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A NEW PERIODICITY CONCEPT FOR TIME SCALES

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ABSTRACT. By means of the shift operators we introduce a new periodicity concept on time scales. This new approach will enable researchers to investigate periodicity notion on a large class of time scales whose members may not satisfy the condition

there exists a $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$,

which is being currently used. Therefore, the results of this paper open an avenue for the investigation of periodic solutions of q-difference equations and more.

1. Introduction

In the last two decades, theory of time scales has become a very useful tool for the unification of difference and differential equations under dynamic equations on time scales (see [1]–[10], and references therein). A time scale, denoted by $\mathbb{T}$, is a non-empty arbitrary subset of real numbers. To be able to investigate the notion of periodicity of the solutions of dynamic equations on time scales researchers had to first introduce the concept of periodic time scales and then define what it meant for a function to be periodic on such a time scale. To be more specific, we restate the following definitions and introductory examples which can be found in [5], [6], and [10].

Definition 1. ([9]) A time scale $\mathbb{T}$ is said to be *periodic* if there exists a $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$. If $\mathbb{T} \neq \mathbb{R}$, the smallest positive P is called the *period* of the time scale.

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Example 1. The following time scales are periodic.

i) $\mathbb{T} = \mathbb{Z}$ has period $P = 1$,

ii) $\mathbb{T} = h\mathbb{Z}$ has period $P = h$,

iii) $\mathbb{T} = \mathbb{R}$,

iv) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [(2i-1)h, 2ih]$, $h > 0$, has period $P = 2h$,

v) $\mathbb{T} = \{t = k - q^m : \ k \in \mathbb{Z}, \ m \in \mathbb{N}_0\}$, where $0 < q < 1$, has period $P = 1$.

Definition 2. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period P. We say that the function $f\colon \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = nP$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Based on the Definitions 1 and 2, periodicity and existence of periodic solutions of dynamic equations on time scales were studied by various researchers and for first papers on the subject we refer to (see for instance [3]–[6], [9]–[11]).

There is no doubt that a time scale $\mathbb{T}$ which is periodic in the sense of Definition 1 must satisfy

$$t \pm P \in \mathbb{T} \qquad \text{for all} \quad t \in \mathbb{T} \tag{1.1}$$

for a fixed $P > 0$. This property obliges the time scale to be unbounded from above and below. However, these two restrictions prevent us from investigating the periodic solutions of q-difference equations since the time scale

$$\overline{q^{\mathbb{Z}}} = \{q^n : \ q > 1 \ \text{is constant and} \ n \in \mathbb{Z}\} \cup \{0\}$$

which is neither closed under the operation $t \pm P$ for a fixed $P > 0$ nor unbounded below.

The main purpose of this paper is to introduce a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm P$ for a fixed $P > 0$ or to be unbounded. We define our new periodicity concept with the aid of shift operators which are first defined in [1] and then generalized in [2].

2. Shift operators

Next, we give a generalized version of shift operators (see [2]). A limited version of shift operators can be found in [1]. Hereafter, we use the notation $[a, b]_{\mathbb{T}}$ to indicate the time scale interval $[a, b] \cap \mathbb{T}$. The intervals $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$, and $(a, b)_{\mathbb{T}}$ are similarly defined.

Definition 3. Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_\pm \colon [t_0, \infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:

P.1. The functions $\delta_\pm$ are strictly increasing with respect to their second arguments, i.e., if

$$(T_0, t), (T_0, u) \in \mathcal{D}_\pm := \left\{(s,t) \in [t_0, \infty)_\mathbb{T} \times \mathbb{T}^* : \ \delta_\pm(s,t) \in \mathbb{T}^*\right\},$$

then

$$T_0 \le t < u \implies \delta_\pm(T_0, t) < \delta_\pm(T_0, u).$$

P.2. If $(T_1, u), (T_2, u) \in \mathcal{D}_-$ with $T_1 < T_2$, then

$$\delta_-(T_1, u) > \delta_-(T_2, u),$$

and if $(T_1, u), (T_2, u) \in \mathcal{D}_+$ with $T_1 < T_2$, then

$$\delta_+(T_1, u) < \delta_+(T_2, u).$$

P.3. If $t \in [t_0, \infty)_\mathbb{T}$, then $(t, t_0) \in \mathcal{D}_+$ and $\delta_+(t, t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0, t) \in \mathcal{D}_+$ and $\delta_+(t_0, t) = t$ holds.

