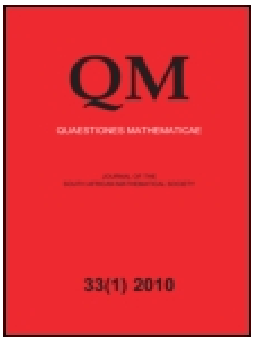

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QUADRATIC PENCIL OF DIFFERENCE EQUATIONS: JOST SOLUTIONS, SPECTRUM, AND PRINCIPAL VECTORS

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To my father Ali Akın Adıvar

Abstract. In this paper, a quadratic pencil of Schrödinger type difference operator L_λ is taken under investigation to provide a general perspective for the spectral analysis of non-selfadjoint difference equations of second order. Introducing Jost-type solutions, structure and quantitative properties of the spectrum of L_λ are investigated. Therefore, a discrete analog of the theory in [6] and [7] is developed. In addition, several analogies are established between difference and q-difference cases. Finally, the principal vectors of L_λ are introduced to lay a groundwork for the spectral expansion.

Mathematics Subject Classification (2000): 39A10, 39A12, 39A13.
Key words: Eigenvalue, Jost solution, principal function, quadratic pencil of difference equation, q-difference equation, spectral analysis, spectral singularity.

1. Introduction. Let L_λ denote the quadratic pencil of difference operator generated in $\ell^2(\mathbb{Z})$ by the difference expression

$$\ell_\lambda y_n := \Delta\left(a_{n-1}\Delta y_{n-1}\right) + \left(q_n + 2\lambda p_n + \lambda^2\right) y_n, \ \ n \in \mathbb{Z},$$

where Δ is forward difference operator, λ is spectral parameter, $\{a_n\}_{n\in\mathbb{Z}}$, $\{p_n\}_{n\in\mathbb{Z}}$, and $\{q_n\}_{n\in\mathbb{Z}}$ are complex sequences satisfying

$$\textstyle\sum_{n\in\mathbb{Z}} |n| \left\{|1-a_n| + |p_n| + |q_n|\right\} < \infty, \tag{1.1}$$

and $a_n \neq 0$ for all $n \in \mathbb{Z}$.

Evidently, Schrödinger type difference equation

$$\Delta(a_{n-1}\Delta y_{n-1}) + (q_n - \lambda)y_n = 0 \ \ n \in \mathbb{Z} \tag{1.2}$$

and the difference equation

$$\Delta(a_{n-1}\Delta y_{n-1}) + (q_n - \lambda)^2 y_n = 0, \quad n \in \mathbb{Z} \tag{1.3}$$

of Klein–Gordon type are special cases of the equation

$$\Delta\left(a_{n-1}\Delta y_{n-1}\right)+\left(q_n+2\lambda p_n+\lambda^2\right)y_n=0,\ \ n\in\mathbb{Z}. \tag{1.4}$$

Observe that the dependence on spectral parameter λ in (1.3) and (1.4) is nonlinear while it is linear in (1.2). Also, since the sequences $\{a_n\}_{n\in\mathbb{Z}}$, $\{p_n\}_{n\in\mathbb{Z}}$, and $\{q_n\}_{n\in\mathbb{Z}}$ are allowed to take complex values, Equation (1.4) is non-selfadjoint.

Note that, the equation (1.2) can be rewritten as

$$a_ny_{n+1}+b_ny_n+a_{n-1}y_n=\lambda y_n, \tag{1.5}$$

where

$$b_n=q_n-a_n-a_n.$$

In [11], Guseinov studied the inverse problem of scattering theory for the Equation (1.5), where $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are real sequences satisfying $a_n>0$ and

$$\textstyle\sum_{n=1}^{\infty}|n|\left(|1-a_n|+|b_n|\right)<\infty.$$

In [1] and [2], the authors investigated spectral properties of the difference operator associated with Equation (1.5) in the case when $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ are complex sequences satisfying

$$\textstyle\sum_{n\in\mathbb{Z}}|n|\left(|1-a_n|+|b_n|\right)<\infty. \tag{1.6}$$

To the best of author's knowledge, quantitative properties of spectrum of the non-selfadjoint difference operators corresponding to Equation (1.3) and Equation (1.4) have not been treated elsewhere before.

In recent years, quantum calculus and q-difference equations has taken a prominent attention in the literature including [3], [4], [8], [18]. In particular, [3] and [4] are concerned with the spectral analysis of q-difference equation

$$\left(a(t)\,u^{\Delta}(t)\right)^{\Delta\rho}+\left(b(t)-\lambda\right)u(t)=0,\ t=q^n \text{ and } n\in\mathbb{Z}. \tag{1.7}$$

However, there is lack of literature on the spectral analysis of quadratic pencil of q-difference equation

$$\left(a(t)\,u^{\Delta}(t)\right)^{\Delta\rho}+\left(b(t)+2\mu c(t)+\mu^2\right)u(t)=0,\ t=q^n \text{ and } n\in\mathbb{Z} \tag{1.8}$$

which includes Equation (1.7) as a particular case.

This paper aims to investigate quantitative properties of spectrum of quadratic pencil difference operator L_λ. This will provide a wide perspective on spectral analysis of second order difference Equations (1.2) and (1.3) and avoid deriving results separately. The remainder of the manuscript is organized as follows: In Section 2, we proceed by the procedure, which has been developed by Naimark, Lyance, and others, consisting of the following steps:

- Formulation of Jost solutions.
- Determination of the resolvent operator.

- Description of the sets of eigenvalues and spectral singularities in terms of singular points of the kernel of the resolvent.
- Use of boundary uniqueness theorems of analytic functions to provide sufficient conditions guaranteeing finiteness of eigenvalues and spectral singularities.

Section 3 is concerned with applications of acquired results in Section 2. This section also contains a brief subsection to show how the obtained results might be extended to quadratic pencil of q-difference Equation (1.8). The latter section introduces principal functions of the operator L_λ.

Therefore, we improve and generalize the results given in [1, 2, 3, 4].

2. Spectrum.

Hereafter, we assume (1.1) unless otherwise stated.

2.1. Jost solutions of Equation (1.4).

The structure of Jost solutions plays a substantial role in spectral analysis of difference and differential operators. By the next theorem and several lemmas in this section, we provide an extensive information about the structure of Jost solutions of Equation (1.4).

