Model-based robust chaotification using sliding mode control

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Abstract: Chaos is a complex behavior of dynamical nonlinear systems that is undesirable in most applications and should be controlled; however, it is desirable in some situations and should be generated. In this paper, a robust chaotification scheme based on sliding mode control is proposed for model based chaotification. A continuous time single input observable system is considered such that it is subject to parameter uncertainties, nonlinearities, noises, and disturbances, which are all additive to the input and can be modeled as an unknown function but bounded by a known function. The designed dynamical state feedback control law forces the system to match a reference chaotic system in finite time irrespective of the mentioned uncertainties, noises, and disturbances, as provided by the developed sliding mode control scheme. Simulation results are provided to illustrate the robustness of the proposed scheme against parameter uncertainties and noises. The results are compared with those of other model-based methods and Lyapunov exponents are calculated to show whether the closed-loop control systems exhibit chaotic behavior or not.

Key words: Anticontrol, dynamical feedback, robust chaotification, sliding mode control

1. Introduction

Chaos is a complex behavior of dynamical nonlinear systems that exhibits 3 main properties: sensitivity to initial conditions, topological transitivity, and denseness of unstable periodic orbits [1, 2, 3]. Chaos is a behavior to be avoided in most applications and thus should be controlled; however, it is thought to be useful in nature and in some engineering applications and so it should not be suppressed and even should be generated. Generation of chaos from a nonchaotic dynamical system is the process of chaotification (also called anticontrol of chaos or chaotization). In contrast, controlling chaos is the process of directing a chaotic system to exhibit a desired behavior. Stabilizing an equilibrium, tracking a desired nonchaotic trajectory, e.g., a periodic solution, and modification of chaotic behavior in some ways are examples of chaos control applications [4, 5, 6, 7, 8, 9].

Controlling chaos can be achieved via feedforward (open loop) and feedback (closed loop) methods. Feedforward methods rely generally on a properly chosen input function or external excitations as expressed in [4, 5]. It is beneficial for its quick response and its simplicity due to the absence of the feedback part. The OGY method [10], Pyragas's time-delayed method [11], the sliding mode control method [12, 13, 14, 15, 16, 17, 18], frequency domain methods [19], adaptive control methods [4], the speed gradient method [4], neural network and fuzzy-based methods [4, 20], and other nonlinear methods [4, 6, 21] are among the feedback methods used for chaos control in the literature. In addition to these feedback methods applied to the discrete and continuous time

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integer order dynamical systems, controlling chaos in the fractional order systems [22] and designing fractional order controllers [23] are attracting growing interest in the literature. Among the chaos control methods, sliding mode chaos control [12, 13, 14, 15, 16, 17, 18], which is indeed a discontinuous feedback method, has a distinguishing feature of applying a high frequency switching feedback in order to eliminate chaotic behavior even under parameter uncertainties, noises, disturbances, and nonlinearities. On the other hand, anticontrol of chaos is emerged as an interesting and potential area of research due to the many indications about the usefulness of chaos in nature and in engineering practice. Several efficient chaotification methods employing feedback control techniques have been introduced in the literature for both discrete and continuous time systems. Chaotification methods for discrete time systems [24] are mainly based on a proper feedback law yielding chaos in the overall system in the sense of Devaney [2] and/or Li-Yorke [25]. Many chaotification methods for continuous time systems have been developed in the literature [26, 27]. Some of them can be categorized as the Vanecek–Celikovsky method [28], time-delay feedback [29, 30, 31], impulsive control [27], and model-based static feedback chaotification [32, 33]. In addition to these methods, sliding mode control-based chaotification methods have been introduced in [34, 35]. The method in [34] can be categorized as a synchronization method because it is designed to follow the states of a reference chaotic system. In [35] a sliding mode control-based chaotification method designed for nonlinear discrete time systems is proposed.