P.4. If $(s,t) \in \mathcal{D}_\pm$, then $(s, \delta_\pm(s,t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s, \delta_\pm(s,t)) = t$, respectively.

P.5. If $(s,t) \in \mathcal{D}_\pm$ and $(u, \delta_\pm(s,t)) \in \mathcal{D}_\mp$, then $(s, \delta_\mp(u,t)) \in \mathcal{D}_\pm$ and $\delta_\mp(u, \delta_\pm(s,t)) = \delta_\pm(s, \delta_\mp(u,t))$, respectively.

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be *backward and forward shift operators* on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_\mathbb{T}$ in $\delta_\pm(s,t)$ is called the shift size. The values $\delta_+(s,t)$ and $\delta_-(s,t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets $\mathcal{D}_\pm$ are the domains of the shift operators $\delta_\pm$, respectively.

Hereafter, we shall denote by $\mathbb{T}^*$ the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm \colon [t_0, \infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Example 2. Let $\mathbb{T} = \mathbb{R}$ and $t_0 = 1$. The operators

$$\delta_-(s,t) = \begin{cases} t/s & \text{if } t \ge 0, \\ st & \text{if } t < 0, \end{cases} \qquad \text{for} \quad s \in [1, \infty) \tag{2.1}$$

and

$$\delta_+(s,t) = \begin{cases} st & \text{if } t \ge 0, \\ t/s & \text{if } t < 0, \end{cases} \qquad \text{for} \quad s \in [1, \infty) \tag{2.2}$$

are backward and forward shift operators (on the set $\mathbb{R}^* = \mathbb{R} - \{0\}$) associated with the initial point $t_0 = 1$. In the table below, we state different time scales with their corresponding shift operators.

$\mathbb{T}$	t_0	$\mathbb{T}^*$	$\delta_-(s,t)$	$\delta_+(s,t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t-s$	$t+s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t-s$	$t+s$
$q^{\mathbb{Z}}\cup\{0\}$	1	$q^{\mathbb{Z}}$	$\frac{t}{s}$	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$\sqrt{t^2-s^2}$	$\sqrt{t^2+s^2}$

The proof of the next lemma is a direct consequence of Definition 3.

Lemma 1. *Let δ_- and δ_+ be the shift operators associated with the initial point t_0. We have*

(i) $\delta_-(t,t)=t_0$ *for all* $t\in[t_0,\infty)_{\mathbb{T}}$.

(ii) $\delta_-(t_0,t)=t$ *for all* $t\in\mathbb{T}^*$.

(iii) *If* $(s,t)\in\mathcal{D}_+$, *then* $\delta_+(s,t)=u$ *implies* $\delta_-(s,u)=t$. *Conversely, if* $(s,u)\in\mathcal{D}_-$, *then* $\delta_-(s,u)=t$ *implies* $\delta_+(s,t)=u$.

(iv) $\delta_+(t,\delta_-(s,t_0))=\delta_-(s,t)$ *for all* $(s,t)\in\mathcal{D}_+$ *with* $t\geq t_0$.

(v) $\delta_+(u,t)=\delta_+(t,u)$ *for all* $(u,t)\in([t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}})\cap\mathcal{D}_+$.

(vi) $\delta_+(s,t)\in[t_0,\infty)_{\mathbb{T}}$ *for all* $(s,t)\in\mathcal{D}_+$ *with* $t\geq t_0$.

(vii) $\delta_-(s,t)\in[t_0,\infty)_{\mathbb{T}}$ *for all* $(s,t)\in([t_0,\infty)_{\mathbb{T}}\times[s,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$.

(viii) *If* $\delta_+(s,\cdot)$ *is* Δ*-differentiable in its second variable, then* $\delta_+^{\Delta_t}(s,\cdot)>0$.