To show the structural differences between Jost solutions in continuous and discrete cases, we first consider the Jost solutions of the differential equations

$$-y'' + \left[q(x) + 2\lambda p(x) - \lambda^2\right] y = 0\,, \quad x \in \mathbb{R}_+ \tag{2.1}$$

and

$$-y'' + \left[q(x) - \lambda\right] y = 0\,, \quad x \in \mathbb{R}_+. \tag{2.2}$$

While a Jost solution of the quadratic pencil of differential Equation (2.1) is given by

$$e(x,\lambda) = e^{iw(x)+i\lambda x} + \int_x^\infty A(x,t)e^{i\lambda t}dt, \quad \operatorname{Im}\lambda \geq 0, \tag{2.3}$$

where $w(x) = \int\limits_x^\infty p(t)dt$ (see [13]), the Jost solution of Equation (2.2) is obtained as

$$f(x,\lambda) = e^{i\sqrt{\lambda}x} + \int_x^\infty B(x,t)e^{i\sqrt{\lambda}t}dt, \quad \operatorname{Im}\sqrt{\lambda} \geq 0 \tag{2.4}$$

[17]. Note that, $w(x)$ does not appear in (2.4) since $p(x) = 0$ in (2.2). The term $w(x)$ in (2.3) makes the spectral analysis of (2.1) quite challenging. For one thing, the set of eigenvalues of Equation (2.2) lies only in $\mathbb{C}_+$ (see [14]) while that of Equation (2.1) resides both in $\mathbb{C}_+$ and $\mathbb{C}_-$ (see [6] and [12]), where $\mathbb{C}_+$ and $\mathbb{C}_-$ indicates the open upper and lower half-planes, respectively.

One of the main achievements of this paper is to introduce Jost solutions of Equation (1.4) in a simple structure and to show that there is no such a difficulty in discrete case. This will enable us to investigate the spectral analysis of (1.4) as it is done for (1.2) in which dependence on λ is linear. In discrete case, the Jost solutions of Equation (1.2) takes the form

$$e_n^{\pm}(z) = \beta_n^{\pm} e^{\pm inz} + \sum_{m\in\mathbb{Z}^{\pm}}^{\infty} A_{n,m}^{\pm} e^{\pm imz}, \quad n \in \mathbb{Z} \tag{2.5}$$

where $\lambda = 2\cos z$, $\operatorname{Im} z \geq 0$, and $\mathbb{Z}^{\pm}$ denotes the sets of positive and negative integers, respectively (see [1] and [11]).

Despite the fact that the dependence on spectral parameter λ is non-linear in Equation (1.4), the next theorem offers Jost solutions of the form

$$f_n^{+}(z) = \alpha_n^{+} e^{inz} \left(1 + \textstyle\sum_{m=1}^{\infty} K_{n,m}^{+} e^{imz/2}\right), \quad n \in \mathbb{Z} \tag{2.6}$$

and

$$f_n^{-}(z) = \alpha_n^{-} e^{-inz} \left(1 + \textstyle\sum_{-\infty}^{m=-1} K_{n,m}^{-} e^{-imz/2}\right), \quad n \in \mathbb{Z} \tag{2.7}$$

which have similar structure to (2.5), i.e., there is no additional function ω of n in the exponent of first terms.

Theorem 1. *For $\lambda = 2\cos(z/2)$ and $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, (2.6) and (2.7) solve Equation (1.4). The coefficients $\alpha_n^{\pm}$ and the kernels $K_{n,m}^{\pm}$ are uniquely expressed in terms of $\{a_n\}_{n\in\mathbb{Z}}$, $\{p_n\}_{n\in\mathbb{Z}}$, and $\{h_n\}_{n\in\mathbb{Z}}$ (where $h_n = 2 - a_n - a_{n-1} + q_n$) as follows:.*

$$\begin{aligned}
\alpha_n^{+} &= \left(\textstyle\prod_{r=n}^{\infty}(-a_r)\right)^{-1}, \\
K_{n,1}^{+} &= 2\textstyle\sum_{r=n+1}^{\infty} p_r, \\
K_{n,2}^{+} &= \textstyle\sum_{r=n+1}^{\infty} \left(h_r + 2p_r K_{r,1}^{+}\right), \\
K_{n,3}^{+} &= \textstyle\sum_{r=n+1}^{\infty} h_r K_{r,1}^{+} + 2p_r \left(K_{r,2}^{+} + 1\right), \\
K_{n,4}^{+} &= \textstyle\sum_{r=n+1}^{\infty} \left(1 - a_r^2\right) + h_r K_{r,2}^{+} + 2p_r \left(K_{r,3}^{+} + K_{r,1}^{+}\right),
\end{aligned}$$

$$K_{n,m+4}^{+} = K_{n,m}^{+} + \sum_{r=n+1}^{\infty} \left(1 - a_r^2\right) K_{r+1,m}^{+} h_r K_{r,m+2}^{+} + 2p_r \left(K_{r,m+1}^{+} + K_{r,m+3}^{+}\right),$$

for $m = 1, 2, \ldots;\ n \in \mathbb{Z}$, and

$$\begin{aligned}
\alpha_n^{-} &= \left(\textstyle\prod_{-\infty}^{r=n-1}(-a_r)\right)^{-1}, \\
K_{n,-1}^{-} &= 2\textstyle\sum_{-\infty}^{r=n-1} p_r, \\
K_{n,-2}^{-} &= \textstyle\sum_{-\infty}^{r=n-1} \left(h_r + 2p_r K_{r,-1}^{-}\right), \\
K_{n,-3}^{-} &= \textstyle\sum_{-\infty}^{r=n-1} h_r K_{r,-1}^{-} + 2p_r \left(K_{r,-2}^{-} + 1\right), \\
K_{n,-4}^{-} &= \textstyle\sum_{-\infty}^{r=n-1} \left(1 - a_{r-1}^2\right) + h_r K_{r,-2}^{-} + 2p_r \left(K_{r,-3}^{-} + K_{r,-1}^{-}\right),
\end{aligned}$$

$$K_{n,m-4}^{-} = K_{n,m}^{-} + \sum_{-\infty}^{r=n-1} \left(1 - a_{r-1}^2\right) K_{r-1,m}^{-} + h_r K_{r,m-2}^{-} + 2p_r \left(K_{r,m-1}^{-} + K_{r,m-3}^{-}\right),$$

for $m = -1, -2, \ldots;\ n \in \mathbb{Z}$.

Proof. Using $\lambda = 2\cos(z/2)$, Equation (1.4) can be rewritten as

$$a_n y_{n+1} + a_{n-1} y_{n-1} + \left(h_n + 2\left(e^{iz/2} + e^{-iz/2}\right) p_n + e^{iz} + e^{-iz}\right) y_n = 0.$$