This paper proposes a model-based robust chaotification scheme using sliding mode control in a manner different to that in [34, 35]. The proposed chaotification method yields a dynamical state feedback in order to match all system states to a reference chaotic system. In contrast to this, another model-based dynamical feedback method [3] introduces extra states and matches to a higher dimensional reference chaotic system. The proposed method can be applied to any single input, observable and input state linearizable system subject to parameter uncertainties, nonlinearities, noises, and disturbances. It is assumed that parameter uncertainties, nonlinearities, noises, and disturbances are all additive to the input and they can be modeled as an unknown function having a bound specified by a known function. The discontinuous feedback control law of the sliding mode chaotification method provides robustness against noise and disturbances. The robustness of the proposed method makes the chaotification immune in the sense that the resulting system remains in chaos for a wide range of system parameters and also under noise and disturbances. The matching of the considered system to the reference chaotic system is always achieved in finite time, which can be made arbitrarily small by modifying a parameter changing the control input.

The proposed method needs reference chaotic systems in the normal form. Therefore, in Section 2, transformation of reference chaotic systems into the normal form and obtaining higher dimensional chaotic systems in the normal form are described. In Section 3, a sliding mode control-based robust chaotification scheme in which a nonlinear sliding manifold and a dynamical feedback law are determined appropriately to match all states of controllable linear and input state linearizable nonlinear systems to reference chaotic systems in the normal form is described. Section 4 presents simulation results for a linear system and an input state linearizable nonlinear system respectively subject to parameter uncertainties and uniformly distributed random noise.

2. Normal form of reference chaotic systems

A great number of chaotic systems are in the normal form [32, 33, 36, 37] given as

$$\dot{z} = A_c z + b_c g_c(z), \tag{1}$$

where $g_c : R^n \to R$ is a nonlinear function, $A_c \in R^{n \times n}$ is a constant matrix, and $b_c \in R^n$ is a constant vector. A_c and b_c are in the following controllable canonical form:

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \\ a_0 & a_1 & \dots & \dots & a_{n-1} \end{pmatrix}, b_c = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

Including the linear terms weighted by a_i's into $G_c(z)$, the system in (1) can be reformulated as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= G_c(z), \end{aligned} \tag{3}$$

where $G_c(z) = g_c(z) + a_0 z_1 + a_1 z_2 + \cdots + a_{n-1} z_n$.

For $n = 3$, $G_c(z)$ represents the jerk function and $g_c(z)$ represents the nonlinearity in the jerk function. In the rest of the paper, $G_c(z)$ will be called a jerk function for arbitrary dimension $n \geq 3$. There exists a large class of systems in the form of jerk equations [36, 37]. In addition, a great number of chaotic systems can be formulated as

$$\dot{x} = Ax + bg_c(x), \tag{4}$$

where $A \in R^{n \times n}$ is a constant matrix, $b \in R^n$ is a constant vector, and $g_c : R^n \to R$ represents the nonlinear part of the chaotic system. Any system in the form of (4) can be transformed into the normal form as in (1) via the linear transformation $z = Tx$ if the controllability matrix $C^{n x n}$ has rank n, and hence it is invertible [32].

$$C = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

The transformation matrix is given as

$$T = \begin{pmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{pmatrix} \tag{5}$$

where q^T is the nth row of C^{-1}, and in the transformed system $A_c = TAT^{-1}$ and $b_c = Tb$ have the form in 2.

Example 1 (Transformation of Chua's circuit with cubic nonlinearity into the normal form):
State space of Chua's circuit with cubic nonlinearity [38] is in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha[x_2 - f(x_1)] \\ x_1 - x_2 + x_3 \\ -\beta x_2 \end{pmatrix} \tag{6}$$

where $f(x_1) = m_0 x_1^3 + m_1 x_1$ is the cubic nonlinearity of the Chua's circuit. This system can be written in the form of (4) as follows:

$$\dot{x} = \begin{pmatrix} 0 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} [-\alpha f(x_1)] \tag{7}$$

There exists an invertible linear transformation

$$z = Tx = \begin{pmatrix} 0 & 0 & -\beta^{-1} \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} x \tag{8}$$

yielding a normal form as in (1),

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \alpha - \beta & -1 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} [-\alpha f(\beta z_1 + z_2 + z_3)] \tag{9}$$

and can be reformulated as in (3) for $n = 3$, where

$$G_c(z) = (\alpha - \beta)z_2 - z_3 - \alpha f(\beta z_1 + z_2 + z_3). \tag{10}$$

In [39] the conditions to transform a 3-dimensional nonlinear system into at least 1 equivalent jerk equation are introduced. In [40] it is proved that chaotic systems in the strict-feedback form can be transformed into the normal form. There exists a great number of 3-dimensional chaotic systems that either exist in the normal form [36] or can be transformed into the normal form via global diffeomorphisms on R^n as described above.

