(0,0,1,0) - \sum_{m=0}^{k-1} g(m,0,2,0).$$

The other probabilities can be treated similarly and the proof is completed. □

Corollary 2.1. *The reliabilities of series* $(C, n, n : G)$ *and parallel* $(C, 1, n : G)$ *systems under the hypothesis* H_1 *are given, respectively, by*:

$$R^1_{n,n} = g(n_1, n_2, 0, 0)$$
$$R^1_{n,1} = 1 - g(0,0,n_1,n_2).$$

Example 2.1. Let

$$F_X(x) = 1 - \exp\{-\theta x\},$$
$$F^{(1)}(x) = 1 - \exp\{-\alpha x\}, \quad F^{(2)}(x) = 1 - \exp\{-\beta x\}, \ \ x > 0.$$

Then the function defined by (5) can be represented by the following double summation:

$$g(a,b,c,d) = \sum_{i=0}^{c}\sum_{j=0}^{d} (-1)^i (-1)^j \binom{c}{i}\binom{d}{j} \frac{\theta}{\theta + \alpha(a+i) + \beta(b+j)}. \quad (6)$$

Example 2.2. Consider the consecutive 2-out-of-3 : G system whose structure function is given by

$$\phi(\xi_1, \xi_2, \xi_3) = \max(\min(\xi_1, \xi_2), \min(\xi_2, \xi_3)).$$

Suppose that $n_1 = 2$, $n_2 = 1$ and stress and strength distributions are, as in Example 2.1. Then,

$$R^1_{3,2} = 1 - g(0,0,1,1) - g(1,0,1,1) - g(0,1,1,0)$$
$$= \theta\left(\frac{1}{\theta + 2\alpha} + \frac{1}{\theta + \alpha + \beta} - \frac{1}{\theta + 2\alpha + \beta}\right).$$

3. Estimation of Reliability

If θ, α, and β are known then $R^1_{n,k}$ is simply calculated as above, otherwise we need to estimate the reliability. In this section, we provide the minimum variance unbiased (MVU) estimator both parametric and nonparametric of the reliability of $(C, k, n : G)$ system in the multicomponent stress-strength model described above.

3.1. *Parametric MVU Estimator*

For the parametric estimation we assume that the stress and strength distributions are exponential and

$$F_X(x) = 1 - \exp\{-\theta x\},$$
$$F^{(1)}(x) = 1 - \exp\{-\alpha x\}, \quad F^{(2)}(x) = 1 - \exp\{-\beta x\}, \; x > 0.$$

MVU estimation of $g(a, b, c, d)$ will be enough to estimate the reliability since it is a linear combination of $g(a, b, c, d)$. Thus, considering (6) we only need to estimate

$$\varphi_{a_1,a_2}(\theta, \alpha, \beta) = \frac{\theta}{\theta + \alpha a_1 + \beta a_2}.$$

Theorem 3.1. *Let* $X_1, X_2, \ldots, X_m$, $Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_{m_1}^{(1)}$ *and* $Y_1^{(2)}, Y_2^{(2)}, \ldots, Y_{m_2}^{(2)}$ *be independent random samples from*

$$F_X(x) = 1 - \exp\{-\theta x\},$$
$$F^{(1)}(x) = 1 - \exp\{-\alpha x\}, \quad F^{(2)}(x) = 1 - \exp\{-\beta x\},$$

respectively. Then the MVU estimate of $\varphi_{a_1,a_2}(\theta, \alpha, \beta)$ *is given by*

$$\varphi_{a_1,a_2}\widehat{(\theta, \alpha, \beta)} = \begin{cases} Q_1(a_1, a_2; V_1, V_2; m, m_1, m_2) & \text{if } a_1V_1 \le 1 \text{ and } a_2V_2 \le 1 \\ Q_{(a_2V_2)^{-1}}(a_1, a_2; V_1, V_2; m, m_1, m_2) & \text{if } a_1V_1 \le 1 \text{ and } a_2V_2 > 1 \\ Q_{(a_1V_1)^{-1}}(a_1, a_2; V_1, V_2; m, m_1, m_2) & \text{if } a_1V_1 > 1 \text{ and } a_2V_2 \le 1 \\ Q_{\min((a_1V_1)^{-1},(a_2V_2)^{-1})}(a_1, a_2; V_1, V_2; m, m_1, m_2) & \text{if } a_1V_1 > 1 \text{ and } a_2V_2 > 1 \end{cases},$$

where $V_1 = T/T^{(1)}$, $V_2 = T/T^{(2)}$, $T = \sum_{i=1}^{m} X_i$, $T^{(1)} = \sum_{i=1}^{m_1} Y_i^{(1)}$, $T^{(2)} = \sum_{i=1}^{m_2} Y_i^{(2)}$ *and*

$$Q_p(a_1, a_2; V_1, V_2; m, m_1, m_2) = \int_0^p (1 - a_1V_1s)^{m_1-1}(1 - a_2V_2s)^{m_2-1}(m-1)(1-s)^{m-2}ds.$$