Substituting $f_n^+(z)$ for y_n in this equation and comparing the coefficients of $e^{i(n-1)z}$, $e^{i(n-\frac{1}{2})z}$, e^{inz}, $e^{i(n+\frac{1}{2})z}$, and $e^{i(n+1)z}$, we find

$$\begin{aligned} a_{n-1}\alpha_{n-1}^+ + \alpha_n^+ &= 0, \\ K_{n,1}^+ - K_{n-1,1}^+ + 2p_n &= 0, \\ K_{n,2}^+ - K_{n-1,2}^+ + 2p_n K_{n,1}^+ + h_n &= 0, \\ K_{n,3}^+ - K_{n-1,3}^+ + 2p_n\left(K_{n,2}^+ + 1\right) + h_n K_{n,1}^+ &= 0, \\ K_{n,4}^+ - K_{n-1,4}^+ + \left(1 - a_n^2\right) + h_n K_{n,2}^+ + 2p_n\left(K_{n,1}^+ + K_{n,3}^+\right) &= 0, \end{aligned}$$

respectively. From remaining terms we have the following recurrence relation

$$\begin{aligned} K_{n-1,m+4}^+ - K_{n,m+4}^+ = {} & K_{n,m}^+ - a_n^2 K_{n+1,m}^+ + h_n K_{n,m+2}^+ \\ & + 2p_n\left(K_{n,m+1}^+ + K_{n,m+3}^+\right) \end{aligned}$$

for $m = 1, 2, ...$ and $n \in \mathbb{Z}$. Similarly we obtain

$$\begin{aligned} a_{n-1}\alpha_{n+1}^- + \alpha_n^- &= 0, \\ K_{n,-1}^- - K_{n+1,-1}^- + 2p_n &= 0, \\ K_{n,-2}^- - K_{n+1,-2}^- + 2p_n K_{n,-1}^- + h_n &= 0, \\ K_{n,-3}^- - K_{n+1,-3}^- + 2p_n\left(K_{n,-2}^- + 1\right) + h_n K_{n,-1}^- &= 0, \\ K_{n,-4}^- - K_{n+1,4}^- + \left(1 - a_n^2\right) + h_n K_{n,-2}^- + 2p_n\left(K_{n,-1}^- + K_{n,-3}^-\right) &= 0, \end{aligned}$$

from the coefficients of $e^{-i(n+1)z}$, $e^{-i(n+\frac{1}{2})z}$, e^{-inz}, $e^{-i(n-\frac{1}{2})z}$, and $e^{-i(n-1)z}$, respectively. Use of remaining terms yields

$$\begin{aligned} K_{n+1,m-4}^- - K_{n,m-4}^- = {} & K_{n,m}^- - a_n^2 K_{n-1,m}^- + h_n K_{n,m-2}^- \\ & + 2p_n\left(K_{n,m-1}^- + K_{n,m-3}^-\right) \end{aligned}$$

for $m = -1, -2, ...$ and $n \in \mathbb{Z}$. Above difference equations give the desired result, whereas convergence of the coefficients $\alpha_n^\pm$ and the kernels $K_{n,m}^\pm$ is immediate from the condition (1.1). □

Different than the solutions (2.5) of (1.2) the coefficients $K_{n,m}^\pm$ in (2.6-2.7) are determined by recurrence relations depending on first four terms $K_{n,m}^\pm$, $n \in \mathbb{Z}$, $m = \pm 1, \pm 2, \pm 3, \pm 4$ (the ones for $A_{n,m}^\pm$ depend on $A_{n,m}^\pm$, $n \in \mathbb{Z}$, $m = \pm 1, \pm 2$). In addition to these, the exponential function in the second terms in (2.5) and (2.6) contains the half of the complex variable z because of the transformation $\lambda = 2\cos(z/2)$.

In the following lemma, we list some properties of Jost solutions $f^{\pm}(z) = \{f_n^{\pm}(z)\}$:

Lemma 1. *i. The kernels $K_{n,m}^{\pm}$ satisfy*

$$\left|K_{n,m}^{+}\right| \leq c\sum\nolimits_{r=n+[m/2]}^{\infty}\left(|1-a_r|+|p_r|+|q_r|\right),$$
$$\left|K_{n,m}^{-}\right| \leq C\sum\nolimits_{-\infty}^{r=n+[m/2]+1}\left(|1-a_r|+|p_r|+|q_r|\right),$$

where $[m/2]$ is the integer part of $m/2$, and c and C are positive constants,

ii. $f_n^{\pm}(z)$ are analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, continuous in $\overline{\mathbb{C}}_+$,

iii. For $z \in \overline{\mathbb{C}}_+$, $f_n^{\pm}(z)$ satisfy the following estimates

$$f_n^{+}(z) = \exp(inz)\left[1+o(1)\right] \text{ as } n \to \infty, \tag{2.8}$$
$$f_n^{-}(z) = \exp(-inz)\left[1+o(1)\right] \text{ as } n \to -\infty, \tag{2.9}$$

and

$$f_n^{\pm}(z) = \alpha_n^{\pm}\exp(\pm inz)\left[1+o(1)\right] \text{ as } \operatorname{Im} z \to \infty \text{ for } n \in \mathbb{Z}. \tag{2.10}$$

Proof. The proof follows from (1.1), (2.6), and (2.7). □

Let $g^{\pm}(z) = \{g_n^{\pm}(z)\}$ denote the solutions of Equation (1.4) satisfying

$$\lim\nolimits_{n\to\pm\infty} g_n^{\pm}(z)\,e^{inz} = 1,$$

respectively. By making use of Theorem 1, (2.8) and (2.9) we have the next result:

Lemma 2. *i. For $z \in \mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$*

$$g^{\pm}(z) = \left\{f_n^{\pm}(-z)\right\}_{n\in\mathbb{Z}}$$

holds.

ii. $g_n^{\pm}(z)$ are analytic with respect to z in $\mathbb{C}_-$, and continuous in $\overline{\mathbb{C}}_-$.

iii. For $\zeta \in \mathbb{R}$,

$$W\left[f^{\pm}(\zeta), g^{\pm}(\zeta)\right] = \mp 2i\sin\zeta$$

holds, where the Wronskian of two solutions u and v of Equation (1.4) is defined by

$$W[u,v] = a_n\left(u_n v_{n+1} - u_{n+1}v_n\right).$$

iv. For $\zeta \in \mathbb{R}\backslash\{n\pi : n \in \mathbb{Z}\}$ and $\lambda = 2\cos(\zeta/2)$

$$f_n^{+}(\zeta) = \psi(\zeta)\, f_n^{-}(\zeta) + \mu(\zeta)\, g_n^{-}(\zeta), \tag{2.11}$$

where

$$\psi(\zeta) = \frac{W\left[f^{+}(\zeta), g^{-}(\zeta)\right]}{2i\sin\zeta},\ \mu(\zeta) = -\frac{W\left[f^{+}(\zeta), f^{-}(\zeta)\right]}{2i\sin\zeta}.$$

It is worth noting that the function μ has an analytic continuation to the open upper half-plane $\mathbb{C}_+$.

2.2. Resolvent and discrete spectrum. The set of values $\lambda \in \mathbb{C}$ such that $R_\lambda(L_\lambda) = L_\lambda^{-1}$ exists as a bounded operator on $\ell^2(\mathbb{Z})$ is said to be the resolvent set $\rho(L_\lambda)$ of L_λ.

