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On The Bondage Number of Middle Graphs*

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Abstract—Let $G = (V(G), E(G))$ be a simple graph. A subset S of $V(G)$ is a dominating set of G if, for any vertex $v \in V(G) - S$, there exists some vertex $u \in S$ such that $uv \in E(G)$. The domination number, denoted by $\gamma(G)$, is the cardinality of a minimal dominating set of G. There are several types of domination parameters depending upon the nature of domination and the nature of dominating set. These parameters are bondage, reinforcement, strong-weak domination, strong-weak bondage numbers. In this paper, we first investigate the strong-weak domination number of middle graphs of a graph. Then several results for the bondage, strong-weak bondage number of middle graphs are obtained.

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1. INTRODUCTION

There has been an explosive growth between graph theory and computer science, Operations Research, chemical applications, electrical and communication engineering in the last 50 years. In view of many varied applications in the fields of communication networks, algorithm designs, computational complexity, etc., the study of several domination parameters is the fastest growing area in graph theory. There are several types of domination depending upon the nature of domination and the nature of dominating set. In the following, we present the notions of strong and weak domination number, bondage number, and strong and weak bondage number in connected graphs.

In a graph $G = (V(G), E(G))$, a subset $S \subseteq V(G)$ of vertices is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ is the minimal cardinality of a dominating set. In graph theory, the concept of domination with its numerous variations is well studied.

If $uv \in E(G)$, then u and v dominate each other. The notions of strong and weak domination in graphs were first introduced by Sampathkumar and Pushpa Latha [1]. For two adjacent vertices of G, u and v, if $\deg(u) \geq \deg(v)$, then u *strongly dominates* v. Similarly, if $\deg(v) \geq \deg(u)$, then u *weakly dominates* v.

A set $S \subseteq V(G)$ is *strong-dominating set* (sd-*set*) of G if every vertex in $V(G) - S$ is strongly dominated by at least one vertex in S. Similarly, if every vertex in $V(G) - S$ is weakly dominated by at least one vertex in S, then S is a *weak-dominating set* (wd-*set*).

Applications of strong and weak domination are seen in certain practical situations. For instance, in a road network in which a number of locations are connected, the degree of a vertex v is the number of roads meeting at v. If $\deg(u) \geq \deg(v)$, then, naturally, the traffic at u is heavier than that at v, and

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vice versa. If the traffic between u and v is considered, preference should be given to the vehicles going from u to v. Thus, u strongly dominates v and v weakly dominates u.

The *strong domination number* $\gamma_{\mathrm{s}}(G)$ and the *weak domination number* $\gamma_{\mathrm{w}}(G)$ of G are defined similarly as the domination number $\gamma(G)$ of G.

When investigating the domination number of any given graph G, one may want to learn the answer of the following questions: How does the domination number decrease in a graph G? How does the domination number increase in a graph G? One of the vulnerability parameters known as bondage number in a graph G answers the second question, while the other vulnerability measure known as reinforcement number gives the answer of the first question. The bondage number was introduced by Bauer, Harary, Nieminen and Suffel [2]; and has been further studied by Fink, Jacobson, Kinch and Roberts [3], Hartnell and Rall [4] and others. The *bondage number* $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$.

The *strong bondage number* of G, denoted by $b_{\mathrm{s}}(G)$, is the minimal cardinality among all sets of edges $E' \subseteq E(G)$ such that $\gamma_{\mathrm{s}}(G-E') > \gamma_{\mathrm{s}}(G)$ [5]. The *weak bondage number* of G, denoted by $b_{\mathrm{w}}(G)$, is the minimal cardinality among all sets of edges $E' \subseteq E(G)$ such that $\gamma_{\mathrm{w}}(G-E') > \gamma_{\mathrm{w}}(G)$; and we deal with the strong bondage number of a nonempty graph G [6]. In this paper, we investigate, first of all, strong-weak domination number of the middle graph of a graph. Then several results for the bondage and strong-weak bondage of middle graphs are obtained.

Throughout this paper, for any graph G, $\alpha(G)$ and $\beta(G)$ are, respectively, the covering number and the independence number of G.

2. BASIC RESULTS

In this section, we will review some of the known results.