(ix) $\delta_+(\delta_-(u,s),\delta_-(s,v))=\delta_-(u,v)$ *for all* $(s,v)\in([t_0,\infty)_{\mathbb{T}}\times[s,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$ *and* $(u,s)\in([t_0,\infty)_{\mathbb{T}}\times[u,\infty)_{\mathbb{T}})\cap\mathcal{D}_-$.

(x) *If* $(s,t)\in\mathcal{D}_-$ *and* $\delta_-(s,t)=t_0$, *then* $s=t$.

Proof.

(i) is obtained from P.3–5 since

$$\delta_-(t,t)=\delta_-(t,\delta_+(t,t_0))=t_0 \qquad \text{for all} \quad t\in[t_0,\infty)_{\mathbb{T}}.$$

(ii) is obtained from P.3–P.4 since

$$\delta_-(t_0,t)=\delta_-(t_0,\delta_+(t_0,t))=t \qquad \text{for all} \quad t\in\mathbb{T}^*.$$

Let $u:=\delta_+(s,t)$. By P.4 we have $(s,u)\in\mathcal{D}_-$ for all $(s,t)\in\mathcal{D}_+$, and hence,

$$\delta_-(s,u)=\delta_-(s,\delta_+(s,t))=t.$$

The latter part of (iii) can be done similarly.

We have (iv) since P.3 and P.5 yield

$$\delta_+(t,\delta_-(s,t_0))=\delta_-(s,\delta_+(t,t_0))=\delta_-(s,t).$$

P.3 and P.5 guarantee that

$$t = \delta_+(t, t_0) = \delta_+(t, \delta_-(u, u)) = \delta_-(u, \delta_+(t, u))$$

for all $(u, t) \in ([t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}) \cap \mathcal{D}_+$.

Using (iii) we have

$$\delta_+(u, t) = \delta_+(u, \delta_-(u, \delta_+(t, u))) = \delta_+(t, u).$$

This proves (v).

To prove (vi) and (vii) we use P.1–2 to get

$$\delta_+(s, t) \geq \delta_+(t_0, t) = t \geq t_0$$

for all $(s, t) \in ([t_0, \infty) \times [t_0, \infty)_{\mathbb{T}}) \cap \mathcal{D}_+$ and

$$\delta_-(s, t) \geq \delta_-(s, s) = t_0$$

for all $(s, t) \in ([t_0, \infty)_{\mathbb{T}} \times [s, \infty)_{\mathbb{T}}) \cap \mathcal{D}_-$.

Since $\delta_+(s, t)$ is strictly increasing in its second variable we have (viii) by [8: Corollary 1.16].

(ix) is proven as follows: from P.5 and (v) we have

$$\begin{aligned} \delta_+(\delta_-(u, s), \delta_-(s, v)) &= \delta_-(s, \delta_+(v, \delta_-(u, s))) \\ &= \delta_-(s, \delta_-(u, \delta_+(v, s))) \\ &= \delta_-(s, \delta_+(s, \delta_-(u, v))) \\ &= \delta_-(u, v) \end{aligned}$$

for all $(s, v) \in ([t_0, \infty)_{\mathbb{T}} \times [s, \infty)_{\mathbb{T}}) \cap \mathcal{D}_-$ and $(u, s) \in ([t_0, \infty)_{\mathbb{T}} \times [u, \infty)_{\mathbb{T}}) \cap \mathcal{D}_-$.

Suppose $(s, t) \in \mathcal{D}_- = \{(s, t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_-(s, t) \in \mathbb{T}^*\}$ and $\delta_-(s, t) = t_0$. Then by P.4 we have

$$t = \delta_+(s, \delta_-(s, t)) \in \delta_+(s, t_0) = s.$$

This is (x). The proof is complete. □

Notice that the shift operators $\delta_\pm$ are defined once the initial point $t_0 \in \mathbb{T}^*$ is known. For instance, we choose the initial point $t_0 = 0$ to define shift operators $\delta_\pm(s, t) = t \pm s$ on $\mathbb{T} = \mathbb{R}$. However, if we choose $\lambda \in (0, \infty)$ as the initial point, then the new shift operators associated with λ are defined by $\widetilde{\delta}_\pm(s, t) = t \mp \lambda \pm s$. In terms of $\delta_\pm$ the new shift operators $\widetilde{\delta}_\pm$ can be given as follows

$$\widetilde{\delta}_\pm(s, t) = \delta_\mp(\lambda, \delta_\pm(s, t)).$$

Example 3. In the following, we give some particular time scales with shift operators associated with different initial points to show the change in the formula of shift operators as the initial point changes.