Similar to the one in [1], we formulate the resolvent set $\rho(L_\lambda)$ and the resolvent operator $R_\lambda(L_\lambda)$ as follows:

$$\rho(L_\lambda) = \left\{\lambda = 2\cos\frac{z}{2} : z \in \mathbb{C}_+ \text{ and } \Phi(z) \neq 0\right\} \tag{2.12}$$

and

$$R_\lambda(L_\lambda)\phi_n = \sum_{m\in\mathbb{Z}} \mathcal{G}_{n,m}(z)\phi_m$$

for $\lambda = 2\cos\frac{z}{2} \in \rho(L_\lambda)$ and $\phi := \{\phi_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$, where

$$\mathcal{G}_{n,m}(z) = \left\{ \begin{array}{c} \frac{f_m^-(z) f_n^+(z)}{\Phi(z)}, \quad m = n-1, n-2, ... \\ \frac{f_m^+(z) f_n^-(z)}{\Phi(z)}, \quad m = n, n+1, ... \end{array} \right. , \tag{2.13}$$

and

$$\Phi(z) := 2i\sin z\mu(z) = W\left[f^-(z), f^+(z)\right]. \tag{2.14}$$

Notice that the function Φ is 4π periodic, analytic in $\mathbb{C}_+$, and continuous in $\overline{\mathbb{C}}_+$. The zeros of the function Φ play a substantial role in the formulation of the sets of eigenvalues and spectral singularities of L_λ. From (2.11-2.14) and definition of eigenvalues we arrive at the following conclusion.

LEMMA 3. (Eigenvalues) *Let $\sigma_d(L_\lambda)$ denote the set of eigenvalues of the operator L_λ. Then we have*

$$\sigma_d(L_\lambda) = \left\{\lambda = 2\cos\frac{z}{2} : z \in P^+ \text{ and } \Phi(z) = 0\right\},$$

where

$$P^+ = \{z = \eta + i\varphi : \eta \in [-\pi, 3\pi] \text{ and } \varphi > 0\}.$$

Note that L_λ has no eigenvalues on the real line. This is because, if $\lambda_0 = 2\cos(z_0/2) \in \mathbb{R}$ is an eigenvalue, then the corresponding solution $y_n(z_0)$ will satisfy the estimate $y_n(z_0) = c_1 e^{inz_0} + c_2 e^{-inz_0} + o(1)$ as $n \to \infty$ contradicting the fact that $y_n \in \ell^2(\mathbb{Z})$.

2.3. Continuous spectrum and the spectral singularities. To obtain the continuous spectrum of the operator L_λ we shall resort to the following lemma.

LEMMA 4. *For every $\delta > 0$, there is a positive number c_δ such that*

$$\|R_\lambda(L_\lambda)\| \geq \frac{c_\delta}{|\Phi(z)|\sqrt{1 - \exp(-2\operatorname{Im} z)}}$$

for $\lambda = 2\cos\frac{z}{2}$, $z \in \mathbb{C}_+$, and $\operatorname{Im} z > \delta$. Hence, $\|R_\lambda(L_\lambda)\| \to \infty$ as $\operatorname{Im} z \to 0$.

Proof. Let $\delta > 0$, $z \in \mathbb{C}_+$ and $\operatorname{Im} z > \delta$. Define the function h^{m_0} by

$$h_n^{m_0}(z) := \begin{cases} \overline{f_n^-(z)}, & n = m_0 - 1,\ m_0 - 2, ... \\ 0, & n = m_0,\ m_0 + 1, ... \end{cases}.$$

Evidently, $h^{m_0}(z) \in \ell^2(\mathbb{Z})$ and

$$\begin{aligned} R_\lambda(L_\lambda)\, h_n^{m_0}(z) &= \textstyle\sum_{-\infty}^{m=m_0-1} \mathcal{G}_{n,m}(z)\, \overline{f_m^-}(z) \\ &= \frac{f_n^+(z)}{\Phi(z)} \left\| h^{m_0} \right\|^2 \end{aligned}$$

for $m_0 < n$. By (310) we have $f_n^+(z) = e^{inz} + o(1)$ as $\operatorname{Im} z \to \infty$ for $n \in \mathbb{Z}$. Thus, we can choose $m_0 = m_0(\delta)$ sufficiently large so that $m_0 < n$, $\operatorname{Im} z > \delta$, and the inequality

$$\left| f_n^+(z) \right| > \frac{1}{2} e^{-n \operatorname{Im} z}$$

holds. Therefore, we get

$$\left\| f^+(z) \right\|^2 \geq \frac{\exp(-2m_0 \operatorname{Im} z)}{4\left(1 - \exp(-2 \operatorname{Im} z)\right)}.$$

Hence, we arrive at the following inequality:

$$\frac{\left\| R_\lambda(L_\lambda)\, h^{m_0} \right\|^2}{\left\| h^{m_0} \right\|^2} \geq \frac{c_\delta^2}{\left(1 - \exp(-2 \operatorname{Im} z)\right) \left| \Phi(z) \right|^2}$$

as desired, where

$$c_\delta = \frac{\left\| h^{m_0} \right\|}{2 \exp(m_0 \operatorname{Im} z)}.$$

□

We shall need the following theorem at several occasions in our further work.

Theorem 2. $\sigma_c(L_\lambda) = [-2, 2]$, *where* $\sigma_c(L_\lambda)$ *denotes the continuous spectrum of the operator* L_λ.

Proof. By (2.12), for any $\lambda \in \rho(L_\lambda)$ there is a corresponding $z \in \mathbb{C}_+$ such that $\lambda = 2\cos\frac{z}{2}$ and $\Phi(z) \neq 0$. Let $\lambda_0 = 2\cos\frac{z_0}{2} \in \sigma_c(L_\lambda)$. Then $\left\| R_\lambda(L_\lambda) \right\| \to \infty$ as $\lambda \to \lambda_0$. This shows that $\Phi(z) = 2i \sin z \mu(z) \to 0$ as $z \to z_0$. Continuity of μ and $\mu(z_0) \neq 0$ yield $\sin z \to 0$ and $\operatorname{Im} z \to 0$. On the other hand, we have $\operatorname{Im} z \to \operatorname{Im} z_0$ since $\lambda \to \lambda_0$. It follows that $\operatorname{Im} z_0 = 0$, i.e., $\lambda_0 = 2\cos\frac{z_0}{2} \in [-2, 2]$. Conversely, from Lemma 4 $\left\| R_\lambda(L_\lambda) \right\| \to \infty$ for $\lambda = 2\cos\frac{z}{2} \in [-2, 2]$. Now, we have to show that the range $\mathcal{R}(L_\lambda)$ of values of the operator L_λ is dense in the space $\ell^2(\mathbb{Z})$. It is obvious that the orthogonal complement of $\mathcal{R}(L_\lambda)$ coincides with the space of solutions $y \in \ell^2(\mathbb{Z})$ of Equation $L_\lambda^* y = 0$, where L_λ^* denotes the adjoint operator.