Proof. The proof is based on Rao–Blackwell method. An unbiased estimate of $\varphi_{a_1,a_2}(\theta, \alpha, \beta)$ is:

$$h\big(X_1, Y_1^{(1)}, Y_1^{(2)}\big) = \begin{cases} 1 & \text{if } Y_1^{(1)} > a_1X_1 \text{ and } Y_1^{(2)} > a_2X_1 \\ 0 & \text{otherwise} \end{cases}.$$

Because $\mathbf{T} = (T, T^{(1)}, T^{(2)})$ is complete sufficient statistic, the unique MVU estimate of $\varphi_{a_1,a_2}(\theta, \alpha, \beta)$ is

$$\varphi_{a_1,a_2}\widehat{(\theta, \alpha, \beta)} = E\big[h\big(X_1, Y_1^{(1)}, Y_1^{(2)}\big) \mid \mathbf{T}\big] = P\big\{Y_1^{(1)} > a_1X_1, Y_1^{(2)} > a_2X_1 \mid \mathbf{T}\big\}.$$

Letting $S = X_1/T$, $S^{(1)} = Y_1^{(1)}/T^{(1)}$, $S^{(2)} = Y_1^{(2)}/T^{(2)}$ and $V_1 = T/T^{(1)}$, $V_2 = T/T^{(2)}$ we have

$$\varphi_{a_1,a_2}\widehat{(\theta,\alpha,\beta)} = P\{S^{(1)} > a_1V_1S, S^{(2)} > a_2V_2S \mid \mathbf{T}\}.$$

$(S, S^{(1)}, S^{(2)})$ is independent of $\mathbf{T}$ and

$$f_{S,S^{(1)},S^{(2)}}(s, s_1, s_2 \mid \mathbf{T}=\mathbf{t}) = (m-1)(m_1-1)(m_2-1)(1-s)^{m-2}(1-s_1)^{m_1-2}(1-s_2)^{m_2-2}$$
$$0 < s < 1,\ \ 0 < s_i < 1,\ \ i = 1, 2. \tag{7}$$

Using (7) we have the following.

For the case $a_1V_1 \le 1$ and $a_2V_2 \le 1$,

$$\begin{aligned}\varphi_{a_1,a_2}\widehat{(\theta,\alpha,\beta)} &= P\{S^{(1)} > a_1V_1S, S^{(2)} > a_2V_2S \mid \mathbf{T}\}\\ &= \int_0^1 (1-a_1V_1s)^{m_1-1}(1-a_2V_2s)^{m_2-1}(m-1)(1-s)^{m-2}ds,\end{aligned}$$

if $a_1V_1 \le 1$ and $a_2V_2 > 1$

$$\varphi_{a_1,a_2}\widehat{(\theta,\alpha,\beta)} = \int_0^{(a_2V_2)^{-1}} (1-a_1V_1s)^{m_1-1}(1-a_2V_2s)^{m_2-1}(m-1)(1-s)^{m-2}ds,$$

if $a_1V_1 > 1$ and $a_2V_2 \le 1$,

$$\varphi_{a_1,a_2}\widehat{(\theta,\alpha,\beta)} = \int_0^{(a_1V_1)^{-1}} (1-a_1V_1s)^{m_1-1}(1-a_2V_2s)^{m_2-1}(m-1)(1-s)^{m-2}ds,$$

and if $a_1V_1 > 1$ and $a_2V_2 > 1$,

$$\varphi_{a_1,a_2}\widehat{(\theta,\alpha,\beta)} = \int_0^{\min((a_1V_1)^{-1},(a_2V_2)^{-1})} (1-a_1V_1s)^{m_1-1}(1-a_2V_2s)^{m_2-1}(m-1)(1-s)^{m-2}ds.$$

The estimation of $g(a, b, c, d)$ can be readily obtained since the MVU of a linear combination of related parameters is the linear combinations of the MVUs of those parameters; see, e.g., Rao (1973, p. 318). □

Under the hypothesis H_0, i.e., $F^{(1)}(x) = F^{(2)}(x) = 1 - \exp\{-\alpha x\}$, it is enough to estimate

$$\lambda_a = g(a, 0, 0, 0) = \frac{\theta}{\theta + \alpha \cdot a}.$$

The MVU estimate of λ_a can be obtained as a special case of the previous theorem:

Corollary 3.1. *Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_{m_1}$ be independent random samples from*

$$F_X(x) = 1 - e^{-\theta x} \ \ \textit{and} \ \ F^{(1)}(x) = 1 - e^{-\alpha x}, \quad x > 0,$$

respectively. Then the MVU estimate of λ_a *is given by*

$$\hat{\lambda}_a = \begin{cases} Q_1(a, 0; V, 0; m, m_1, 1) & \text{if } aV \leq 1 \\ Q_{(aV)^{-1}}(a, 0; V, 0; m, m_1, 1) & \text{if } aV > 1 \end{cases}.$$

where $V = T/T^{(1)}$, $T = \sum_{i=1}^{m} X_i$, $T^{(1)} = \sum_{i=1}^{m_1} Y_i$.

Now, using $\hat{\lambda}_a$ in (3) the MVU estimation of the reliability under the hypothesis H_0 is obtained.

3.2. *Nonparametric MVU Estimator*

In the following, we provide the nonparametric MVU estimator of reliability when $F^{(1)} = F^{(2)}$ (there is no change in strength). The corresponding reliability is $R^0_{n,k}$. Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_{m_1}$ be independent random samples from F_X and $F^{(1)}$, where the unknown distribution functions F_X and $F^{(1)}$ are assumed to be continuous.

Bhattacharya and Johnson (1975) derived the MVU estimator of s-out-of-n system, consisting of n components, functions when at least s $(1 \leq s \leq n)$ of the components survive a common shock. If $R_{n,s}$ denotes the reliability of such a system then its MVU estimator as a generalized U-statistics is given by

$$\widehat{R}_{n,s} = \frac{1}{m\binom{m_1}{n}} \sum_{i=n-s+1}^{m_1-s+1} \binom{i-1}{n-s} \binom{m_1-i}{s-1} (s_{(i)} - i), \quad (8)$$

where $s_{(1)} \leq s_{(2)} \leq \cdots \leq s_{(m_1)}$ are the ordered ranks of the Y's in the combined sample; see, e.g., Johnson (1988, p. 39).

As seen from Eq. (3), it is enough to estimate λ_s which is in fact the reliability of series system consists of s components (s-out-of-s system). So that using (8) the nonparametric MVU estimator of λ_s is

$$\hat{\lambda}_s = \frac{1}{m\binom{m_1}{s}} \sum_{i=1}^{m_1-s+1} \binom{m_1-i}{s-1} (s_{(i)} - i). \quad (9)$$

Now, the nonparametric MVU estimator of $R^0_{n,k}$ is obtained using (9) in (3).

4. Numerical Illustrations

In this section, we compare maximum likelihood estimate (MLE) and parametric MVU estimates of $R^0_{n,k}$ and $R^1_{n,k}$. The MLE of (θ, α, β) is given by

$$\tilde{\theta} = \frac{1}{\overline{X}_m}, \quad \tilde{\alpha} = \frac{1}{\overline{Y}_{m_1}}, \quad \text{and} \quad \tilde{\beta} = \frac{1}{\overline{Y}_{m_2}}.$$

Using $(\tilde{\theta}, \tilde{\alpha}, \tilde{\beta})$ in (6) and employing the invariance property of MLE and expression given in Theorem 2.1, the MLE of $R^1_{n,k}$ is obtained. Denote by $\widetilde{R}^{\cdot}_{n,k}$ and $\widehat{R}^{\cdot}_{n,k}$ MLE and MVU estimates, respectively.

Table 1
Mean squared error of reliability estimates of $(C, 2, 3 : G)$ system, $m = m_1 = m_2 = 10$

θ	α	β	$R^1_{3,2}$	$\widehat{R}^1_{3,2}$	$\widetilde{R}^1_{3,2}$	$MSE(\widehat{R}^1_{3,2})$	$MSE(\widetilde{R}^1_{3,2})$
4	2	3	0.5808	0.5796	0.5793	0.01250	0.01170
4	2	2	0.6000	0.5973	0.5972	0.01210	0.01120
3	1	2	0.6714	0.6712	0.6660	0.01110	0.01020
4	1	2	0.7381	0.7391	0.7288	0.00868	0.00841
10	1	2	0.8883	0.8888	0.8795	0.00245	0.00262
20	2	1	0.9029	0.9034	0.8954	0.00179	0.00201

Table 2
Mean squared error of reliability estimates of $(C, 2, 3 : G)$ system, $m = m_1 = m_2 = 20$