A method to generate chaotic systems for $n > 3$ in the form of (1), which have chaotic attractors qualitatively similar to lower dimensional chaotic systems of the same form, is described in [33]. In order to obtain a 4-dimensional chaotic system via modifying a 3-dimensional chaotic system in the form of (3), the following system is considered:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \hat{G}_c(z_1, z_2, z_3) \end{aligned} \tag{11}$$

The system in (11) is modified in order to obtain a higher dimensional system in the following form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \hat{G}_c(z_1, z_2, z_3) + z_4 \\ \dot{z}_4 &= -\gamma z_4 \end{aligned} \tag{12}$$

where $\gamma > 0$ is an arbitrary constant. It is well known that $z_4(t) = z_4(0)e^{\gamma t} \to 0$ as $t \to \infty$. The system in (12) exhibits a chaotic attractor qualitatively similar to the lower dimensional chaotic model in (11). By the procedure described in [33], the system in (12) can be reformulated into the normal form as follows:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= G_c(z) \end{aligned} \tag{13}$$

where

$$G_c(z_1, z_2, z_3, z_4) = \frac{d}{dt}(\hat{G}_c(z_1, z_2, z_3)) - \gamma[z_4 - \hat{G}_c(z_1, z_2, z_3)]. \tag{14}$$

Furthermore, an arbitrary dimensional chaotic system can be obtained by introducing a new state variable at each step, provided that $\hat{G}_c$ is sufficiently smooth. The details of the procedure may be found in [33].

Example 2 (Obtaining a 4-dimensional chaotic system by modifying a 3-dimensional chaotic system with a quadratic nonlinearity): A 3-dimensional chaotic system with a quadratic nonlinearity is considered here in order to obtain a 4-dimensional chaotic system as described above. The 3-dimensional chaotic system with quadratic nonlinearity is given in the state space form as in (11), where $\hat{G}_c = \alpha z_3 + \beta z_2 + z_1^2 - 1$ [37]. This chaotic system can be modified in order to obtain a 4-dimensional chaotic system as described in (12) and it can be written into the normal form as in (13) by choosing $\gamma = 1$:

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= G_c(z)
\end{aligned}
\tag{15}
$$

where

$$
G_c(z) = (\alpha - 1)z_4 + (\alpha + \beta)z_3 + (\beta + 2z_1)z_2 + z_1^2 - 1.
\tag{16}
$$

3. Sliding mode chaotifying control laws for matching input state linearizable systems to reference chaotic systems

Continuous time single input observable systems subject to uncertainties, nonlinearities, noises, and disturbances are considered in the paper as the systems to be chaotified

$$
\dot{x} = f(x) + g(x)[u + \delta(t, x, u)],
\tag{17}
$$

where $x \in R^n$ is the state, $u \in R$ is the scalar control input, $f : R^n \to R^n$ and $g : R^n \to R^n$ are sufficiently smooth functions, and the function $\delta(t, x, u)$ with $\delta : R \times R^n \times R \to R$ is an unknown real valued function describing the uncertainties, nonlinearities, noises, and disturbances additive to the input. It is assumed that the unknown function $\delta(t, x, u)$ is bounded by a known function. For this system, a feedback law such that the resulting closed-loop system exhibits chaotic behavior after a finite time under uncertainties, nonlinearities, noises, and disturbances additive to the input is proposed. All the states of the systems of (17) are assumed available or can be obtained in an indirect way since the systems are considered observable.

A sliding mode feedback control law yielding the desired chaotic behavior for the closed loop system is provided in Section 3.1 for the controllable linear system case of (17). Section 3.2 provides the feedback law in the same way for the input state feedback linearizable case of (17).