Since Equation $L_\lambda^* y = 0$ has no any eigenvalue on the real line, the orthogonal complement of the set $\mathcal{R}(L_\lambda)$ consists only of the zero element. This completes the proof. □

REMARK 1. If $p_n = 0$, the difference equation

$$\Delta(a_{n-1}\Delta y_{n-1}) + \left(q_n + 2\lambda p_n + \lambda^2\right) y_n = 0, \quad n \in \mathbb{Z},$$

turns into

$$a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1} = \widetilde{\lambda} y_n, \quad n \in \mathbb{Z}, \tag{2.15}$$

where

$$b_n = 2 + q_n - a_n - a_{n-1} \quad \text{and} \quad \widetilde{\lambda} = 2 - \lambda^2.$$

Namely, in the case $p_n = 0$, (1.1) is equivalent to (1.6) and $\widetilde{\lambda} = 2 - \lambda^2$ becomes the new spectral parameter. Thus, Theorem 2 implies that the continuous spectrum of the difference operator corresponding to (2.15) is $[-2, 2]$. This result was obtained in [1, Theorem 3.1].

Spectral singularities are poles of the kernel of the resolvent operator and are imbedded in the continuous spectrum ([15, Definition 1.1.]). Analogous to the quadratic pencil of Schrödinger operator [6], from Theorem 2 and (2.13) we obtain the set of spectral singularities of the operator L_λ as follows:

COROLLARY 1. (Spectral singularities) *Let $\sigma_{ss}(L_\lambda)$ denote the set of spectral singularities. We have*

$$\sigma_{ss}(L_\lambda) = \left\{\lambda = 2\cos\frac{z}{2} : z \in P_0 \text{ and } \Phi(z) = 0\right\},$$

where $P_0 := (-\pi, 3\pi) \setminus \{0, \pi, 2\pi\}$.

Hereafter, we discuss the quantitative properties of eigenvalues and spectral singularities. For this purpose, we will make use of boundary uniqueness theorems of analytic functions [9]. Lemma 3 and Corollary 1 show that the problem of investigation of quantitative properties of eigenvalues and spectral singularities can be reduced to the investigation of quantitative properties of zeros of the function Φ in the semi-strip $P^+ \cup P_0$.

2.4. Quantitative properties of eigenvalues and spectral singularities. Now, we investigate the structure of discrete spectrum and the set of spectral singularities. In this context, the sets of eigenvalues and spectral singularities are analyzed in terms of boundedness, closedness, being countable, etc.

Theorem 1 and (2.10) imply the next result.

COROLLARY 2. *Let Φ be defined by (2.14). Then*

$$\Phi(z) = \left(\prod_{r\in Z}(-a_r)\right)^{-1} e^{-iz}\left[1 + o(1)\right] \quad \text{for } z \in P^+ \text{as } \operatorname{Im} z \to \infty.$$

Let M_1 and M_2 denote the sets of zeros of the function Φ in P^+ and P_0, respectively, i.e.,

$$M_1 = \left\{z \in P^+ : \Phi(z) = 0\right\},$$
$$M_2 = \left\{z \in P_0 : \Phi(z) = 0\right\}.$$

Corollary 2 shows boundedness of the set M_1. Since Φ is a 4π periodic function and is analytic in P^+, the set M_1 has at most countable number of elements. By uniqueness of analytic functions, we figure out that the limit points of the set M_1 lie in the closed interval $[-\pi, 3\pi]$. Moreover, we can obtain the closedness and the property of having zero Lebesgue measure of the set M_2 as a natural consequence of boundary uniqueness theorems of analytic functions [9]. Hence, from Lemma 3 and Corollary 1 we conclude the following.

Lemma 5. *The set of eigenvalues $\sigma_d(L_\lambda)$ is bounded and countable, and its accumulation points lie on the closed interval $[-2, 2]$. The set of spectral singularities $\sigma_{ss}(L_\lambda)$ is closed and its Lebesgue measure is zero.*

Hereafter, we call the multiplicity of a zero of the function Φ in $P^+ \cup P_0$ the multiplicity of corresponding eigenvalue or spectral singularity.

In the next two theorems, we shall employ the following conditions and show that each of them guarantees finiteness of eigenvalues, spectral singularities, and their multiplicities.

Condition 1. $\sup_{n \in Z}\{\exp(\varepsilon|n|)(|1-a_n| + |p_n| + |q_n|)\} < \infty$ for some $\varepsilon > 0$.

Condition 2. $\sup_{n \in Z}\left\{\exp\left(\varepsilon|n|^\delta\right)(|1-a_n| + |p_n| + |q_n|)\right\} < \infty$ for some $\varepsilon > 0$ and $\frac{1}{2} \le \delta < 1$.

Note that Condition 2 is weaker than Condition 1. If Condition 1 holds, then we get by (i) of Lemma 1 that

$$\left|K^{\pm}_{n,m}\right| \le c_{1,2}\exp\left(\mp(\varepsilon/4)\, m\right) \text{ for } n = 0, 1 \text{ and } m = \pm1, \pm2, ...,$$

where $c_{1,2}$ are positive constants. That is, the function Φ has an analytic continuation to the lower half-plane $\operatorname{Im} z > -\varepsilon/2$. Since Φ is a 4π periodic function, this analytic continuation implies that the bounded sets M_1 and M_2 have no any limit point on the real line. Hence, we have the finiteness of the zeros of the function Φ in $P^+ \cup P_0$. These results are complemented by the next theorem.

Theorem 3. *Under Condition 1, the operator L_λ has finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. The proof follows from Lemma 5 and the fact that the sets $\sigma_d(L_\lambda)$ and $\sigma_{ss}(L_\lambda)$ have no limit points. □

Under Condition 2, Φ has no any analytic continuation, so finiteness of eigenvalues and spectral singularities cannot be proven in a similar way to that of Theorem 3.

The following lemma can be proved similar to that of [2, Lemma 2.2]:

LEMMA 6. *If Condition 2 holds, then we have*

$$\left|\frac{d^k\Phi}{d\lambda^k}(z)\right| \leq A_k, \;\; z \in P^+, \;\; k = 0, 1, ...,$$

where

$$A_k \leq C4^k + Dd^k k! k^{k(1/\delta - 1)}$$

and C, D, and d are positive constants depending on ε and δ.

THEOREM 4. *If Condition 2 holds, then eigenvalues and spectral singularities of the operator L_λ are finite, and each of them is of finite multiplicity.*

Proof. Let M_3 and M_4 denote the sets of limit points of the sets M_1 and M_2, respectively, and M_5 the set of zeros in P^+ of the function Φ with infinite multiplicity. From the boundary uniqueness theorem of analytic functions we have the relations

$$M_1 \cap M_5 = \emptyset, \;\; M_3 \subset M_2, \;\; M_5 \subset M_2,$$

and using continuity of all derivatives of Φ on $[-\pi, 3\pi]$ we get that

$$M_3 \subset M_5, \;\; M_4 \subset M_5.$$

Combining Lemma 6 and the uniqueness theorem (see [2, Theorem 2.3]), we conclude that

$$M_5 = \emptyset.$$

Thus, countable and bounded sets M_1 and M_2 have no limit points. The proof is complete. □

3. Applications to special cases. In this section, the spectral results obtained for Equation (1.4) are applied to the following particular cases:

Case 1. $p_n = 0$

Case 2. $p_n = -v_n$ and $q_n = v_n^2$,

in which we obtain the equations (1.2) and (1.3), respectively.