θ	α	β	$R^1_{3,2}$	$\widehat{R}^1_{3,2}$	$\widetilde{R}^1_{3,2}$	$MSE(\widehat{R}^1_{3,2})$	$MSE(\widetilde{R}^1_{3,2})$
4	2	3	0.5808	0.5807	0.5793	0.00636	0.00611
4	2	2	0.6000	0.5995	0.5975	0.00582	0.00589
3	1	2	0.6714	0.6701	0.6668	0.00513	0.00495
4	1	2	0.7381	0.7381	0.7335	0.00410	0.00402
10	1	2	0.8883	0.8882	0.8840	0.00115	0.00120
20	2	1	0.9029	0.9033	0.8988	0.00086	0.00089

Table 3
Mean squared error of reliability estimates of $(C, 1, 2 : G)$ system, $m = m_1 = 5$

θ	α	$R^0_{2,1}$	$\widehat{R}^0_{2,1}$	$\widetilde{R}^0_{2,1}$	$MSE(\widehat{R}^0_{2,1})$	$MSE(\widetilde{R}^0_{2,1})$
10	1	0.9848	0.9844	0.9735	0.00059	0.00119
8	1	0.9778	0.9779	0.9640	0.00097	0.00180
5	1	0.9524	0.9528	0.9318	0.00384	0.00508
5	2	0.8730	0.8733	0.8451	0.01396	0.01391

Table 4
Nonparametric MVU estimator of $(C, 2, 4 : G)$ system

θ	α	$R^0_{4,2}$	$\widehat{R}^0_{4,2}$	$MSE(\widehat{R}^0_{4,2})$
5	3	0.6494	0.6488	0.0113
10	2	0.8929	0.8926	0.0044
10	1	0.9615	0.9606	0.0012

Table 5
Nonparametric MVU estimator of $(C, 3, 4 : G)$ system

θ	α	$R^0_{4,3}$	$\widehat{R}^0_{4,3}$	$MSE(\widehat{R}^0_{4,3})$
5	3	0.4202	0.4182	0.0125
10	2	0.6944	0.6920	0.0128
10	1	0.8242	0.8265	0.0083

A simulation study is performed to compare two parametric MLE and MVU estimates of the reliability of consecutive 2-out-of-3 : G system when $n_1 = 2$ and $n_2 = 1$. We simulated 5,000 realizations of $\widetilde{R}^1_{3,2}$ and $\widehat{R}^1_{3,2}$ for $m = m_1 = m_2 = 10$ and $m = m_1 = m_2 = 20$. The mean squared errors of $\widetilde{R}^1_{3,2}$ and $\widehat{R}^1_{3,2}$ as well as the estimated reliabilities are computed and the results are presented in Tables 1 and 2. It was seen that the MSE of both estimates appear to be nearly equal. It seems that either the MLE or the MVU estimates of reliability can be used. However, one may think that these numerical results may be sensitive to the design values of k and n as well as the size of random samples m, m_1, m_2. To investigate this, we repeat the same simulation study for the reliability of consecutive 1 -out-of-2 : G system (parallel system with two components) when there is no change in strength and $m = m_1 = m_2 = 5$. The results for this system are reported in Table 3.

For an illustration, in Tables 4 and 5 we provide some simulation results for nonparametric MVU estimator of reliability of consecutive 2-out-of-4 : G and consecutive 3-out-of-4 : G systems. We generate the random samples $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_{m_1}$ from exponential distributions with parameters θ and α, respectively, and compute 5,000 realizations of $\widehat{R}^0_{4,2}$ and $\widehat{R}^0_{4,3}$ for $m = m_1 = 20$.

5. Conclusions

In this article, we have considered the problem of computing and estimating the stress-strength reliability under the formation of consecutive k-out-of-$n : G$ system which is of special importance in reliability engineering applications. We investigated both cases in which there is a change and no change in strength. Some exact and computable formulas were provided for the reliability.

Estimation of reliability was discussed by different methods. Parametric minimum variance unbiased and maximum likelihood estimators were obtained when the stress and strength distributions are exponential. From Tables 1 and 2, it was observed that the MSEs of both of the parametric estimators are nearly equal under the assumption of exponential stress and strength distributions and the performance of the maximum likelihood estimator is quite satisfactory for $(C, 2, 3 : G)$ system. However, as can be observed from Table 3, the minimum variance unbiased estimator might be superior especially for small sample sizes. There is a considerable difference between the MSEs for $(C, 1, 2 : G)$ system especially when the reliability is large. Thus, the performance of estimators depends on sample sizes and system design values of k and n.

We have also presented the nonparametric minimum variance unbiased estimation of reliability. Computation of this estimator is quite easy (MATLAB program is avaliable on request from the author) and it can be used if there is any doubt about distributions.

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