3.1. Linear system case

An n-dimensional single input observable linear system subject to parameter uncertainties, nonlinearities, noises, and disturbances additive to the input is considered in the following form:

$$
\dot{x} = Ax + b[u + \delta(t, x, u)]
\tag{18}
$$

Assuming that the linear system is controllable, the system can be transformed via a linear transformation $z = Tx$, similarly given in (5), into the following normal form:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ a_0 & a_1 & \ldots & \ldots & a_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [u + \delta(t, T^{-1}z, u)] \tag{19}$$

where $\delta(t, T^{-1}z, u)$ represents uncertainties, nonlinearities, noises, and disturbances additive to the input. The dynamical feedback law is chosen as

$$\begin{aligned} u &= -a_0 z_1 - a_1 z_2 - \cdots - a_{n-2} z_{n-1} - a_{n-1} z_n + \dot{z}_n - \ddot{z}_n \\ &\quad + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} \dot{z}_n + v, \end{aligned} \tag{20}$$

where $G_c(z_1, z_2, \cdots, z_n) : R^n \to R$ is a nonlinear function chosen to be the jerk function of the reference chaotic system in the normal form of (3). Then the system becomes

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} [v + \dot{z}_n - \ddot{z}_n + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} \dot{z}_n + \hat{\delta}(t, z, v)], \tag{21}$$

where v is the new control input and $\hat{\delta}(t, z, v)$ is the uncertainty, nonlinearity, and noise rewritten in terms of z and v. By defining a new state variable $\dot{z}_n = z_{n+1}$ with $\ddot{z}_n = \dot{z}_{n+1} = \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v + \hat{\delta}(t, z, v)$, the $n+1$ dimensional state space form of the system can be obtained as in (22). The amplitude of $\hat{\delta}(t, z, v)$ is assumed to be bounded by a known function: $|\hat{\delta}(t, z, v)| \le \hat{p}(t, z) + k \parallel v \parallel_\infty$ for $\hat{p}(t, z) > 0$ and $0 \le k < 1$.

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= z_{n+1} \\ \dot{z}_{n+1} &= \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v + \hat{\delta}(t, z, v) \end{aligned} \tag{22}$$

Now, the sliding manifold is specified as $s = z_{n+1} - G_c(z_1, z_2, \cdots, z_n)$, where $G_c(z_1, z_2, \cdots, z_n)$ is chosen to be the jerk function of the reference chaotic system in the normal form of (3). Then one can apply the sliding mode control where $\eta > 0$ is a scalar to adjust finite reaching time to the sliding manifold.

$$v = -\frac{\eta + \hat{p}(t, z)}{1 - k} sign(s) \tag{23}$$

After reaching the sliding manifold, s becomes zero and so $z_{n+1} = G_c(z_1, z_2, \cdots z_n)$. Therefore, the first n states of the system (22) can be seen to be matched to the reference chaotic system (3) with the jerk function $G_c(z_1, z_2, \cdots, z_n)$.

In order to show that the matching can be achieved in finite time, one can choose the Lyapunov function as $V = \frac{1}{2}s^2$. It is shown below that time derivative $\dot{V}$ of this Lyapunov function along trajectories of the system in (22) is not greater than $-\eta|s|$. Therefore, reaching to the sliding manifold is of finite duration rather than asymptotical [41].

$$\dot{V} = s\dot{s} = s(\nabla_{\bar{z}}s)^T\dot{\bar{z}} =$$

$$s\begin{bmatrix} -\frac{\partial G_c}{\partial z_1} \\ -\frac{\partial G_c}{\partial z_2} \\ \vdots \\ -\frac{\partial G_c}{\partial z_n} \\ 1 \end{bmatrix}^T \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_{n+1} \\ \frac{\partial G_c}{\partial z_1}z_2 + \frac{\partial G_c}{\partial z_2}z_3 + \cdots + \frac{\partial G_c}{\partial z_n}z_{n+1} + v + \hat{\delta}(t,z,v) \end{bmatrix} = s[v + \hat{\delta}(t,z,v)], \tag{24}$$

where $\nabla_{\bar{z}}s$ is the gradient of s manifold with respect to $\bar{z} = [z^T \, z_{n+1}]^T$.