In addition, we explore some analogies between Equation (1.4) and its q-analog (1.8) using some transformations and the results obtained in theorems 1-4. Finally, we deduce the main results of [3, Theorem 5]-[4] in the special case.

In the next we cover the first case.

3.1. Sturm-Liouville type difference equation. It is evident that the substitution $p_n = 0$ in (1.4) yields the Sturm-Liouville type difference equation (2.15) whose spectral parameter is $\widetilde{\lambda} = 2 - \lambda^2$. Denote by Λ the difference operator corresponding to Equation (2.15). In [1], the authors show that the operator Λ has finitely many eigenvalues and spectral singularities provided that

$$\sup_{n\in\mathbb{Z}} \{\exp(\varepsilon |n|) (|1 - a_n| + |b_n|)\} < \infty \tag{3.1}$$

holds for some $\varepsilon > 0$. Afterwards, a relaxation of the condition (3.1) is given by [2, Theorem 2.5] as follows

$$\sup_{n\in\mathbb{Z}} \left\{\exp\left(\varepsilon |n|^{\delta}\right) (|1 - a_n| + |b_n|)\right\} < \infty, \qquad \frac{1}{2} \le \delta < 1. \tag{3.2}$$

Note that Conditions 1 and 2 turn into the conditions (3.1) and (3.2), respectively. That is, the results [1, Theorem 4.2] and [2, Theorem 2.5] can be obtained from Theorem 3 and Theorem 4 as corollaries.

The second case is handled in the following.

3.2. Klein-Gordon type difference equation. Setting $p_n = -v_n$ and $q_n = v_n^2$ in Equation (1.4), we obtain the Klein-Gordon type non-selfadjoint difference equation

$$\Delta(a_{n-1}\Delta y_{n-1}) + (v_n - \lambda)^2 y_n = 0, \quad n \in \mathbb{Z}. \tag{3.3}$$

Observe that Equation (3.3) is more general than the discrete analog of the differential equation

$$-y'' + (p(x) - \lambda)^2 y = 0. \tag{3.4}$$

Let Γ denote the difference operator corresponding to Equation (3.3). The set of eigenvalues of differential operator corresponding to (3.4) was determined by Degasperis [**7**], in the case that p is real, analytic and vanishes rapidly for $x \to \infty$ (for non-selfadjoint case see also [5]). However, finiteness of eigenvalues and spectral singularities of difference operator Γ has not been shown elsewhere before. As a consequence of Theorem 3 and Theorem 4, we derive this result as a corollary.

COROLLARY 3. *If for some* $\varepsilon > 0$

$$\sup_{n\in Z} \left\{\exp\left(\varepsilon |n|^{\delta}\right) (|1 - a_n| + |v_n|)\right\} < \infty, \qquad \frac{1}{2} \le \delta \le 1$$

holds, then the difference operator Γ *has finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

3.3. *q*-difference case. We suppose $q > 1$ and use the following notations throughout this section:

$$q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\},$$
$$q^{-\mathbb{N}} = \left\{q^{-n} : n \in \mathbb{N}\right\},$$
$$q^{\mathbb{Z}} = \{q^n : n \in \mathbb{Z}\}.$$

The q-difference equation is an equation which contains q-derivative of its unknown function. The q-derivative is given by

$$y^{\Delta}(t)=\frac{y(qt)-y(t)}{(q-1)t},\ t\in q^{\mathbb{Z}},$$

and the q-integral is defined by

$$\int_a^b f(t)\,\Delta_q t=(q-1)\sum_{t\in[a,b)\cap q^{\mathbb{Z}}} tf(t).$$

We shall denote by $\ell^2\left(q^{\mathbb{Z}}\right)$ the Hilbert space of square integrable functions with the norm

$$\|f\|_q^2=\int_{q^{\mathbb{Z}}}|f(t)|^2\,\Delta_q t.$$

Consider the quadratic pencil of Schrödinger type q-difference operator L_λ^q corresponding to the equation

$$\left(a(t)\,u^{\Delta}(t)\right)^{\Delta\rho}+\left(b(t)+2\lambda c(t)+\lambda^2\right)u(t)=0,\ t\in q^{\mathbb{Z}} \tag{3.5}$$

where a, b, and c are complex valued functions, λ is spectral parameter and ρ is the backward jump operator defined by

$$u(t)^{\rho}=u(t/q),\quad t\in q^{\mathbb{Z}}.$$

Multiplying Equation (3.5) by $\sqrt{t/q}$ we arrive at

$$\begin{aligned}
0&=\sqrt{\frac{t}{q}}\left\{\left(a(t)\,u^{\Delta}(t)\right)^{\Delta\rho}+\left(b(t)+2\lambda c(t)+\lambda^2\right)u(t)\right\}\\
&=\sqrt{\frac{t}{q}}\left\{a(t)\frac{u(qt)-u(t)}{(q-1)t}\right\}^{\Delta\rho}+\sqrt{\frac{t}{q}}\left(b(t)+2\lambda c(t)+\lambda^2\right)u(t)\\
&=\sqrt{\frac{t}{q}}\left\{\frac{1}{(q-1)t}\left[a(qt)\frac{u(q^2t)-u(qt)}{(q-1)qt}-a(t)\frac{u(qt)-u(t)}{(q-1)t}\right]\right\}^{\rho}\\
&\quad+\sqrt{\frac{t}{q}}\left(b(t)+2\lambda c(t)+\lambda^2\right)u(t)\\
&=\sqrt{\frac{t}{q}}\left\{\frac{1}{(q-1)t/q}\left[a(t)\frac{u(qt)-u(t)}{(q-1)t}-a(\frac{t}{q})\frac{u(t)-u(t/q)}{(q-1)t/q}\right]\right\}\\
&\quad+\sqrt{\frac{t}{q}}\left(b(t)+2\lambda c(t)+\lambda^2\right)u(t)\\
&=\frac{a(t)}{(q-1)^2t^2}\sqrt{qt}u(qt)+\frac{a(t/q)}{(q-1)^2t^2/q^2}\sqrt{\frac{t}{q}}u(\frac{t}{q})\\
&\quad+\left\{\left[\frac{b(t)}{\sqrt{q}}-\sqrt{q}\frac{a(t)}{(q-1)^2t^2}-\frac{1}{\sqrt{q}}\frac{a(t/q)}{(q-1)^2t^2/q^2}\right]\right.\\
&\qquad\left.+2\left(\frac{\lambda}{q^{1/4}}\right)\frac{c(t)}{q^{1/4}}+\left(\frac{\lambda}{q^{1/4}}\right)^2\right\}\sqrt{t}u(t)
\end{aligned}$$