	$\mathbb{T}=\mathbb{N}^{1/2}$		$\mathbb{T}=h\mathbb{Z}$		$\mathbb{T}=2^{\mathbb{N}}$	
t_0	0	λ	0	$h\lambda$	1	2^{λ}
$\delta_-(s,t)$	$\sqrt{t^2-s^2}$	$\sqrt{t^2+\lambda^2-s^2}$	$t-s$	$t+h\lambda-s$	t/s	$2^{\lambda}ts^{-1}$
$\delta_+(s,t)$	$\sqrt{t^2+s^2}$	$\sqrt{t^2-\lambda^2+s^2}$	$t+s$	$t-h\lambda+s$	ts	$2^{-\lambda}ts$

where $\lambda\in\mathbb{Z}_+$, $\mathbb{N}^{1/2}=\left\{\sqrt{n}:\ n\in\mathbb{N}\right\}$, $2^{\mathbb{N}}=\{2^n:\ n\in\mathbb{N}\}$, and $h\mathbb{Z}=\{hn:\ n\in\mathbb{Z}\}$.

3. Periodicity

In the following we propose a new periodicity notion which does not oblige the time scale to be closed under the operation $t\pm P$ for a fixed $P>0$ or to be unbounded.

Definition 4 (Periodicity in shifts)**.** Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0\in\mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be *periodic in shifts* $\delta_\pm$ if there exists a $p\in(t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t)\in\mathcal{D}_\mp$ for all $t\in\mathbb{T}^*$. Furthermore, if

$$P:=\inf\left\{p\in(t_0,\infty)_{\mathbb{T}^*}:\ (p,t)\in\mathcal{D}_\mp\ \text{ for all }\ t\in\mathbb{T}^*\right\}\neq t_0,$$

then P is called the *period* of the time scale $\mathbb{T}$.

The following example indicates that a time scale, periodic in shifts, does not have to satisfy (1.1). That is, a time scale periodic in shifts may be bounded.

Example 4. The following time scales are not periodic in the sense of Definition 1 but periodic with respect to the notion of shift operators given in Definition 4.

(1) $\mathbb{T}_1=\left\{\pm n^2:\ n\in\mathbb{Z}\right\}$,

$$\delta_\pm(P,t)=\begin{cases}\left(\sqrt{t}\pm\sqrt{P}\right)^2 & \text{if } t>0,\\ \pm P & \text{if } t=0,\\ -\left(\sqrt{-t}\pm\sqrt{P}\right)^2 & \text{if } t<0,\end{cases}\qquad P=1,\quad t_0=0,$$

(2) $\mathbb{T}_2=\overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t)=P^{\pm1}t$, $P=q$, $t_0=1$,

(3) $\mathbb{T}_3=\overline{\bigcup_{n\in\mathbb{Z}}[2^{2n},2^{2n+1}]}$, $\delta_\pm(P,t)=P^{\pm1}t$, $P=4$, $t_0=1$,

(4) $\mathbb{T}_4 = \left\{ \frac{q^n}{1+q^n} : \ q > 1 \text{ is constant and } n \in \mathbb{Z} \right\} \cup \{0,1\}$,

$$\delta_{\pm}(P,t) = \frac{q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}{1+q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}, \qquad P = \frac{q}{1+q}.$$

Notice that the time scale $\mathbb{T}_4$ in Example 4 is bounded above and below and $\mathbb{T}_4^* = \left\{ \frac{q^n}{1+q^n} : \ q > 1 \text{ is constant and } n \in \mathbb{Z} \right\}$.

