

## ON WEAK SUBDIFFERENTIALS, DIRECTIONAL DERIVATIVES, AND RADIAL EPIDERIVATIVES FOR NONCONVEX FUNCTIONS\*

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**Abstract.** In this paper we study relations between the directional derivatives, the weak subdifferentials, and the radial epiderivatives for nonconvex real-valued functions. We generalize the well-known theorem that represents the directional derivative of a convex function as a pointwise maximum of its subgradients for the nonconvex case. Using the notion of the weak subgradient, we establish conditions that guarantee equality of the directional derivative to the pointwise supremum of weak subgradients of a nonconvex real-valued function. A similar representation is also established for the radial epiderivative of a nonconvex function. Finally the equality between the directional derivatives and the radial epiderivatives for a nonconvex function is proved. An analogue of the well-known theorem on necessary and sufficient conditions for optimality is drawn without any convexity assumptions.

**Key words.** weak subdifferential, radial epiderivative, directional derivative, nonconvex analysis, optimality condition

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**1. Introduction.** The notion of the directional derivative plays an important role in optimization. Many optimality conditions for convex and nonconvex optimization problems are established by using this notion. There are many attempts to generalize the differentiability property by introducing different notions of differentials, subdifferentials, generalized subdifferentials, etc. (see, for example, [7]). The notion of a subgradient became a very convenient tool in convex analysis.

There are many difficulties in the investigation of nonconvex problems. The main ingredient is the method of supporting the nonconvex set under consideration. Nonconvexity may arise in many different forms, and each case may require a special approach.

In recent years many problems of nonconvex optimization have been studied in the framework of the so-called abstract convexity. Abstract convexity suggests a variety of approaches which can be used to analyze different nonconvex problems. It generalizes the existing supporting philosophy for convex sets and suggests different ways to support nonconvex sets by using a suitable class of real functions alternatively to the class of linear functions used in convex analysis. Abstract convexity has found many applications in the study of problems of mathematical analysis and optimization. The books by Pallaschke and Rolewicz [12] and by Singer [17] contain detailed presentations of many results of the abstract convex analysis, which are concentrated around notions of subdifferentials and dualities. Some special nonlinear analogues of Lagrange and penalty functions in nonconvex single-objective optimization are studied in the

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book by Rubinov [13] (see also [14, 15, 16]). These investigations demonstrate the importance of finding a concrete class of functions defining special nonlinear supporting surfaces which are suitable to analyze a given nonconvex problem.

The purpose of this paper is to investigate differentiability properties of a wide class of nonconvex functions by using the weak subdifferentials. The concept of the weak subdifferential is introduced by Azimov and Gasimov [1, 2]. This notion allows one to analytically express the special class of conic surfaces which are supporting to the given nonconvex set, and thus to investigate some differentiability and optimality properties in the nonconvex case.

By using the similar idea, an efficient derivative-free algorithm was developed in [8]. This algorithm has been generalized then by Gasimov and Rubinov [9].

The theorem on the representation of the directional derivative of a convex function as a pointwise maximum of subgradients of this function is one of the central theorems in convex analysis (see, for example, [5, 6]).

In this paper we generalize this theorem to the nonconvex case by using the notion of the weak subdifferential. We introduce a special class of directionally differentiable functions (without any convexity assumptions) and express the directional derivative as a pointwise supremum of the weak subgradients. It is also remarkable that the presented class of functions contains convex functions.

The relation obtained between the directional derivative and the weak subdifferentials is strengthened by using a notion of the radial epiderivative, which turns out to be a natural generalization of the directional derivative for the considered class of functions.

In this paper we use the concept of the radial epiderivative introduced in [11], where using this notion, some characterization theorems for proper and weak minimizers in set-valued optimization are presented without any convexity and boundedness conditions. In [11] the radial epiderivatives of real-valued functions are also investigated and some relation with the weak subdifferential is established. Note that in the convex case similar relations are obtained by using the well-known theorems about the existence of supporting hyperplanes for convex sets and consequently the subdifferentiability property of convex functions [10]. However, when the function under consideration is not convex, one cannot guarantee the existence of a supporting hyperplane to the epigraph of such a function and as a result cannot guarantee the subdifferentiability of this function.

Optimality conditions in nonconvex set-valued optimization was earlier investigated by Flores-Bazan in [3, 4] with the help of a notion of radial epiderivative introduced by the author in different settings. By using these definitions, Flores-Bazan obtained optimality conditions for ideal and weak-minimal points. The definitions of the radial epiderivatives used by Flores-Bazan in [3] and [4] assume the existence of the infimum of values of the set-valued map under consideration. The main characterization theorem given in [3] is proved under the assumption that the cone  $P$  defining a partial ordering on the space  $Y$  is a convex pointed cone with the property  $P \cup (-P) = Y$  (see [3, Theorem 3.9]). Although this assumption was considered as restrictive by the author (see [4]), in the special case when the main assumptions made by Flores-Bazan are satisfied, the definitions of the radial epiderivatives given in [3] and [11] coincide (see [3, Theorem 3.6] and [11, Theorem 3.5]).

In this paper we continue the investigations begun in [11], where the classical supporting philosophy based on hyperplanes is generalized by using the special class of conic surfaces. These conic surfaces are analytically represented as graphs of some superlinear functions.

The theorem representing the radial epiderivative as a pointwise supremum of weak subgradients is established. This theorem leads to the equality between the directional derivatives and the radial epiderivatives for a nonconvex function. As a result of these representations, the well-known theorem on necessary and sufficient conditions for optimality is drawn without any convexity assumption.

The paper is organized as follows. The definition of the weak subdifferentials is presented in the following section. In this section we give some important properties of the weak subdifferentials and their relation with the directional derivatives. The definition and properties of the radial epiderivatives are given in section 3. The main results on the representation of the directional derivatives and the radial epiderivatives as a pointwise supremum of the weak subgradients, and the necessary and sufficient optimality conditions are presented in section 4. Finally, section 5 presents some conclusions.

**2. Weak subdifferentials.** The notion of the weak subdifferential, which is a generalization of the classic subdifferential, was introduced by Azimov and Gasimov [1, 2]. By using this notion, a collection of zero duality gap conditions for a wide class of nonconvex optimization problems were derived. In this section we give some important properties of the weak subdifferentials and study some relationships between the weak subdifferentials and the directional derivatives in the nonconvex case. We recall the concept of the supporting cones and the weak subdifferentials (see [1, 2, 11]).