$$\dot{V} \leq sv + |s||\hat{\delta}(t,z,v)| \tag{25}$$

Under the assumption of $|\hat{\delta}(t,z,v)| \leq \hat{p}(t,z) + k \parallel v \parallel_\infty$, (25) becomes as follows:

$$\dot{V} \leq sv + [\hat{p}(t,z) + k \parallel v \parallel_\infty]|s| \tag{26}$$

Now, substituting v in (23) into (26), an upper bound for $\dot{V}$ is obtained as follows:

$$\dot{V} \leq -\frac{\eta + \hat{p}(t,z)}{1-k}|s| + [\hat{p}(t,z) + k\frac{\eta + \hat{p}(t,z)}{1-k}]|s| = -\eta|s| \tag{27}$$

The differential inequality in (27) can be solved first by dividing both sides by $|s|$ and then integrating them. Thus, one can get $|s(t)| - |s(0)| \leq -\eta t$ for the initial time, i.e. $t_0 = 0$. It means that the time needed to reach sliding manifold $s = 0$ should be finite and has an upper bound as $t_{reach} \leq \frac{|s(0)|}{\eta}$, i.e. $t_0 = 0$ [42]. Therefore, (22) with (23) becomes (28) when $s = 0 \Rightarrow z_{n+1} = G_c(z_1, z_2, \cdots, z_n)$.

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= G_c(z_1, z_2, \cdots, z_n) \\ \dot{z}_{n+1} &= \dot{G}_c(z_1, z_2, \cdots, z_n) \end{aligned} \tag{28}$$

The first n states of the system (28) can be seen to be matched to the reference chaotic system (3) with the jerk function $G_c(z_1, z_2, \cdots, z_n)$. $|z_{n+1}| = |G_c(z_1, z_2, \cdots, z_n)| < \infty$ since z_i's are bounded for a chaotic trajectory and the continuous function $G_c(z_1, z_2, \cdots, z_n)$ maps a bounded set into a bounded set.

3.2. Input state linearizable nonlinear system case

An n-dimensional single input observable and input state linearizable system subject to parameter uncertainties, nonlinearities, noises, and disturbances additive to the input is considered in the form given below:

$$\dot{x} = f(x) + g(x)[u + \delta(t,x,u)], \tag{29}$$

where $x \in R^n$ is the state, $u \in R$ is the scalar control input, $f : R^n \to R^n$ and $g : R^n \to R^n$ are sufficiently smooth functions, and $\delta(t, x, u)$ with $\delta : R \times R^n \times R$ is an unknown real valued function describing the uncertainties, nonlinearities, noises, and disturbances additive to the input. The $\delta(t, x, u)$ is assumed to be bounded by a known function. Since the system (29) is assumed to be input state linearizable then there should exist a smooth scalar function $h(x)$ satisfying $L_g L_f^i h(x) = 0$ for $i = 0, \cdots, n - 2$ and $L_g L_f^{n-1} h(x) \neq 0$ for all $x \in D$ [43].

Under the above input state linearizability assumption, the system in (29) can be transformed with the state transformation $z = T(x) = \left(h(x) L_f h(x) \cdots L_f^{n-1} h(x) \right)^T$,

which is a diffeomorphism over $D \in R^n$, into the normal form [43] as

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))[u + \delta(t, T^{-1}(z), u)]
\end{aligned}
\tag{30}
$$

The dynamical feedback law is chosen as

$$
\begin{aligned}
u = \frac{1}{L_g L_f^{n-1} h(T^{-1}(z))} [\dot{z}_n - \ddot{z}_n - L_f^n h(T^{-1}(z)) + \frac{\partial G_c}{\partial z_1} z_2 \\
+ \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} \dot{z}_n + v],
\end{aligned}
\tag{31}
$$

where $G_c(z_1, z_2, \cdots, z_n) : R^n \to R$ is a nonlinear function chosen to be the jerk function of the reference chaotic system in the normal form of (3). Then the system in (30) becomes