Remark 1. Let $\mathbb{T}$ be a time scale that is periodic in shifts with the period P. Thus, by P.4 of Definition 3 the mapping $\delta_+^P \colon \mathbb{T}^* \to \mathbb{T}^*$ defined by $\delta_+^P(t) = \delta_+(P,t)$ is surjective. On the other hand, we know by P.1 of Definition 3 that shift operators $\delta_\pm$ are strictly increasing in their second arguments. That is, the mapping $\delta_+^P(t) := \delta_+(P,t)$ is injective. Hence, δ_+^P is an invertible mapping with the inverse $\left(\delta_+^P\right)^{-1} = \delta_-^P$ defined by $\delta_-^P(t) := \delta_-(P,t)$.

In next two results, we suppose that $\mathbb{T}$ is a periodic time scale in shifts $\delta_\pm$ with period P and show that the operators $\delta_\pm^P : \mathbb{T}^* \to \mathbb{T}^*$ are commutative with the forward jump operator $\sigma \colon \mathbb{T} \to \mathbb{T}$ given by

$$\sigma(t) := \inf \{ s \in \mathbb{T} : \ s > t \}.$$

That is,

$$\left(\delta_\pm^P \circ \sigma\right)(t) = \left(\sigma \circ \delta_\pm^P\right)(t) \qquad \text{for all} \quad t \in \mathbb{T}^*. \tag{3.1}$$

Lemma 2. *The mapping* $\delta_+^T \colon \mathbb{T}^* \to \mathbb{T}^*$ *preserves the structure of the points in* $\mathbb{T}^*$. *That is,*

$$\sigma(\widehat{t}\,) = \widehat{t} \implies \sigma(\delta_+(P,\widehat{t}\,)) = \delta_+(P,\widehat{t}\,).$$
$$\sigma(\widehat{t}\,) > \widehat{t} \implies \sigma(\delta_+(P,\widehat{t}\,) > \delta_+(P,\widehat{t}\,).$$

Proof. By definition we have $\sigma(t) \geq t$ for all $t \in \mathbb{T}^*$. Thus, by P.1

$$\delta_+(P,\sigma(t)) \geq \delta_+(P,t).$$

Since $\sigma(\delta_+(P,t))$ is the smallest element satisfying

$$\sigma(\delta_+(P,t)) \geq \delta_+(P,t),$$

we get

$$\delta_+(P,\sigma(t)) \geq \sigma(\delta_+(P,t)) \qquad \text{for all} \quad t \in \mathbb{T}^*. \tag{3.2}$$

If $\sigma(\widehat{t}\,) = \widehat{t}$, then (3.2) implies

$$\delta_+(P,\widehat{t}\,) = \delta_+(P,\sigma(\widehat{t}\,)) \geq \sigma(\delta_+(P,\widehat{t}\,)).$$

That is,

$$\delta_+(P,\widehat{t}\,) = \sigma(\delta_+(P,\widehat{t}\,)) \qquad \text{provided} \quad \sigma(\widehat{t}\,) = \widehat{t}.$$

If $\sigma(\widehat{t}) > \widehat{t}$, then by definition of σ we have

$$(\widehat{t}, \sigma(\widehat{t}))_{\mathbb{T}^*} = \varnothing \tag{3.3}$$

and by P.1

$$\delta_+(P, \sigma(\widehat{t})) > \delta_+(P, \widehat{t}).$$

Suppose contrary that $\delta_+(P, \widehat{t})$ is right dense, i.e., $\sigma(\delta_+(P, \widehat{t})) = \delta_+(P, \widehat{t})$. This along with (3.2) implies

$$(\delta_+(P, \widehat{t}), \delta_+(P, \sigma(\widehat{t})))_{\mathbb{T}^*} \neq \varnothing.$$

Pick one element $s \in \big(\delta_+(P, \widehat{t}), \delta_+(P, \sigma(\widehat{t}))\big)_{\mathbb{T}^*}$. Since $\delta_+(P, t)$ is strictly increasing in t and invertible there should be an element $t \in (\widehat{t}, \sigma(\widehat{t}))_{\mathbb{T}^*}$ such that $\delta_+(P, t) = s$. This contradicts (3.3). Hence, $\delta_+(P, \widehat{t})$ must be right scattered, i.e., $\sigma(\delta_+(P, \widehat{t})) > \delta_+(P, \widehat{t})$. The proof is complete. □

Corollary 1. *We have*

$$\delta_+(P, \sigma(t)) = \sigma(\delta_+(P, t)) \qquad \textit{for all} \quad t \in \mathbb{T}^*. \tag{3.4}$$