Theorem 1 ([6]). *If G is a nonempty graph with a unique minimal dominating set, then $b(G) = 1$.*

Theorem 2 ([6]). *The strong bondage number of*

(a) *the complete graph K_n, $n \ge 2$, is $b_{\mathrm{s}}(K_n) = \lceil n/2 \rceil$;*

(b) *the cycle C_n, $n \ge 3$, is*

$$b_{\mathrm{s}}(C_n) = \begin{cases} 3 & \text{if } n \equiv 1 \pmod 3, \\ 2 & \text{otherwise;} \end{cases}$$

(c) *the path P_n, $n \ge 3$, is*

$$b_{\mathrm{s}}(P_n) = \begin{cases} 2 & \text{if } n \equiv 1 \pmod 3, \\ 1 & \text{otherwise;} \end{cases}$$

(d) *the wheel W_n, $n \ge 4$, is $b_{\mathrm{s}}(W_n) = 1$;*

(e) *the complete bipartite graph $K_{r,t}$, $4 \le r \le t$, is*

$$b_{\mathrm{s}}(K_{r,t}) = \begin{cases} 2r & \text{if } t = r+1, \\ r & \text{otherwise.} \end{cases}$$

Theorem 3 ([6]). *The weak bondage number of*

(f) *the complete graph K_n, $n \ge 2$, is $b_{\mathrm{w}}(K_n) = 1$;*

(g) *the cycle* C_n, $n \geq 3$, *is*

$$b_{\mathrm{w}}(C_n) = \begin{cases} 2 & \text{if } n \equiv 1 \pmod 3, \\ 1 & \text{otherwise}; \end{cases}$$

(h) *the path* P_n, $n \geq 3$, *is*

$$b_{\mathrm{w}}(P_n) = \begin{cases} 2 & \text{if } n = 3 \text{ or } 5, \\ 1 & \text{otherwise}; \end{cases}$$

(i) *the wheel* W_n, $n \geq 4$, *is*

$$b_{\mathrm{w}}(W_n) = \begin{cases} 2 & \text{if } n \equiv 2 \pmod 3, \\ 1 & \text{otherwise}; \end{cases}$$

(j) *the complete bipartite graph* $K_{r,t}$, $4 \leq r \leq t$, *is* $b_{\mathrm{w}}(K_{r,t}) = t$.

Theorem 4 ([6]). *If* T *is a nontrivial tree, then* $b_{\mathrm{s}}(T) \leq 3$.

Theorem 5 ([6]). *If any vertex of tree* T *is adjacent with two or more end-vertices, then* $b_{\mathrm{s}}(T) = 1$.

Theorem 6 ([6]). *If* T *is a nontrivial tree, then* $b_{\mathrm{w}}(T) \leq \Delta(T)$, *where* $\Delta(T)$ *denotes the maximum vertex degree of* G.

3. SOME EXACT VALUES FOR THE BONDAGE NUMBER OF MIDDLE GRAPH

In this section, we first give the definition of the middle graph of a graph. Then, the domination number and bondage number of middle graphs are calculated.

Definition 7 ([7]). The *middle* graph $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G.

We can see that middle graphs $M(G)$ have a higher stability than the given graphs G. So, in this paper, we study bondage number, strong bondage number, and weak domination number for middle graphs $M(G)$ of any given graph G.

Theorem 8. *Let* G *be a nonempty graph of order* n *and* $M(G)$ *be the middle graph of* G. *Then,*

$$\gamma_{\mathrm{w}}(M(G)) = n = |V(G)|.$$

Proof. The vertices which are added to G to construct $M(G)$ are adjacent to at least one vertex of G and the degrees of these vertices are at least greater by 1 than the degrees of the vertices of G. Thus, the necessary and sufficient condition to dominate weakly the entire set of vertices of $M(G)$ is to include the entire set of vertices of G in the wd-set of $M(G)$. Furthermore, the wd-set of $M(G)$ is unique and includes only the entire set of vertices of G. Hence we conclude that $\gamma_{\mathrm{w}}(M(G)) = n = |V(G)|$. □