Let  $(X, \|\cdot\|_X)$  be a real normed space, and let  $X^*$  be the topological dual of  $X$ . Let  $(x^*, c) \in X^* \times \mathbb{R}_+$ , where  $\mathbb{R}_+$  is the set of nonnegative real numbers. We define the conic surface  $C(\bar{x}; x^*, c) \subset X$  with vertex at  $\bar{x} \in X$  as follows:

$$(2.1) \quad C(\bar{x}; x^*, c) = \{x \in X : \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| = 0\}.$$

Then the corresponding upper- and lower-conic halfspaces are, respectively, defined as

$$(2.2) \quad C^+(\bar{x}; x^*, c) = \{x \in X : \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| \leq 0\}$$

and

$$(2.3) \quad C^-(\bar{x}; x^*, c) = \{x \in X : \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| \geq 0\}.$$

Note that if  $c = 0$ , the conic surface  $C(\bar{x}; x^*, c)$  becomes a hyperplane. Hence the supporting cone defined below is a simple generalization of the supporting hyperplane.

**DEFINITION 2.1.**  $C(\bar{x}; x^*, c)$  is called the supporting cone to the set  $S \subset X$  if  $S \subset C^+(\bar{x}; x^*, c)$  (or  $S \subset C^-(\bar{x}; x^*, c)$ ) and  $\text{cl}(S) \cap C(\bar{x}; x^*, c) \neq \emptyset$ .

It is clear that the lower-conic halfspace  $C^-(\bar{x}; x^*, c)$  is a convex cone with vertex at  $\bar{x}$ .

**DEFINITION 2.2.** Let  $F : X \rightarrow \mathbb{R}$  be a single-valued function, and let  $\bar{x} \in X$  be the given point where  $F(\bar{x})$  is finite. A pair  $(x^*, c) \in X^* \times \mathbb{R}_+$  is called the weak subgradient of  $F$  at  $\bar{x}$  if

$$(2.4) \quad F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| \text{ for all } x \in X.$$

The set

$$\partial^w F(\bar{x}) = \{(x^*, c) \in X^* \times \mathbb{R}_+ : F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| \text{ for all } x \in X\}$$

of all weak subgradients of  $F$  at  $\bar{x}$  is called the weak subdifferential of  $F$  at  $\bar{x}$ . If  $\partial^w F(\bar{x}) \neq \emptyset$ , then  $F$  is called weakly subdifferentiable at  $\bar{x}$ . If (2.4) is satisfied only

for  $x \in S$ , where  $S \subset X$ , then we say that  $F$  is weakly subdifferentiable at  $\bar{x}$  on  $S$ . The weak subdifferential of  $F$  at  $\bar{x}$  on  $S$  will be denoted by  $\partial_S^w F(\bar{x})$ .

*Remark 2.3.* It is obvious that, when  $F$  is subdifferentiable at  $\bar{x}$  (in the classical sense), then  $F$  is also weakly subdifferentiable at  $\bar{x}$ ; that is, if  $x^* \in \partial F(\bar{x})$ , then by definition  $(x^*, c) \in \partial^w F(\bar{x})$  for every  $c \geq 0$ . It follows from Definition 2.2 that the pair  $(x^*, c) \in X^* \times \mathbb{R}_+$  is a weak subgradient of  $F$  at  $\bar{x} \in X$  if there is a continuous (superlinear) concave function

$$(2.5) \quad g(x) = \langle x^*, x - \bar{x} \rangle + F(\bar{x}) - c \|x - \bar{x}\|$$

such that  $g(x) \leq F(x)$  for all  $x \in X$  and  $g(\bar{x}) = F(\bar{x})$ . The set  $\text{hypo}(g) = \{(x, \alpha) \in X \times \mathbb{R} : g(x) \geq \alpha\}$  is a closed convex cone in  $X \times \mathbb{R}$  with vertex at  $(\bar{x}, F(\bar{x}))$ . Indeed,

$$\begin{aligned} & \text{hypo}(g) - (\bar{x}, F(\bar{x})) \\ &= \{(x - \bar{x}, \alpha - F(\bar{x})) \in X \times \mathbb{R} : \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| \geq \alpha - F(\bar{x})\} \\ &= \{(u, \beta) \in X \times \mathbb{R} : \langle x^*, u \rangle - c \|u\| \geq \beta\}. \end{aligned}$$

Thus, it follows from (2.4) and (2.5) that

$$\text{graph}(g) = \{(x, \alpha) \in X \times \mathbb{R} : g(x) = \alpha\}$$

is a conic surface which is a supporting cone to

$$\text{epi}(F) = \{(x, \alpha) \in X \times \mathbb{R} : F(x) \leq \alpha\}$$

at the point  $(\bar{x}, F(\bar{x}))$  in the sense that

$$\text{epi}(F) \subset \text{epi}(g), \text{ and } \text{cl}(\text{epi}(F)) \cap \text{graph}(g) \neq \emptyset.$$

The following theorem describes an important property of the weak subdifferential.

**THEOREM 2.4.** *Let the weak subdifferential  $\partial^w F(\bar{x})$  of the function  $F : X \rightarrow \mathbb{R}$  not be empty. Then the set  $\partial^w F(\bar{x})$  is convex and closed.*

*Proof.* Let  $\{(x_n^*, c_n)\} \subset \partial^w F(\bar{x})$ , and let  $(x_n^*, c_n) \rightarrow (x_0^*, c_0)$ . Suppose to the contrary that  $(x_0^*, c_0) \notin \partial^w F(\bar{x})$ . Then there exist an element  $z \in X$  and a positive number  $\varepsilon$  such that

$$x_0^*(z - \bar{x}) - c_0 \|z - \bar{x}\| = F(z) - F(\bar{x}) + \varepsilon,$$

and by the inclusion  $\{(x_n^*, c_n)\} \subset \partial^w F(\bar{x})$ ,

$$x_n^*(z - \bar{x}) - c_n \|z - \bar{x}\| \leq F(z) - F(\bar{x}).$$

By subtracting the second expression from the first one, we obtain

$$(x_0^* - x_n^*)(z - \bar{x}) - (c_0 - c_n) \|z - \bar{x}\| \geq \varepsilon,$$

which implies by letting to the limit as  $n \rightarrow \infty$  that  $0 \geq \varepsilon$ . The convexity of  $\partial^w F(\bar{x})$  is obvious.  $\square$

It follows from Remark 2.3 and from Definition 2.2 that the class of weakly subdifferentiable functions are essentially larger than the class of subdifferentiable functions. Azimov and Gasimov [1] showed that certain subclasses of lower (locally) Lipschitz functions are weakly subdifferentiable.

Now we present the definition of lower Lipschitz functions.

DEFINITION 2.5. A function  $F : X \rightarrow (-\infty, +\infty]$  is called lower locally Lipschitz at  $\bar{x} \in X$  if there exist a nonnegative number  $L$  (Lipschitz constant) and a neighborhood  $N(\bar{x})$  of  $\bar{x}$  such that

$$(2.6) \quad F(x) - F(\bar{x}) \geq -L \|x - \bar{x}\| \quad \text{for all } x \in N(\bar{x}).$$

If the above inequality holds for all  $x \in X$ , then  $F$  is called lower Lipschitz at  $\bar{x}$  with the Lipschitz constant  $L$ .