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= \dot{z}_n - \ddot{z}_n + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} \dot{z}_n + v \\
&\quad + L_g L_f^{n-1} h(T^{-1}(z))[\hat{\delta}(t, z, v)],
\end{aligned}
\tag{32}
$$

where v is the new control input and $\hat{\delta}(t, z, v)$ is the uncertainty, nonlinearity, and noise rewritten in terms of z and v. As done in the linear case, by defining a new state variable $\dot{z}_n = z_{n+1}$ with $\ddot{z}_n = \dot{z}_{n+1} = \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v + L_g L_f^{n-1} h(T^{-1}(z))[\hat{\delta}(t, z, v)]$, an $n + 1$ dimensional state space form of the system can be obtained as in Eq. (33). The amplitude of $\hat{\delta}(t, z, v)$ with $L_g L_f^{n-1} h(T^{-1}(z))$ is assumed to be bounded by a known function: $|L_g L_f^{n-1} h(T^{-1}(z))[\hat{\delta}(t, z, v)]| \leq \hat{p}(t, z) + k \parallel v \parallel_\infty$ for $\hat{p}(t, z) > 0$ and $0 \leq k < 1$.

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_n &= z_{n+1} \\
\dot{z}_{n+1} &= \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \cdots + \frac{\partial G_c}{\partial z_n} z_{n+1} + v \\
&\quad + L_g L_f^{n-1} h(T^{-1}(z))[\hat{\delta}(t, z, v)]
\end{aligned}
\tag{33}
$$

The sliding manifold is chosen as $s = z_{n+1} - G_c(z_1, z_2, \cdots, z_n)$, where $G_c(z_1, z_2, \cdots, z_n)$ is the jerk function of the reference chaotic system in the normal form of (3). Then one can apply the sliding mode control in (34)

where $\eta > 0$ is a scalar used to adjust finite reaching time to the sliding manifold. In a similar way to the linear case, the first n states of the system (33) match the reference chaotic system (3) with the jerk function $G_c(z_1, z_2, \cdots, z_n)$ when the reaching phase is over.

$$v = -\frac{\eta + \hat{p}(t, z)}{1 - k} sign(s) \tag{34}$$

After reaching the sliding manifold, s becomes zero and so $z_{n+1} = G_c(z_1, z_2, \cdots z_n)$. Therefore, the first n states of the system (33) can be seen to be matched to the reference chaotic system (3) with the jerk function $G_c(z_1, z_2, \cdots, z_n)$.

As done in the linear controllable case, $V = \frac{1}{2}s^2$ can be chosen as the Lyapunov function and its time derivative along the trajectories of the system in (33) is shown not to be greater than $-\eta|s|$ to ensure that the system (33) reaches the sliding manifold in finite time.

$$\dot{V} = s\dot{s} = s(\nabla_{\bar{z}}s)^T \dot{\bar{z}} = s[v + L_g L_f^{n-1} h(T^{-1}(z))[\hat{\delta}(t, z, v)], \tag{35}$$

where $\nabla_{\bar{z}}s$ is the gradient of s manifold with respect to $\bar{z} = [z^T z_{n+1}]^T$.

$$\dot{V} \leq sv + |s||L_g L_f^{n-1} h(T^{-1}(z))[\hat{\delta}(t, z, v)]| \tag{36}$$

Under the assumption of $|L_g L_f^{n-1} h(T^{-1}(z))[\hat{\delta}(t, z, v)| \leq \hat{p}(t, z) + k \parallel v \parallel_\infty$, it becomes as follows:

$$\dot{V} \leq sv + [p(x, t) + k \parallel v \parallel_\infty]|s| \tag{37}$$

Now, substituting v in (34) into (37), an upper bound for $\dot{V}$ is obtained as follows:

$$\dot{V} \leq -\frac{\eta + p(x, t)}{1 - k}|s| + [p(x, t) + k\frac{\eta + p(x, t)}{1 - k}]|s| = -\eta|s| \tag{38}$$

The differential inequality in (38) means that the time needed to reach sliding manifold $s = 0$ should be finite and has an upper bound $t_{reach} \leq \frac{|s(0)|}{\eta}$, i.e. $t_0 = 0$ [42]. Therefore, (33) with (34) becomes (39) when $s = 0 \Rightarrow z_{n+1} = G_c(z_1, z_2, \cdots, z_n)$.