Thus,

$$\delta_-(P, \sigma(t)) = \sigma(\delta_-(P, t)) \qquad \textit{for all} \quad t \in \mathbb{T}^*. \tag{3.5}$$

P r o o f. The equality (3.4) can be obtained as we did in the proof of preceding lemma. By (3.4) we have

$$\delta_+(P, \sigma(s)) = \sigma(\delta_+(P, s)) \qquad \text{for all} \quad s \in \mathbb{T}^*.$$

Substituting $s = \delta_-(P, t)$ we obtain

$$\delta_+(P, \sigma(\delta_-(P, t))) = \sigma(\delta_+(P, \delta_-(P, t))) = \sigma(t).$$

This and (iii) of Lemma 1 imply

$$\sigma(\delta_-(P, t)) = \delta_-(P, \sigma(t)) \qquad \text{for all} \quad t \in \mathbb{T}^*.$$

The proof is complete. □

Observe that (3.4) along with (3.5) yields (3.1).

Definition 5 (Periodic function in shifts $\delta_\pm$)**.** Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real valued function f defined on $\mathbb{T}^*$ is *periodic in shifts* $\delta_\pm$ if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T, t) \in \mathcal{D}_\pm \quad \text{and} \quad f(\delta_\pm^T(t)) = f(t) \qquad \text{for all} \quad t \in \mathbb{T}^*, \tag{3.6}$$

where $\delta_\pm^T(t) := \delta_\pm(T, t)$. The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ such that (3.6) holds is called the period of f.

Example 5. By Definition 4 we know that the set of reals $\mathbb{R}$ is periodic in shifts $\delta_\pm$ defined by (2.1)–(2.2) associated with the initial point $t_0 = 1$. The function

$$f(t) = \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi\right), \qquad t \in \mathbb{R}^* := \mathbb{R} - \{0\}$$

is periodic in shifts $\delta_\pm$ defined by (2.1)–(2.2) with the period $T = 4$ since

$$\begin{aligned} f(\delta_\pm(T,t)) &= \begin{cases} f\left(t4^{\pm 1}\right) & \text{if } t \geq 0, \\ f\left(t/4^{\pm 1}\right) & \text{if } t < 0, \end{cases} \\ &= \sin\left(\frac{\ln|t| \pm 2\ln(1/2)}{\ln(1/2)}\pi\right) \\ &= \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi \pm 2\pi\right) \\ &= \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi\right) \\ &= f(t) \end{aligned}$$

for all $t \in \mathbb{R}^*$ (see Figure 1).

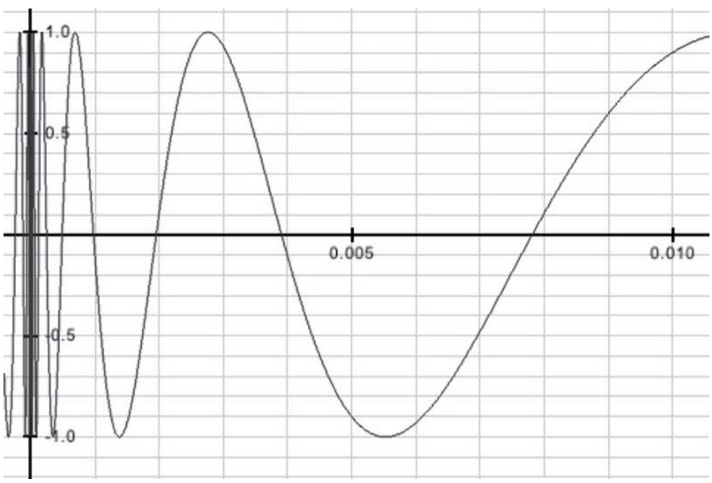

FIGURE 1. Graph of $f(t) = \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi\right)$