The following theorem proved in [2] characterizes some classes of weakly subdifferentiable functions. It follows from the definition that the class of weakly subdifferentiable functions includes those nonconvex functions whose epigraph can be supported from below by a graph of some superlinear function. In the theory of abstract convexity some classes of functions that can be used in nonconvex analysis, alternatively to the class of linear functions used in convex analysis, are studied in a general setting (see, for example, [12, 13]). By using a generalized supporting philosophy, different generalizations of subgradients are presented in the framework of abstract convexity. In this setting, the following theorem can be compared to Proposition 2.1.6 given in [12].

THEOREM 2.6. For any function  $F : X \rightarrow (-\infty, +\infty]$  and any point  $\bar{x}$ , where  $F(\bar{x})$  is finite, the following properties are equivalent to each other:

- (i)  $F$  is weakly subdifferentiable at  $\bar{x}$ ,
- (ii)  $F$  is lower Lipschitz at  $\bar{x}$ , and
- (iii)  $F$  is lower locally Lipschitz at  $\bar{x}$ , and there exist numbers  $p \geq 0$  and  $q$  such that

$$F(x) \geq -p \|x\| + q \quad \text{for all } x \in X.$$

The following two lemmas deal with weak subdifferentiability of positively homogeneous functions which will be used in the proof of main theorems in section 4.

LEMMA 2.7. Let  $F : X \rightarrow \mathbb{R}$  be a positively homogeneous function bounded from below on some neighborhood of zero. Then  $F$  is weakly subdifferentiable at  $0_X$ .

Proof. We will show that under the hypotheses of the lemma,  $F$  is lower Lipschitz at  $0_X$ .

By the hypotheses,  $F$  is bounded from below on the unit sphere: There exists a constant  $l$  such that

$$F(z) \geq l \text{ for all } z \in S_1 = \{z \in X : \|z\| = 1\}.$$

Let  $x \in X, x \neq 0_X$ , be an arbitrary element. Then there exist a nonnegative number  $t$  and an element  $z \in S_1$  such that  $x = tz$ . We have:

$$(2.7) \quad F(x) - F(0) = F(x) = F(tz) = tF(z) \geq tl = tl\|z\| = l\|tz\| = l\|x\|.$$

First consider the case that  $l \geq 0$ . In this case (2.7) implies

$$F(x) - F(0) \geq l\|x\| \geq -L\|x\|,$$

where  $L > 0$  is an arbitrary number.

In the case where  $l < 0$ , by taking  $L = -l > 0$  from (2.7) we have

$$F(x) - F(0) \geq l\|x\| = -L\|x\|.$$

Hence we obtain that  $f$  is lower Lipschitz at  $0_X$ . Then the assertion of the lemma follows from Theorem 2.6.  $\square$

LEMMA 2.8. *Let  $F : X \rightarrow \mathbb{R}$  be a positively homogeneous and continuous function. Then  $F$  is weakly subdifferentiable at  $0_X$ .*

*Proof.* As  $F$  is continuous,  $l = \min_{x \in S_1} F(x)$  is a finite number. Then the assertion of the lemma follows from Lemma 2.7.  $\square$

Remark 2.9. Let  $F : X \rightarrow \mathbb{R}$  be weakly subdifferentiable at  $\bar{x} \in X$ . If  $F$  is directionally differentiable at this point in direction  $h \in X$ , then for an arbitrary  $(x^*, c) \in \partial^w F(\bar{x})$  we have

$$F'(\bar{x})(h) = \lim_{\lambda \rightarrow 0^+} \frac{F(\bar{x} + \lambda h) - F(\bar{x})}{\lambda} \geq \langle x^*, h \rangle - c \|h\|.$$

Thus, it follows that

$$(2.8) \quad F'(\bar{x})(h) \geq \sup\{\langle x^*, h \rangle - c \|h\| : (x^*, c) \in \partial^w F(\bar{x})\}.$$

The following example demonstrates that, for some functions  $F$ , the relation in (2.8) may be satisfied as equality. One of the main goals of the present paper is to investigate such class of functions  $F$ .

Example 2.10. Let  $X = \mathbb{R}$ , and let  $F(x) = -|x|$ . Then  $F'(0)(h) = -|h|$ . On the other hand, it follows from the definition of the weak subdifferential that

$$(2.9) \quad (a, c) \in \partial^w F(0) \Leftrightarrow (a, c) \in \mathbb{R} \times \mathbb{R}_+ \text{ and } -|x| \geq ax - c|x| \text{ for all } x \in \mathbb{R}.$$

Hence, the weak subdifferential can explicitly be written as

$$(2.10) \quad \partial^w F(0) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : |a| \leq c - 1\}.$$

It obviously follows from (2.9) that

$$(2.11) \quad F'(\bar{x})(h) = \max\{ah - c \|h\| : (a, c) \in \partial^w F(0)\} = -|h|.$$

The next example demonstrates the case for which the inequality in (2.8) is strong.

Example 2.11. Let again  $X = \mathbb{R}$ , and let  $F(x) = |x^2 - 1|$ . Then  $F'(1)(h) = 2|h|$  for all  $h \in \mathbb{R}$ , and  $\partial^w F(1) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : -c \leq a \leq c + 2\}$ . While

$$(2.12) \quad \max\{ah - c|h| : (a, c) \in \partial^w F(1)\} = \begin{cases} 0 & \text{if } h < 0, \\ 2h & \text{if } h \geq 0. \end{cases}$$

In section 4 we will present some conditions that guarantee the equality in (2.8).

**3. Radial epiderivatives.** In this section we study properties of radial epiderivatives for real-valued functions.

We first give a definition of the radial epiderivative in a general setting.

DEFINITION 3.1. *Let  $U$  be a nonempty subset of a real normed space  $(Z, \|\cdot\|_Z)$ , and let  $\bar{z} \in \text{cl}(U)$  (closure of  $U$ ) be a given element. The closed radial cone  $R(U, \bar{z})$  of  $U$  at  $\bar{z} \in \text{cl}(U)$  is the set of all  $z \in Z$  such that there are  $\lambda_n > 0$  and a sequence  $(z_n)_{n \in \mathbb{N}} \subset Z$  with  $\lim_{n \rightarrow +\infty} z_n = z$  so that  $\bar{z} + \lambda_n z_n \in U$  for all  $n \in \mathbb{N}$ .*

Note that the closed radial cone can equivalently be defined also as follows.