$$= \widehat{a}(t)\widehat{u}(qt) + \widehat{a}(t/q)\widehat{u}(t/q) + \left\{\widehat{b}(t) + 2\widehat{\lambda}\widehat{c}(t) + \widehat{\lambda}^2\right\}\widehat{u}(t), \quad t \in q^{\mathbb{Z}},$$

and therefore,

$$\widehat{a}(t)\widehat{u}(qt) + \widehat{a}(t/q)\widehat{u}(t/q) + \left\{\widehat{b}(t) + 2\widehat{\lambda}\widehat{c}(t) + \widehat{\lambda}^2\right\}\widehat{u}(t) = 0, \tag{3.6}$$

where $t \in q^{\mathbb{Z}}$ and

$$\widehat{u}(t) = \sqrt{t}u(t), \quad \widehat{\lambda} = q^{-1/4}\lambda \tag{3.7}$$

$$\begin{aligned}\widehat{a}(t) &= \frac{a(t)}{(q-1)^2 t^2},\\ \widehat{b}(t) &= \frac{b(t)}{\sqrt{q}} - \sqrt{q}\widehat{a}(t) - \frac{\widehat{a}(t/q)}{\sqrt{q}},\\ \widehat{c}(t) &= q^{-1/4}c(t).\end{aligned}$$

Using the notations

$$t = q^n,\ \widehat{a}(q^n) = \widehat{a}_n,\ \ \widehat{b}(q^n) = \widehat{b}_n,\ \ \widehat{c}(q^n) = \widehat{c}_n,\ \ \widehat{u}(q^n) = \widehat{u}_n \tag{3.8}$$

we can express Equation (3.6) in the following form

$$\widehat{a}_n\widehat{u}_{n+1} + \widehat{a}_{n-1}\widehat{u}_{n-1} + \left\{\widehat{b}_n + 2\widehat{\lambda}\widehat{c}_n + \widehat{\lambda}^2\right\}\widehat{u}_n = 0, \quad n \in \mathbb{Z}. \tag{3.9}$$

By establishing a linkage between Equations (3.5) and (3.9), the next theorem provides an information about some spectral properties of L_λ^q.

THEOREM 5. *Let the sequence $\widehat{u} = \{\widehat{u}_n\}_{n\in\mathbb{Z}}$ and the value $\widehat{\lambda}$ be defined as in (3.7) and (3.8). The following properties hold:*

i. *$u \in \ell^2(q^{\mathbb{Z}})$ and solves (3.5) if and only if $\widehat{u} \in \ell^2(\mathbb{Z})$ and solves (3.9).*

ii. *λ is an eigenvalue of (3.5) if and only if $\widehat{\lambda} = q^{-1/4}\lambda$ is an eigenvalue of (3.9).*

iii. *For $\lambda = 2q^{1/4}\cos(z/2)$*

$$J^+(t,z) = \alpha^+(t)\exp\left(i\frac{\ln t}{\ln q}z\right)\left\{1 + \int_{r\in q^{\mathbb{N}}} A^+(t,r)\exp\left(i\frac{\ln r}{2\ln q}z\right)\Delta_q r\right\}$$

and

$$J^-(t,z) = \alpha^-(t)\exp\left(-i\frac{\ln t}{\ln q}z\right)\left\{1 + \int_{r\in q^{-\mathbb{N}}} A^-(t,r)\exp\left(-i\frac{\ln r}{2\ln q}z\right)\Delta_q r\right\}$$

are Jost solutions of the Equation (3.5), where $\alpha^{\pm}(t)$ and $A^{\pm}(t,r)$ can be uniquely expressed in terms of $\widehat{a}(t)$, $\widehat{b}(t)$, and $\widehat{c}(t)$ provided that the condition

$$\sum_{t\in q^{\mathbb{Z}}}\left|\frac{\ln t}{\ln q}\right|\left(|1-\widehat{a}(t)| + \left|2-\widehat{b}(t)\right| + |\widehat{c}(t)|\right) < \infty$$

holds.

iv.

$$\sigma_d\left(L_\lambda^q\right)=\left\{\lambda=2q^{1/4}\cos\left(\frac{z}{2}\right):z\in P^+\text{ and }Q(z)=0\right\},$$
$$\sigma_{ss}\left(L_\lambda^q\right)=\left\{\lambda=2q^{1/4}\cos\left(\frac{z}{2}\right):z\in P_0\text{ and }Q(z)=0\right\},$$

where

$$Q(z)=W\left[J^-(t,z),J^+(t,z)\right].$$

v.

$$\sigma_c\left(L_\lambda^q\right)=\left[-2q^{1/4},2q^{1/4}\right],$$

where $\sigma_c\left(L_\lambda^q\right)$ denotes the continuous spectrum of the operator L_λ^q.

vi. If the condition

$$\sup_{t\in q^{\mathbb{Z}}}\left\{\exp\left(\varepsilon\left|\frac{\ln t}{\ln q}\right|^{\delta}\right)\left(\left|1-\widehat{a}(t)\right|+\left|2-\widehat{b}(t)\right|+\left|\widehat{c}(t)\right|\right)\right\}<\infty,\quad\frac{1}{2}\leq\delta\leq1$$

holds for some $\varepsilon>0$, then the quadratic pencil of q-difference operator L_λ^q has finite number of eigenvalues and spectral singularities with finite multiplicity.

Proof. The proof can be done similar to that of related results in preceding sections. For brevity we only give the outlines. From (3.7) we have (i) and (ii.). Using $n=\frac{\ln t}{\ln q}$ and (i), proof of (iii) can be obtained in a similar way to that of Theorem 1. Hence, Lemma 3 along with Corollary 1 implies (iv). Using (3.5), (3.7), (3.9), and Theorem 3 we obtain (v). Finally, combining Theorems 3, 4, and (3.9), we conclude (vi). □

REMARK 2. Theorem 5 not only covers the results of [3, Theorem 5] and [4, Theorem 4] but also derives spectral properties of Klein-Gordon type q-difference equations in the special case $b(t)=v^2(t)$ and $c(t)=-v(t)$.

4. Principal vectors. In this section, we determine the principal vectors of L_λ and discuss their convergence properties. Thus, we will have an information about principal vectors of the operators Λ, Γ, and L_λ^q.