Theorem 9. *Let* $M(P_n)$ *be the middle graph of* P_n. *Then*

$$\gamma_{\mathrm{s}}(M(P_n)) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. In order to obtain $M(P_n)$, $n-1$ new vertices are added to P_n. Two of these new vertices and $n-3$ of these new vertices are of degree 3 and 4, respectively. In order to dominate the end vertices of $M(P_n)$ in the γ_{s}-set, two vertices of degree 3 must be taken. These two vertices exactly dominate four vertices of P_n. Thus, there are $n-4$ vertices of P_n remaining in $M(P_n)$. One of the vertices having maximal vertex degree of $M(P_n)$ dominates at most two vertices in P_n. In order to dominate $n-4$ more vertices, $(n-4)/2$ more new vertices must be included in the γ_{s}-set. In view of the parity of the number of vertices, the obtained value is bounded by $\lceil (n-4)/2 \rceil$. Hence $\gamma_{\mathrm{s}}(M(P_n)) = 2 + \lceil (n-4)/2 \rceil = \lfloor n/2 \rfloor$, the result holds. □

Theorem 10. *Let $M(C_n)$ be the middle graph of C_n. Then*

$$\gamma_s(M(C_n)) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. The graph $M(C_n)$ has a total of $2n$ vertices: n are of degree 2 and n are of degree 4. The proof is similar to that of Theorem 9. Thus, $\lceil n/2 \rceil$ vertices of degree 4 are necessary to strongly dominate the entire set of vertices of the graph. Therefore, $\gamma_s(M(C_n)) = \lceil n/2 \rceil$. This completes the proof. □

Theorem 11. *Let $M(K_{1,n})$ be the middle graph of $K_{1,n}$. Then*

$$\gamma_s(M(K_{1,n})) = n.$$

Proof. To dominate strongly the n end-vertices of $M(K_{1,n})$, n vertices of degree $n+1$ must be included in the strong-dominating set of $M(K_{1,n})$. These n vertices of degree $n+1$ also strongly dominate the vertex of degree n. Thus, the strong-dominating set of $M(K_{1,n})$ of minimal cardinality includes n vertices. We have $\gamma_s(M(K_{1,n})) = n$. Furthermore, the strong-dominating set of $M(K_{1,n})$ which gives the strong-domination number of $M(K_{1,n})$ is unique including only the entire set of vertices that are added to $K_{1,n}$ to form $M(K_{1,n})$. □

Theorem 12. *Let $M(P_n)$ be the middle graph of P_n. Then*

$$b(M(P_n)) = \begin{cases} 1 & \textit{if } n \textit{ is even}, \\ 2 & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. The graph $M(P_n)$ is a connected graph with $|V(M(P_n))| = 2n-1$ vertices and $|E(M(P_n))| = 3n-4$ edges. The domination number of $M(P_n)$ is $\gamma(M(P_n)) = \lceil n/2 \rceil$ [8]. We consider two cases in the proof according to the parity of the number of vertices of P_n.

Case 1: *n is odd.* After the removal of one of the edges of $M(P_n)$, $\gamma(M(P_n))$ remains the same. If two edges, say, the pendant edges of $M(P_n)$ (we denote the set of these edges by $S' = \{e_1, e_2\}$) are removed, then there remains a graph $M(P_n) - S'$ with three components, and two of them are isolated vertices. The largest component of $M(P_n) - S'$ has $n-3$ vertices of degree 4. Taking $(n-1)/2$ vertices is enough to dominate the entire set of vertices of the remaining largest component. Thus,

$$\gamma(M(P_n) - S') = 1 + 1 + \frac{n-1}{2} = \frac{n-3}{2},$$

which is greater by 1 than $\gamma(M(P_n)) = (n+1)/2$.

Consequently, we can say that the removal of the set of edges S' of $M(P_n)$ results in a graph with three components with domination number greater by 1 than that of $M(P_n)$. Hence, for odd n, the bondage number of $M(P_n)$ is $b(M(P_n)) = 2$.

Case 2: *n is even.* In this case, $M(P_n)$ has a unique minimal dominating set with two vertices of degree 2 and $(n-4)/2$ vertices of degree 4. Thus, by Theorem 1, for even n, we have $b(M(P_n)) = 1$. □

Theorem 13. *Let $M(C_n)$ be the middle graph of C_n. Then*

$$b(M(C_n)) = \begin{cases} 2 & \textit{if } n \textit{ is even}, \\ 3 & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. We have two cases according to the number of vertices of C_n.