DEFINITION 3.2. *Let  $U$  be a nonempty subset of a real normed space  $(Z, \|\cdot\|_Z)$ . The closed radial cone  $R(U, \bar{z})$  of  $U$  at  $\bar{z} \in \text{cl}(U)$  is the set of all  $z \in Z$  such that there are  $\lambda_n > 0$  and a sequence  $(z_n)_{n \in \mathbb{N}} \subset U$  with  $\lim_{n \rightarrow +\infty} \lambda_n(z_n - \bar{z}) = z$ .*

It follows from these definitions that

$$(3.1) \quad R(U, \bar{z}) = \text{cl}(\text{cone}(U - \bar{z})),$$

where *cone* denotes the conic hull of a set, which is the smallest closed cone containing  $(U - \bar{z})$ .

DEFINITION 3.3. Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be real normed spaces, let  $S$  be a nonempty subset of  $X$ , and let  $F : S \rightarrow 2^Y$  be a set-valued map.

(i) The set

$$(3.2) \quad \text{graph}(F) = \{(x, y) \in X \times Y : x \in S, y \in F(x)\}$$

is called the graph of  $F$ .

(ii) The set

$$(3.3) \quad \text{dom}(F) = \{x \in X : F(x) \neq \emptyset\}$$

is called the domain of  $F$ .  $F$  is said to be proper if  $\text{dom}(F) \neq \emptyset$ .

(iii) Let  $Y$  be partially ordered by a convex cone  $C \subset Y$ . The set

$$(3.4) \quad \text{epi}(F) = \{(x, y) \in X \times Y \mid x \in S, y \in F(x) + C\}$$

is called the epigraph of  $F$ .

DEFINITION 3.4. Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be real normed spaces, let  $Y$  be partially ordered by a convex cone  $C \subset Y$ , let  $S$  be a nonempty subset of  $X$ , and let  $F : S \rightarrow 2^Y$  be a set-valued map. Let a pair  $(\bar{x}, \bar{y}) \in \text{graph}(F)$  be given. A single-valued map  $D_r F(\bar{x}, \bar{y}) : X \rightarrow Y$  whose epigraph equals the radial cone to the epigraph of  $F$  at  $(\bar{x}, \bar{y})$ , i.e.,

$$(3.5) \quad \text{epi}(D_r F(\bar{x}, \bar{y})) = R(\text{epi}(F), (\bar{x}, \bar{y}))$$

is called the radial epiderivative of  $F$  at  $(\bar{x}, \bar{y})$ .

The following two theorems from [11] provide some properties of the radial epiderivatives and the weak subdifferentials.

THEOREM 3.5. Let  $(X, \|\cdot\|_X)$  be a real normed space, and let  $\bar{x} \in X$  be the given element. Let in addition,  $F : X \rightarrow \mathbb{R}$  be a single-valued function being weakly subdifferentiable at  $\bar{x}$ . Then the radial epiderivative  $D_r F(\bar{x}, \bar{y})$  is given as

$$(3.6) \quad D_r F(\bar{x}, \bar{y})(x) = \min\{y \in \mathbb{R} : (x, y) \in R(\text{epi}(F), (\bar{x}, \bar{y}))\} \quad \text{for all } x \in X,$$

where  $\bar{y} = F(\bar{x})$ .

THEOREM 3.6. Let the assumptions of Theorem 3.5 be satisfied. Then

$$(3.7) \quad D_r F(\bar{x}, \bar{y})(h) \geq \sup\{\langle x^*, h \rangle - c \|h\| : (x^*, c) \in \partial^w F(\bar{x})\} \quad \text{for all } h \in X,$$

where  $\bar{y} = F(\bar{x})$ .

The following lemma establishes the conditions under which the radial epiderivative is positively homogeneous.

LEMMA 3.7. Let  $F : X \rightarrow \mathbb{R}$  be a single-valued function having radial epiderivative  $D_r F(\bar{x}, \bar{y})$  defined by (3.6). Then the radial epiderivative is a positively homogeneous function.

*Proof.* Let  $t$  be a positive real number. Then by (3.6)

$$D_r F(\bar{x}, \bar{y})(tx) = \min\{y \in \mathbb{R} : (tx, y) \in R(\text{epi}(F), (\bar{x}, \bar{y}))\} \quad \text{for all } x \in X,$$

or

$$D_r F(\bar{x}, \bar{y})(tx) = \min\{y \in \mathbb{R} : t(x, y/t) \in R(\text{epi}(F), (\bar{x}, \bar{y}))\} \quad \text{for all } x \in X.$$

Since  $R(\text{epi}(F), (\bar{x}, \bar{y}))$  is a cone, the last relation can be written also as

$$D_r F(\bar{x}, \bar{y})(tx) = \min\{y \in \mathbb{R} : (x, y/t) \in R(\text{epi}(F), (\bar{x}, \bar{y}))\} \quad \text{for all } x \in X.$$

Now, by letting  $z = y/t$ , we obtain

$$D_r F(\bar{x}, \bar{y})(tx) = \min\{zt \in \mathbb{R} : (x, z) \in R(\text{epi}(F), (\bar{x}, \bar{y}))\} \quad \text{for all } x \in X$$

or

$$D_r F(\bar{x}, \bar{y})(tx) = t \cdot \min\{z \in \mathbb{R} : (x, z) \in R(\text{epi}(F), (\bar{x}, \bar{y}))\} \quad \text{for all } x \in X,$$

which leads to the assertion

$$D_r F(\bar{x}, \bar{y})(tx) = t D_r F(\bar{x}, \bar{y})(x) \quad \text{for all } x \in X. \quad \square$$

**4. Main results.** In this section we prove that under some mild conditions, the inequalities in (2.8) and (3.7) are satisfied as equalities. First we present the following theorem, which is a generalization of the theorem on necessary and sufficient conditions for subgradients through directional derivatives of a convex function (see [6, Proposition 3.1.16, page 45]).

**THEOREM 4.1.** *Let  $(X, \|\cdot\|_X)$  be a real normed space, and let function  $F : X \rightarrow \mathbb{R}$  be directionally differentiable at  $\bar{x} \in X$ .*

(a) *If  $(x^*, c) \in \partial^w F(\bar{x})$ , then*

$$(4.1) \quad F'(\bar{x})(x - \bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| \quad \text{for all } x \in X.$$

(b) *If*

$$(4.2) \quad F(x) - F(\bar{x}) \geq F'(\bar{x})(x - \bar{x}) \quad \text{for all } x \in X$$

and

$$(4.3) \quad F'(\bar{x})(x - \bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| \quad \text{for all } x \in X,$$

then  $(x^*, c) \in \partial^w F(\bar{x})$ .