Example 6. The time scale $\overline{q^{\mathbb{Z}}} = \left\{q^n : n \in \mathbb{Z} \text{ and } q > 1\right\} \cup \{0\}$ is periodic in shifts $\delta_\pm(P,t) = P^{\pm 1}t$ with the period $P = q$. The function f defined by

$$f(t) = (-1)^{\frac{\ln t}{\ln q}}, \qquad t \in q^{\mathbb{Z}} \tag{3.7}$$

is periodic in shifts $\delta_{\pm}$ with the period $T = q^2$ since $\delta_{+}(q^2, t) \in \overline{q^{\mathbb{Z}}}^{*} = q^{\mathbb{Z}}$ and

$$f\left(\delta_{\pm}(q^2, t)\right) = (-1)^{\frac{\ln t}{\ln q} \pm 2} = (-1)^{\frac{\ln t}{\ln q}} = f(t)$$

for all $t \in q^{\mathbb{Z}}$. However, f is not periodic in the sense of Definition 2 since there is no any positive number T so that $f(t \pm T) = f(t)$ holds.

In the following, we introduce Δ-periodic function in shifts. For a detailed information on Δ-derivative and Δ-integration we refer to [7] and [8].

Definition 6 (Δ-periodic function in shifts $\delta_{\pm}$)**.** Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period P. We say that a real valued function f defined on $\mathbb{T}^*$ is *Δ-periodic in shifts $\delta_{\pm}$* if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T, t) \in \mathcal{D}_{\pm} \qquad \text{for all} \quad t \in \mathbb{T}^*, \tag{3.8}$$

$$\text{the shifts} \quad \delta_{\pm}^{T} \quad \text{are} \quad \Delta\text{-differentiable with rd-continuous derivatives,} \tag{3.9}$$

and

$$f(\delta_{\pm}^{T}(t))\delta_{\pm}^{\Delta T}(t) = f(t) \tag{3.10}$$

for all $t \in \mathbb{T}^*$, where $\delta_{\pm}^{T}(t) := \delta_{\pm}(T, t)$. The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ such that (3.8)–(3.10) hold is called the period of f.

Notice that Definition 5 and Definition 6 give the classic periodicity definition (i.e. Definition 2) on time scales whenever $\delta_{\pm}^{T}(t) = t \pm T$ are the shifts satisfying the assumptions of Definition 5 and Definition 6.

Example 7. The real valued function $g(t) = 1/t$ defined on $2^{\mathbb{Z}} = \{2^n : \ n \in \mathbb{Z}\}$ is *Δ-periodic in shifts* $\delta_{\pm}(T, t) = T^{\pm 1} t$ with the period $T = 2$ since

$$f\left(\delta_{\pm}(2, t)\right) \delta_{\pm}^{\Delta}(2, t) = \frac{1}{2^{\pm 1} t} 2^{\pm 1} = \frac{1}{t} = f(t).$$

The following result is essential for the proof of next theorem:

Theorem 1 (Substitution)**.** ([7: Theorem 1.98]) *Assume $\nu\colon \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f\colon \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_{a}^{b} g(s)\nu^{\Delta}(s)\,\Delta s = \int_{\nu(a)}^{\nu(b)} g(\nu^{-1}(s))\,\tilde{\Delta} s. \tag{3.11}$$

Theorem 2. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period $P \in (t_0, \infty)_{\mathbb{T}^*}$ and f a Δ-periodic function in shifts $\delta_{\pm}$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_{\pm}^{T}(t_0)}^{\delta_{\pm}^{T}(t)} f(s)\Delta s.$$

Proof. Substituting $v(s) = \delta_+^T(s)$ and $g(s) = f(\delta_+^T(s))$ in (3.11) and taking (3.10) into account we have

$$\int_{\delta_+^T(t_0)}^{\delta_+^T(t)} f(s)\Delta s = \int_{\nu(t_0)}^{\nu(t)} g(\nu^{-1}(s))\Delta s$$

$$= \int_{t_0}^{t} g(s)\nu^{\Delta}(s)\Delta s$$

$$= \int_{t_0}^{t} f(\delta_+^T(s))\delta_+^{\Delta T}(t)\Delta s$$

$$= \int_{t_0}^{t} f(s)\Delta s.$$

The equality

$$\int_{\delta_-^T(t_0)}^{\delta_-^T(t)} f(s)\Delta s = \int_{t_0}^{t} f(s)\Delta s$$

can be obtained similarly. The proof is complete. □

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