Case 1: *n is even.* As $\gamma(M(C_n)) = n/2$ [8], deleting less than two edges will leave γ of the resulting graph unchanged. By deleting the two edges incident to one of the vertices of degree 2, this vertex is isolated and γ of the resulting graph increases. This completes the proof.

Case 2: *n is odd.* In this case (see [8]),

$$\gamma(M(C_n)) = \frac{n-1}{2} + 1.$$

That is, $(n-1)/2$ vertices are not enough to dominate the entire set of vertices of $M(C_n)$; one more vertex must be included in a minimal dominating set. Therefore, after deleting the two edges as in Case 1, in addition to this case, one more edge should be deleted from the resulting graph. This edge must be incident to both one of the vertices of degree 3 and one of the vertices of $M(C_n)$ which is added to C_n in order to construct the middle graph. If we denote the set of the deleted edges by S', then $\gamma(M(C_n) - S') > \gamma(M(C_n))$. As a result, $b(M(C_n)) = 3$. □

Theorem 14. *Let $M(K_{1,n})$ be the middle graph of $K_{1,n}$. Then*

$$b(M(K_{1,n})) = n.$$

Proof. The graph $M(K_{1,n})$ is a connected graph with

$$|V(M(K_{1,n}))| = 2n + 1$$

vertices and

$$|E(M(K_{1,n}))| = 2n + \binom{n}{2}$$

edges. The domination number of $M(K_{1,n})$ is $\gamma(M(K_{1,n})) = n$ [8]. The removal of a set edges S' of $M(K_{1,n})$ of cardinality less than n results in a graph with the same domination number as that of $M(K_{1,n})$.

If n edges of $M(K_{1,n})$, say, the pendant edges, $S' = \{e_1, e_2, \dots, e_n\}$, are removed, then there remains $M(K_{1,n}) - S'$, with $n+1$ components n of which are isolated vertices and the other one is the complete graph K_{n+1}. The domination number of the largest component K_{n+1} is 1. Thus,

$$\gamma(M(K_{1,n}) - S') = n + 1,$$

which is greater by 1 than that of $\gamma(M(K_{1,n})) = n$. Therefore, the minimal cardinality of the set of edges S' of $M(K_{1,n})$ whose removal from $M(K_{1,n})$ results in a graph with domination number greater than that of $M(K_{1,n})$, is n. Hence the bondage number of $M(K_{1,n})$ is $b(M(K_{1,n})) = n$. □

Corollary 15. *Let G be a graph of order n with a star as a spanning subgraph. Then*

$$b(G) = \left\lfloor \frac{m(\deg_{n-1}(v))}{2} \right\rfloor,$$

where $m(\deg_{n-1}(v))$ denotes the number of vertices of degree $n-1$ of G.

Theorem 16. *Let G be a nonempty graph of order n, and let $M(G)$ be the middle graph of G. Then*

$$b_{\mathrm{w}}(M(G)) = 2.$$

Proof. Without loss of generality, by the definition of the middle graph, deleting less than two edges does not change the weak domination number of the graph. Let D be a minimal wd-set. As proved in Theorem 8, the set D consists of the vertices of $M(G)$ which are of minimal degree, i.e., exactly the vertices of G. In order to increase $\gamma_{\mathrm{w}}(M(G))$, one of the vertices of $M(G)$ which are added to G to construct $M(G)$ should be included in D. By the definition of the middle graph, this vertex v_{e_k} must be adjacent to the two vertices of G. Thus, the two vertices in D dominate the vertex v_{e_k}. When the two edges incident to these two vertices are deleted in order not to dominate the vertex v_{e_k}, v_{e_k} must be included in D. □

Theorem 17. *Let $M(P_n)$ be the middle graph of P_n, $n > 7$. Then*

$$b_s(M(P_n)) = \begin{cases} 2 & \text{if } n \text{ is odd}, \\ 1 & \text{if } n \text{ is even}. \end{cases}$$

Proof. By definition, the γ_s-set includes the vertices of maximum degree. In order to increase the cardinality of the γ_s-set, one of the vertices of minimal degree must be included in this set. In order to achieve this, this vertex must dominate itself. This yields deleting the edges to isolate this vertex.