*Proof.* (a) Let function  $F : X \rightarrow \mathbb{R}$  be directionally differentiable at  $\bar{x} \in X$ , and let  $(x^*, c) \in \partial^w F(\bar{x})$ . Then

$$\begin{aligned} F'(\bar{x})(x - \bar{x}) &= \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (F(\bar{x} + \lambda(x - \bar{x})) - F(\bar{x})) \\ &\geq \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\langle x^*, \lambda(x - \bar{x}) \rangle - c \|\lambda(x - \bar{x})\|) = \langle x^*, x - \bar{x} \rangle - c \|x - \bar{x}\| \quad \text{for all } x \in X. \end{aligned}$$

(b) The proof of (b) becomes clear by combining the conditions (4.2) and (4.3).  $\square$

**COROLLARY 4.2.** *Let function  $F : X \rightarrow \mathbb{R}$  be directionally differentiable at  $\bar{x} \in X$ , and let*

$$F(x) - F(\bar{x}) \geq F'(\bar{x})(x - \bar{x}) \quad \text{for all } x \in X.$$



Then  $F$  is weakly subdifferentiable at  $\bar{x}$  if and only if the directional derivative  $F'(\bar{x})$  is weakly subdifferentiable at  $0_X$  and

$$\partial^w F(\bar{x}) = \partial^w (F'(\bar{x}))(0).$$

*Proof.* Let  $(x^*, c) \in \partial^w F(\bar{x})$ . Then, by Theorem 4.1(a), inequality (4.1) is satisfied. By taking  $h = x - \bar{x}$  in this formula and noting that  $F'(\bar{x})(0) = 0$  (by positive homogeneity), we obtain  $(x^*, c) \in \partial^w (F'(\bar{x}))(0)$ .

Now let  $(x^*, c) \in \partial^w (F'(\bar{x}))(0)$ . By the definition of the weak subdifferential, this means

$$F'(\bar{x})(h) - F'(\bar{x})(0) \geq \langle x^*, h \rangle - c \|h\| \quad \text{for all } h \in X.$$

By taking  $h = x - \bar{x}$ , this leads to the relation (4.3), which implies by Theorem 4.1(b) that  $(x^*, c) \in \partial^w F(\bar{x})$ .  $\square$

The following theorem defines a class of weakly subdifferentiable functions which can be represented as a pointwise supremum of weak subgradients.

**THEOREM 4.3.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a positively homogeneous function bounded from below on some neighborhood of zero. Then  $f$  is weakly subdifferentiable at  $0_{\mathbb{R}^n}$  and*

$$(4.4) \quad f(h) = \sup\{\langle x^*, h \rangle - c \|h\| : (x^*, c) \in \partial^w f(0)\} \quad \text{for all } h \in \mathbb{R}^n.$$

*Proof.* The weak subdifferentiability of  $f$  at  $0_{\mathbb{R}^n}$  is guaranteed by Lemma 2.7.

Let  $h \in \mathbb{R}^n$  be an arbitrary element. Since for  $h = 0$  the equality in (4.4) is obvious, we can consider the case where  $h \neq 0$ . By the positive homogeneity of  $f$ , it is sufficient to prove (4.4) only for elements  $h$  with  $\|h\| = 1$ .

Let  $\varepsilon > 0$  be an arbitrary positive number.

The proof of this theorem is rather technical, and therefore a short overview is given first in order to examine the geometry. We fix the element  $h$  and a positive number  $\varepsilon$ . In part A it is shown that, for each nonnegative number  $c$ , there exists an element  $x_c^*$  such that the function  $g_c(x) = \langle x_c^*, x \rangle - c \|x\|$  is everywhere less than or equal to  $f(h) - \varepsilon$  and that  $g_c$  achieves its maximum value on the unit sphere  $S_1 = \{x \in \mathbb{R}^n : \|x\| = 1\}$  at the point  $x = h$ . This allows us to guarantee the existence of a number  $c_\varepsilon$  in part B such that the corresponding pair  $(x_\varepsilon^*, c_\varepsilon)$  is a weak subgradient of  $f$  at zero. Then all these features are used in part C to complete the proof.

A. Given  $\varepsilon$ , we seek a pair  $(x^*, c)$  such that  $c \geq 0$  and the function

$$g(x) = \langle x^*, x \rangle - c \|x\|$$

satisfies the following two conditions:

(i)  $g(h) = f(h) - \varepsilon$ ;

(ii)  $g'(h)(p) = 0$ , for each vector  $p \in \mathbb{H} = \{p \in \mathbb{R}^n : \langle h, p \rangle = 0\}$ , where  $g'(h)(p)$  is a directional derivative of  $g$  at  $h$  in direction  $p$ .

These conditions ensure that

$$g(x) < f(h) - \varepsilon \quad \text{for all } x \neq h, \|x\| = 1.$$

In other words, these conditions ensure that function  $g$  corresponding to the pair  $(x^*, c)$  (that will be constructed) achieves its maximum on the unit sphere  $S_1 = \{x \in \mathbb{R}^n : \|x\| = 1\}$  at the point  $x = h$ .

Note that

$$(4.5) \quad g'(h)(p) = \langle x^*, p \rangle - c \langle h, p \rangle / \|h\| \quad \text{for all } p \in \mathbb{R}^n.$$

Then, the equality  $g'(h)(p) = 0$  on the subspace  $\mathbb{H}$  leads to the relation

$$(4.6) \quad \langle x^*, p \rangle = 0 \quad \text{for all } p \in \mathbb{H}.$$

Thus we obtain that the vector  $x^*$  must be orthogonal to the subspace  $\mathbb{H}$ . Since  $\mathbb{H}$  is an  $(n-1)$ -dimensional subspace of  $\mathbb{R}^n$ , there exists a set of orthonormal basis vectors  $\{e_1, \dots, e_{n-1}\}$  in  $\mathbb{H}$ . Then, by orthogonality of  $x^*$  to the subspace  $\mathbb{H}$  we have

$$(4.7) \quad \langle x^*, e_j \rangle = 0 \quad \text{for all } j = 1, \dots, n-1.$$

Note now that the condition  $g(h) = f(h) - \varepsilon$  leads to the relation  $\langle x^*, h \rangle - c\|h\| = f(h) - \varepsilon$ . By using the equality  $\|h\| = 1$  and combining this equality with the  $n-1$  relations given in (4.7), we obtain  $n$  equations for  $n+1$  unknown parameters  $(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+$  in the following form:

$$(4.8) \quad \langle x^*, h \rangle = c\|h\| + f(h) - \varepsilon,$$

$$(4.9) \quad \langle x^*, e_j \rangle = 0 \quad \text{for all } j = 1, \dots, n-1.$$

Since the vector  $h$  is chosen to be perpendicular to the subspace  $\mathbb{H}$  and the basis vectors  $e_j$ ,  $j = 1, \dots, n-1$  are orthonormal, we obtain that the vectors  $h, e_1, \dots, e_{n-1}$  are linearly independent, and therefore the system of linear equations given by relations (4.8)–(4.9) has a unique solution  $x^*$  for each  $c$ .