Define

$$F_n^+(\lambda):=f_n^+\left(2\arccos\frac{\lambda}{2}\right),\quad n\in\mathbb{Z},$$
$$F_n^-(\lambda):=f_n^-\left(2\arccos\frac{\lambda}{2}\right),\quad n\in\mathbb{Z},$$
$$H(\lambda):=\Phi\left(2\arccos\frac{\lambda}{2}\right).$$

Obviously, $F^{\pm}(\lambda) = \{F_n^{\pm}(\lambda)\}$ solve Equation (1.4), and

$$H(\lambda) = W\left[F^{-}(\lambda), F^{+}(\lambda)\right]$$

is satisfied. Furthermore, $F^{\pm}$ and H are analytic in $\Theta = \mathbb{C} \setminus [-2,2]$ and continuous up to the boundary of Θ. Using Lemma 3 and Corollary 1 we can state the sets $\sigma_d(L_\lambda)$ and $\sigma_{ss}(L_\lambda)$ as the sets of zeros of the function H in Θ and in $[-2,2]$, respectively. Moreover,

$$\sup_{n\in Z}\left\{\exp\left(\varepsilon|n|^{\delta}\right)(|1-a_n|+|p_n|+|q_n|)\right\}<\infty, \qquad \frac{1}{2}\le\delta\le 1, \varepsilon>0$$

is the condition that guarantees finiteness of zeros H in Θ and in $[-2,2]$. Let $\lambda_1, ..., \lambda_s$ denote the zeros of the functions H in Θ (which are the eigenvalues of L_λ) with multiplicities $m_1, ..., m_s$, respectively. Similarly, let $\lambda_{s+1}, ..., \lambda_k$ be zeros of the functions H in $[-2,2]$ (which are the spectral singularities of L_λ) with multiplicities $m_{s+1}, ..., m_k$, respectively. Similar to that of [1, Theorem 5.1] one can prove the next result.

Theorem 6.

$$\left\{\frac{d^r}{d\lambda^r}F_n^{+}(\lambda)\right\}_{\lambda=\lambda_j} = \sum_{v=0}^{r}\begin{pmatrix} r \\ v \end{pmatrix}\beta_{r-v}^{+}\left\{\frac{d^v}{d\lambda^v}F_n^{-}(\lambda)\right\}_{\lambda=\lambda_j}, n\in\mathbb{Z},$$

holds for $r = 0, 1, ..., m_{j-1}$, $j = 1, 2, ..., k$.

Let us introduce the vectors

$$U^{(r)}(\lambda_j) := \left\{U_n^{(r)}(\lambda_j)\right\}_{n\in\mathbb{Z}}, \tag{4.1}$$

for $r = 0, 1, ..., m_{j-1}$, $j = 1, 2, ..., k$, where

$$\begin{aligned} U_n^{(r)}(\lambda_j) &= \frac{1}{r!}\left\{\frac{d^r}{d\lambda^r}F_n^{+}(\lambda)\right\}_{\lambda=\lambda_j} \\ &= \sum_{v=0}^{r}\frac{\beta_{r-v}^{+}}{(r-v)!}\frac{1}{v!}\left\{\frac{d^v}{d\lambda^v}F_n^{-}(\lambda)\right\}_{\lambda=\lambda_j}, n\in\mathbb{Z}. \end{aligned} \tag{4.2}$$

Define the difference expression $\ell_\lambda U^{(n)}$ by

$$\ell_\lambda U^{(n)}(\lambda_j) = \Delta\left(a_{n-1}\Delta U^{(n-1)}(\lambda_j)\right) + \left(q_n + 2\lambda_j p_n + \lambda_j^2\right)U^{(n)}(\lambda_j), \ \ n\in\mathbb{Z}.$$

We get by Equation (1.4) that

$$\begin{aligned} \ell_\lambda U^{(0)}(\lambda_j) &= 0, \\ \ell_\lambda U^{(1)}(\lambda_j) + \frac{1}{1!}\frac{d\ell_\lambda}{d\lambda}U^{(0)}(\lambda_j) &= 0, \\ \ell_\lambda U^{(r)}(\lambda_j) + \frac{1}{1!}\frac{d\ell_\lambda}{d\lambda}U^{(r-1)}(\lambda_j) + \frac{1}{2!}\frac{d^2\ell_\lambda}{d\lambda^2}U^{(r-2)}(\lambda_j) &= 0 \end{aligned}$$

for $r = 2, 3, ..., m_{j-1},\ j = 1, 2, ..., k$. This shows that $U^{(r)}(\lambda_j) := \left\{ U_n^{(r)}(\lambda_j) \right\}_{n \in \mathbb{Z}}$ for $r = 0, 1, ..., m_{j-1},\ j = 1, 2, ..., s$ are principal vectors corresponding to the eigenvalues $\lambda_1, ..., \lambda_s$ of the operator L_λ. The principal vectors corresponding to the spectral singularities $\lambda_{s+1}, ..., \lambda_k$ are found similarly.

Let us define the Hilbert spaces

$$H_p(\mathbb{Z}) := \left\{ y = \{y_n\}_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} (1 + |n|)^{2p} |y_n|^2 < \infty \right\},$$

$$H_{-p}(\mathbb{Z}) := \left\{ y = \{y_n\}_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} (1 + |n|)^{-2p} |y_n|^2 < \infty \right\}$$

for $p = 0, 1, ...$ Evidently $H_0(\mathbb{Z}) = \ell^2(\mathbb{Z})$ and

$$H_{p+1} \subsetneqq H_p \subsetneqq \ell^2(\mathbb{Z}) \subsetneqq H_{-p} \subsetneqq H_{-p-1}.$$

Convergence properties of principal vectors of L_λ are given in the following theorem.

THEOREM 7. *We have*

i. $U^{(r)}(\lambda_j) := \left\{ U_n^{(r)}(\lambda_j) \right\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ *for* $r = 0, 1, ..., m_{j-1},\ j = 1, 2, ..., s,$

ii. $U^{(r)}(\lambda_j) := \left\{ U_n^{(r)}(\lambda_j) \right\}_{n \in \mathbb{Z}} \notin \ell^2(\mathbb{Z})$ *for* $r = 0, 1, ..., m_{j-1},\ j = s + 1, s + 2, ..., k,$

iii. $U^{(r)}(\lambda_j) := \left\{ U_n^{(r)}(\lambda_j) \right\}_{n \in \mathbb{Z}} \in H_{-p_0+1}$ *for* $r = 0, 1, ..., m_{j-1},\ j = s + 1, s + 2, ..., k,$ *where*

$$p_0 = \max\{m_1, m_2, ..., m_s, m_{s+1}, ..., m_k\}.$$

Proof. Use (4.1-4.2) and proceed by a method as in [1, Theorem 5.2] and [1, Lemma 5.1]. □

REMARK 3. Using the substitutions that lead to establishment of Subsections 3.1, 3.2, and 3.3 in the equalities (4.1-4.2), one may derive the principal vectors of the operators Λ, Γ, and L_λ^q easily. Moreover, Theorem 7 enables us to see the convergence properties of principal vectors of the operators Λ, Γ, and L_λ^q.

Open problem: The eigenfunction expansion has not been studied even for the above mentioned particular cases. So, this may be the topic of further studies.

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