Case 1: *n is odd.* Deleting one of the edges will leave the γ_s of the resulting graph unchanged. After removing the two edges incident to the end vertices, the γ_s of the resulting graph increases by 1. This leaves the two vertices isolated. For the other remaining component, except these two, $\lfloor n/2 \rfloor$ of the vertices of maximum degree strongly dominate the component. Therefore, the γ_s of the remaining graph results with $2 + (n-1)/2 = (n+3)/2$, which is 1 more than $\gamma_s(M(P_n))$. This completes the proof.

Case 2: *n is even.* To increase the γ_s, we need to delete the edge incident to an end vertex of $M(P_n)$. This leaves one vertex isolated and it must exactly be in any γ_s-set. For the other component, we need to include $n/2$ of the vertices of maximum degree in the γ_s-set. These suffice to strongly dominate the rest. Thus, the γ_s-set has the cardinality $1 + n/2$, and hence we have the result.

The values of $b_s(M(P_n))$ for $n < 8$ are indicated in the following table.

Table. The values of $b_s(M(P_n))$ for $n < 8$

n	2	3	4	5	6	7
$b_s(M(P_n))$	1	2	3	2	1	1

□

Theorem 18. *Let $M(C_n)$ be the middle graph of C_n. Then*

$$b_s(M(C_n)) = \begin{cases} 2 & \text{if } n \text{ is even}, \\ 3 & \text{if } n \text{ is odd}. \end{cases}$$

Proof. We have two cases according to the number of vertices of C_n.

Case 1: *n is even.* When less than two edges of $M(C_n)$ are deleted, $\gamma_s(M(C_n))$ is unchanged. Deleting the edges incident to one of the vertices of C_n increases this number by 1. Thus,

$$b_s(M(C_n)) = 2.$$

Case 2: *n is odd.* Deleting the same edges as in Case 1, $\gamma_s(M(C_n))$ is unchanged. In addition to these edges, one of the edges of $M(C_n)$ must be deleted. This edge is adjacent to one of the deleted edges and the end vertices of this edge are of degree 3 and 4. Thus, three edges in all are deleted by increasing $\gamma_s(M(C_n))$ by 1. Then, we have $b_s(M(C_n)) = 3$. Hence the proof is complete. □

Theorem 19. *Let $M(K_{1,n})$ be the middle graph of $K_{1,n}$. Then*

$$b_s(M(K_{1,n})) = n.$$

Proof. As $\gamma_s(K_{1,n}) = 1$, the proof is similar to that of Theorem 14. □

Theorem 20. *If G is a graph of order n and if G includes an induced subgraph as a star, then*

$$b_s(G) = \left\lceil \frac{m(\deg_{n-1}(v))}{2} \right\rceil,$$

where $m(\deg_{n-1}(v))$ denotes the number of vertices of G of degree $n-1$.

Proof. For such a graph G, we have $\gamma_s(G) = 1$. In order to increase $\gamma_s(G)$, the edges that are incident to the pairs of vertices of maximum degree, should be deleted, yielding a decrease of 1 in the degree of each vertex of maximum degree. □

Theorem 21. *If G is a nonempty connected graph of order n with $\alpha(G) = 1$, then*

$$b(G) = 1, \qquad b_w(G) = n - 1, \qquad b_s(G) = 1.$$

Proof. If $\alpha(G) = 1$, then, without loss of generality, G has a star as a spanning subgraph $b(G) = b_s(G) = 1$ and $b_w(G) = 1$. Hence we have $b(G) = b_s(G) = 1$ and $b_w(G) = 1$. □

4. CONCLUSIONS

Graph theory has seen an explosive growth due to interaction with areas like computer science. Perhaps the fastest growing area within graph theory is the study of domination. There are several types of domination parameters depending upon the nature of domination and the nature of dominating set. These parameters are strong-weak domination, bondage, strong-weak bondage numbers. We have given the exact values of the strong-weak domination number of the middle graph of any graph. Then several results for the bondage, strong-weak bondage of middle graphs are obtained.

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