We now find a solution to the system of equations (4.8)–(4.9) explicitly. Recall that the vector  $h$  is orthogonal to the subspace  $\mathbb{H}$ . Therefore we can seek a solution to the set of equations (4.9) in the form  $x^* = \lambda h$ , where  $\lambda$  is an unknown coefficient. By substituting this expression for  $x^*$  in (4.8), we obtain  $\lambda = c + f(h) - \varepsilon$ .

Thus we have obtained a pair  $(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+$  satisfying the conditions (i)–(ii) of part A in the form  $x^* = (c + f(h) - \varepsilon)h$  for any given  $c \geq 0$ .

Therefore, the function  $g$  satisfying (i)–(ii) can be defined as

$$(4.10) \quad g_c(x) = (c + f(h) - \varepsilon)\langle h, x \rangle - c\|x\|,$$

where  $c$  is an arbitrary nonnegative number.

B. Now we show that the number  $c$  in the definition of  $g$  can be chosen large enough such that

$$(4.11) \quad g_c(x) \leq f(x) \quad \text{for all } x \in \mathbb{R}^n.$$

For this aim, since  $g_c$  and  $f$  are both positively homogeneous functions, it is sufficient to show (4.11) only for points  $x$  in the unit sphere  $S_1$ .

Suppose to the contrary that there exist sequences  $\{c_k\}$  with  $c_k \rightarrow +\infty$  and  $\{x_k\} \subset S_1$  such that

$$g_{c_k}(x_k) > f(x_k) \quad \text{for all } k = 1, 2, \dots$$

or

$$(4.12) \quad c_k(\langle h, x_k \rangle - 1) + f(x_k)(\langle h, x_k \rangle - 1) - \varepsilon \langle h, x_k \rangle > 0 \quad \text{for all } k = 1, 2, \dots$$

Without loss of generality we can assume that  $x_k$  is a convergent sequence. Consider two cases.

(C1) Let  $x_k \rightarrow \tilde{x} \neq h$ . In this case, since both  $h$  and  $\tilde{x}$  are in a unit circle, we have  $\langle h, \tilde{x} \rangle - 1 < 0$ . Then, due to the boundedness from below of  $f$  on the unit sphere (by the hypothesis), the relation (4.12) leads to a contradiction for  $k \rightarrow \infty$ .

(C2) Let  $x_k \rightarrow h$ . Since  $\langle h, x_k \rangle - 1 \leq 0$ , for all  $k$ , it follows from (4.12) that

$$f(x_k)(\langle h, x_k \rangle - 1) - \varepsilon \langle h, x_k \rangle > 0.$$

Now, since  $f$  is bounded from below on the unit sphere, by letting to the limit as  $k \rightarrow \infty$ , we obtain  $-\varepsilon > 0$ , which is a contradiction.

Thus (4.11) is proved, and it is shown that given any  $\varepsilon > 0$ , there exists a number  $c_\varepsilon > 0$  such that the function  $g_{c_\varepsilon}$  corresponding to the pair  $(x_\varepsilon^*, c_\varepsilon) = ((c_\varepsilon + f(h) - \varepsilon)h, c_\varepsilon)$ , defined as in (4.10), that is,

$$g_{c_\varepsilon}(x) = (c_\varepsilon + f(h) - \varepsilon)\langle h, x \rangle - c_\varepsilon \|x\|$$

satisfies the following conditions

$$g_{c_\varepsilon}(x) \leq f(x) \text{ for all } x \in \mathbb{R}^n,$$

$$g_{c_\varepsilon}(h) = f(h) - \varepsilon.$$

The first relation, in particular, means that  $(x_\varepsilon^*, c_\varepsilon) \in \partial^w f(0)$ .

C. Recalling that  $\|h\| = 1$  we have

$$\begin{aligned} & \sup\{\langle x^*, h \rangle - c\|h\| : (x^*, c) \in \partial^w f(0)\} \\ & \geq (c_\varepsilon + f(h) - \varepsilon)\langle h, h \rangle - c_\varepsilon \|h\| \\ & = c_\varepsilon + f(h) - \varepsilon - c_\varepsilon = f(h) - \varepsilon. \end{aligned}$$

Since this inequality holds for every  $\varepsilon > 0$ , we conclude that

$$\sup\{\langle x^*, h \rangle - c\|h\| : (x^*, c) \in \partial^w f(0)\} \geq f(h).$$

On the other hand,

$$f(h) \geq \langle x^*, h \rangle - c\|h\| \quad \text{for all } (x^*, c) \in \partial^w f(0),$$

by the definition of the weak subdifferential, which leads to the required equality (4.4). Thus, the proof of the theorem is completed.  $\square$

For the presentation of main theorems of this section, we use the following standard assumption.

*Assumption 4.4.* Let

- function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be directionally differentiable at  $\bar{x} \in \mathbb{R}^n$ ,
- the directional derivative  $F'(\bar{x})$  of  $F$  at  $\bar{x}$  be bounded from below on some neighborhood of  $0_{\mathbb{R}^n}$ , and
- the following apply:

$$(4.13) \quad F(x) - F(\bar{x}) \geq F'(\bar{x})(x - \bar{x}) \quad \text{for all } x \in \mathbb{R}^n.$$

THEOREM 4.5. *Let Assumption 4.4 be satisfied for function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then  $F$  is weakly subdifferentiable at  $\bar{x} \in \mathbb{R}^n$  and*

$$(4.14) \quad F'(\bar{x})(h) = \sup\{\langle x^*, h \rangle - c \|h\| : (x^*, c) \in \partial^w F(\bar{x})\} \quad \text{for all } h \in \mathbb{R}^n.$$

*Proof.* By Assumption 4.4,  $F$  is directionally differentiable at  $\bar{x} \in \mathbb{R}^n$  and the directional derivative  $F'(\bar{x})$  is bounded from below on some neighborhood of  $0_{\mathbb{R}^n}$ . By positive homogeneity of the directional derivative, Theorem 4.3 implies that  $F'(\bar{x})$  is weakly subdifferentiable at  $0_{\mathbb{R}^n}$ , and

$$(4.15) \quad F'(\bar{x})(h) = \sup\{\langle x^*, h \rangle - c \|h\| : (x^*, c) \in \partial^w(F'(\bar{x}))(0)\} \quad \text{for all } h \in \mathbb{R}^n.$$

On the other hand, by Corollary 4.2,  $\partial^w(F'(\bar{x}))(0) = \partial^w F(\bar{x})$ , and therefore the proof is completed.  $\square$

Theorem 4.5 generalizes the well-known result on the representation of the directional derivative of a convex function as a pointwise maximum of subgradients of this function (see, for example, [6, Theorem 3.1.8 (Max formula)]).

The use of the superlinear functions in the definition of weak subgradients, which is given by the sum of a linear functional and a negative multiple of the norm of  $\mathbb{R}^n$ , extends the usual definition of subgradient and thus allows us to investigate nonconvex cases.

It is well known that any convex function in finite-dimensional space  $\mathbb{R}^n$  can be completely characterized by a family of functions that depend on  $n$  parameters. Similar to this, the presented approach shows that a certain class of nonconvex functions can be characterized by a family of functions that depend on  $n + 1$  parameters.

The next theorem demonstrates that, under some mild conditions, the representation similar to that given for the directional derivative is also valid for the radial epiderivative.

THEOREM 4.6. *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a proper function,  $\bar{x} \in \mathbb{R}^n$ , and  $\bar{y} = F(\bar{x})$  be finite. If  $F$  is weakly subdifferentiable at  $\bar{x}$ , then it has a radial epiderivative  $D_r F(\bar{x}, \bar{y})$  at  $(\bar{x}, \bar{y})$  that can be represented by (3.6); that is,*

$$(4.16) \quad D_r F(\bar{x}, \bar{y})(x) = \min\{y \in \mathbb{R} : (x, y) \in R(\text{epi}(F), (\bar{x}, \bar{y}))\} \quad \text{for all } x \in \mathbb{R}^n.$$

*Conversely, if  $F$  has a radial epiderivative  $D_r F(\bar{x}, F(\bar{x}))$  given by (4.16), then  $F$  is weakly subdifferentiable at  $\bar{x}$  and*

$$(4.17) \quad D_r F(\bar{x}, \bar{y})(h) = \sup\{\langle x^*, h \rangle - c \|h\| : (x^*, c) \in \partial^w F(\bar{x})\} \quad \text{for all } h \in \mathbb{R}^n.$$

*Proof.* If  $F$  is weakly subdifferentiable at  $\bar{x}$ , then the existence of a radial epiderivative at  $(\bar{x}, F(\bar{x}))$  and its expression in the form of (4.16) are given by Theorem 3.5.

Now suppose that the radial epiderivative  $D_r F(\bar{x}, \bar{y})$  exists and is given by relation (4.16). It follows from this relation that  $D_r F(\bar{x}, \bar{y})$  is bounded from below on some neighborhood of  $0_{\mathbb{R}^n}$  and that it is a positively homogeneous function (see Lemma 3.7). Therefore, by Lemma 2.7, it is weakly subdifferentiable at  $0_{\mathbb{R}^n}$ .

On the other hand, it follows from the definition of the radial cone that

$$(x - \bar{x}, F(x) - F(\bar{x})) \in R(\text{epi}(F), (\bar{x}, F(\bar{x}))).$$

Therefore, we obtain the following by (4.16):

$$(4.18) \quad D_r F(\bar{x}, F(\bar{x}))(x - \bar{x}) \leq F(x) - F(\bar{x}).$$

Now we show that  $F$  is weakly subdifferentiable at  $\bar{x}$  and

$$(4.19) \quad \partial^w(D_r F(\bar{x}, \bar{y}))(0) \subset \partial^w F(\bar{x}).$$

Let  $(x^*, c) \in \partial^w(D_r F(\bar{x}, \bar{y}))(0)$ . Then

$$D_r F(\bar{x}, \bar{y})(h) \geq \langle x^*, h \rangle - c\|h\| \quad \text{for all } h \in \mathbb{R}^n,$$

or

$$D_r F(\bar{x}, \bar{y})(x - \bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\| \quad \text{for all } x \in \mathbb{R}^n.$$

By (4.18) this implies

$$F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|, \quad \text{for all } x \in \mathbb{R}^n,$$

which means that  $(x^*, c) \in \partial^w F(\bar{x})$ , and hence the relation (4.19) is established.

By Theorem 4.3,

$$D_r F(\bar{x}, \bar{y})(h) = \sup\{\langle x^*, h \rangle - c\|h\| : (x^*, c) \in \partial^w(D_r F(\bar{x}, \bar{y}))(0)\} \quad \text{for all } h \in \mathbb{R}^n.$$

By using the inclusion (4.19) in the last equality, we obtain

$$D_r F(\bar{x}, \bar{y})(h) \leq \sup\{\langle x^*, h \rangle - c\|h\| : (x^*, c) \in \partial^w F(\bar{x})\} \quad \text{for all } h \in \mathbb{R}^n.$$

On the other hand, by Theorem 3.6 the inverse inequality is also true, and therefore the proof is completed.  $\square$

Theorems 4.5 and 4.6 establish the conditions that guarantee the equality between the directional derivative and the radial epiderivative. This is summarized in the following theorem.

**THEOREM 4.7.** *Let function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be given, and let Assumption 4.4 be satisfied. Assume also that the radial epiderivative  $D_r F(\bar{x}, \bar{y})$  of  $F$  exists and is given by (4.16), where  $\bar{y} = F(\bar{x})$ . Then  $F$  is weakly subdifferentiable at  $\bar{x}$  and*

$$(4.20) \quad F'(\bar{x})(h) = D_r F(\bar{x}, \bar{y})(h) = \sup\{\langle x^*, h \rangle - c\|h\| : (x^*, c) \in \partial^w F(\bar{x})\}$$

for all  $h \in \mathbb{R}^n$ .

*Proof.* The proof is immediate from Theorems 4.5 and 4.6.  $\square$

Now we can provide some necessary and sufficient optimality conditions in the nonconvex case. First we give a definition of the starshaped set.

**DEFINITION 4.8.** *A nonempty subset  $S$  of a real linear space is called starshaped with respect to some  $\bar{x} \in S$  if for all  $x \in S$ ,*

$$(4.21) \quad \lambda x + (1 - \lambda)\bar{x} \in S \quad \text{for all } \lambda \in [0, 1].$$

**THEOREM 4.9.** *Let  $S$  be a nonempty subset of  $\mathbb{R}^n$  starshaped with respect to  $\bar{x} \in S$ , and let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a given function. Suppose that  $F$  has a directional derivative at  $\bar{x}$  in every direction  $x - \bar{x}$  with arbitrary  $x \in S$  and that*

$$F(x) - F(\bar{x}) \geq F'(\bar{x})(x - \bar{x}) \quad \text{for all } x \in S.$$

(a) *If  $\bar{x} \in S$  is a minimal point of  $F$  on  $S$ , then*

$$(4.22) \quad \sup\{\langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\| : (x^*, c) \in \partial_S^w F(\bar{x})\} \geq 0 \quad \text{for all } x \in S.$$

(b) If for some  $\bar{x} \in S$  the inequality (4.22) is satisfied, then  $\bar{x}$  is a minimal point of  $F$  on  $S$ .

*Proof.* The proof follows immediately from Theorem 4.7. Either of the equalities

$$(4.23) \quad F'(\bar{x})(x - \bar{x}) = \sup\{(x^*, x - \bar{x}) - c \|x - \bar{x}\| : (x^*, c) \in \partial^w F(\bar{x})\} \text{ for all } x \in S$$

or

$$(4.24) \quad D_r F(\bar{x}, \bar{y})(x - \bar{x}) = \sup\{(x^*, x - \bar{x}) - c \|x - \bar{x}\| : (x^*, c) \in \partial^w F(\bar{x})\} \text{ for all } x \in S$$

given in (4.20) can be used for the proof.

The proof of part (a)—first way. Take any  $x \in S$ . Since  $F$  has a directional derivative at  $\bar{x}$  in direction  $x - \bar{x}$ , we have

$$F'(\bar{x})(x - \bar{x}) = \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (F(\bar{x} + \lambda(x - \bar{x})) - F(\bar{x})).$$

The point  $\bar{x}$  is assumed to be a minimal point of  $F$  on  $S$ , and therefore we get, for sufficiently small  $\lambda > 0$ ,

$$F(\bar{x} + \lambda(x - \bar{x})) \geq F(\bar{x}).$$

Consequently, we obtain

$$(4.25) \quad F'(\bar{x})(x - \bar{x}) \geq 0.$$

Now by using the equality (4.23), we obtain the desired relation.

The proof of part (a)—second way. Since the point  $\bar{x}$  is assumed to be a minimal point of  $F$  on  $S$ , by Theorem 3.6 in [11],  $\bar{x}$  is also a minimal point of  $D_r F(\bar{x}, \bar{y})(x - \bar{x})$  on  $S$ , which implies

$$D_r F(\bar{x}, \bar{y})(x - \bar{x}) \geq 0 \text{ for all } x \in S.$$

Then, by using Theorem 4.7 or the equality (4.24), we obtain the desired relation.

The proof of part (b) is immediate from the hypothesis

$$F(x) - F(\bar{x}) \geq F'(\bar{x})(x - \bar{x}) \text{ for all } x \in S$$

and the relation (4.23).  $\square$

**5. Conclusions.** In this paper, a well-known theorem on the representation of the directional derivative of a convex function as a pointwise maximum of subgradients of this function is generalized to a nonconvex case by using the notion of a weak subgradient.

We introduce a special class of directionally differentiable functions (without convexity assumption) and express the directional derivative as a pointwise supremum of weak subgradients for these functions. It is also remarkable that the presented class includes the class of convex functions.

The main idea behind the weak subgradient is that the classic supporting philosophy based on linear functionals is extended to a special class of superlinear functions. The level set of such a function is a cone, and the use of these cones as supporting surfaces leads to the notion of a weak subgradient. Thus, the use of superlinear functions in the definition of weak subgradients, which is given by the sum of a linear

functional and a negative multiple of the norm, extends the usual definition of the subgradient and hence allows us to investigate nonconvex cases.

It is well known that any convex function can completely be characterized by a family of linear functions. In the case of finite-dimensional space  $\mathbb{R}^n$ , this family depends on  $n$  parameters. The approach presented in this paper shows that some class of nonconvex functions in  $\mathbb{R}^n$  can be characterized by a family of “simple” functions depending on  $n + 1$  parameters.

The representation similar to that obtained for directional derivatives is also established for radial epiderivatives. This representation leads to the equality relation between the directional derivatives and the radial epiderivatives and allows us to claim that the notion of the radial epiderivative is a generalization of the directional derivative for the class of functions introduced in this paper.

Finally, as an application of these representations, a theorem on the necessary and sufficient optimality conditions is drawn without any convexity assumptions.

## REFERENCES

- [1] A. Y. AZIMOV AND R. N. GASIMOV, *On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization*, Int. J. Appl. Math., 1 (1999), pp. 171–192.
- [2] A. Y. AZIMOV AND R. N. GASIMOV, *Stability and duality of nonconvex problems via augmented Lagrangian*, Cybernet. Systems Anal., 3 (2002), pp. 120–130.
- [3] F. FLORES-BAZAN, *Optimality conditions in non-convex set-valued optimization*, Math. Methods Oper. Res., 53 (2001), pp. 403–417.
- [4] F. FLORES-BAZAN, *Radial epiderivatives and asymptotic functions in nonconvex vector optimization*, SIAM J. Optim., 14 (2003), pp. 284–305.
- [5] J. BORWEIN, *A note on the existence of subgradients*, Math. Program., 24 (1982), pp. 225–228.
- [6] J. BORWEIN AND A. S. LEWIS, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, CMS Books in Mathematics, Springer, New York, 2000.
- [7] V. F. DEMYANOV AND A. M. RUBINOV, *The Fundamentals of Nonsmooth Analysis and Quasi-Differentiable Calculus*, Nauka, Moscow, 1990.
- [8] R. N. GASIMOV, *Augmented Lagrangian duality and nondifferentiable optimization methods in nonconvex programming*, J. Global Optim., 24 (2002), pp. 187–203.
- [9] R. N. GASIMOV AND A. M. RUBINOV, *On augmented Lagrangians for optimization problems with a single constraint*, J. Global Optim., 28 (2004), pp. 153–173.
- [10] J. JAHN AND R. RAUH, *Contingent epiderivatives and set-valued optimization*, Math. Methods Oper. Res., 46 (1997), pp. 193–211.
- [11] R. KASIMBEYLI, *Radial epiderivatives and set-valued optimization*, Optimization, 58 (2009), pp. 519–532.
- [12] D. PALLASCHKE AND S. ROLEWICZ, *Foundations of Mathematical Optimization: Convex Analysis without Linearity*, Math. Appl., Kluwer, Dordrecht, 1997.
- [13] A. M. RUBINOV, *Abstract Convexity and Global Optimization*, Kluwer, Dordrecht, 2000.
- [14] A. M. RUBINOV AND R. N. GASIMOV, *The nonlinear and augmented Lagrangians for nonconvex optimization problems with a single constraint*, Appl. Comput. Math., 1 (2002), pp. 142–157.
- [15] A. M. RUBINOV AND R. N. GASIMOV, *Strictly increasing positively homogeneous functions with application to exact penalization*, Optimization, 52 (2003), pp. 1–28.
- [16] A. M. RUBINOV, X. Q. YANG, A. M. BAGIROV, AND R. N. GASIMOV, *Lagrange-type functions in constrained optimization*, J. Math. Sci., 115 (2003), pp. 2437–2505.
- [17] I. SINGER, *Abstract Convex Analysis*, John Wiley and Sons, New York, 1997.