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_n &= G_c(z_1, z_2, \cdots, z_n) \\
\dot{z}_{n+1} &= \dot{G}_c(z_1, z_2, \cdots, z_n)
\end{aligned} \tag{39}$$

The first n states of the system (39) can be seen to be matched to the reference chaotic system (3) with the jerk function $G_c(z_1, z_2, \cdots, z_n)$. $|z_{n+1}| = |G_c(z_1, z_2, \cdots, z_n)| < \infty$ since z_i's are bounded for a chaotic trajectory and the continuous function $G_c(z_1, z_2, \cdots, z_n)$ maps a bounded set into a bounded set.

4. Simulation results

4.1. Linear system application

A single input linear system with parameter uncertainty is considered in the form of (40)

$$\dot{x} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (u + \delta(x)) \tag{40}$$

where $\delta(x)$ is the parameter uncertainty and defined as $\delta(x) = \Delta a_0 x_1 + \Delta a_1 x_2 + \Delta a_2 x_3$. $\delta(x)$ is obviously bounded by a known function $|\delta(x)| \le p(x) = |\Delta a_0||x_1| + |\Delta a_1||x_2| + |\Delta a_2||x_3|$. The system can be transformed into the controllable canonical form by a linear transform as in (5):

$$z = Tx = \begin{pmatrix} 0 & 0.5 & -0.25 \\ 0 & -0.5 & 0.75 \\ 1 & 0.5 & -1.25 \end{pmatrix} x \tag{41}$$

The transformed system is in the normal form as in (19):

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{pmatrix} \dot{z} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (u + \delta(T^{-1}z)) \tag{42}$$

To focus on the effect of parameter uncertainty, the system can also be written as

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 + \triangle w_0 & -4 + \triangle w_1 & -3 + \triangle w_2 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \tag{43}$$

where $\triangle w_0 = \triangle a_0 + 3\triangle a_1 + 2\triangle a_2$, $\triangle w_1 = 2\triangle a_0 + \triangle a_1 + 2\triangle a_2$, and $\triangle w_0 = \triangle a_0$.

In order to chaotify the system in (42), $G_c(z_1, z_2, z_3)$ is taken to be the jerk function of Chua's circuit with cubic nonlinearity as given in (10) and the dynamical feedback law is chosen as follows with $\alpha = 15.6$, $\beta = 28.58$, $m_0 = 0.0659179490$, $m_1 = -0.1671315463$ where the parameters are taken from [44].

$$\begin{aligned} u &= 4z_1 + 4z_2 + 3z_3 + \dot{z}_3 - \ddot{z}_3 + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} \dot{z}_3 + v \\ &= 4z_1 + 4z_2 + 3z_3 + \dot{z}_3 - \ddot{z}_3 - \alpha 3 m_0 (\beta z_1 + z_2 + z_3)^2 (\beta z_2 + z_3 + \dot{z}_3) \\ &\quad -\alpha m_1 \beta z_2 + (\alpha - \beta - \alpha m_1) z_3 - (1 + \alpha m_1) \dot{z}_3 + v \end{aligned} \tag{44}$$

By inserting the control input in (44) into the system in (42) and by defining a new state variable $\dot{z}_3 = z_4$ with $\ddot{z}_3 = \dot{z}_4 = \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + v + \hat{\delta}(t, z, v)$, the 4-dimensional state space form of the system can be obtained as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + v + \hat{\delta}(t, z, v) \end{aligned} \tag{45}$$

The sliding manifold is specified as $s = z_4 - G_c(z_1, z_2, z_3)$ and the switching control input (v) described in (23) with setting the parameters of it as k=0, $\eta = 1$, and $\hat{p}(z) = p(T^{-1}z) = |\triangle a_0||z_1 + 2z_2 + z_3| + |\triangle a_1||3z_1 + z_2| +$

$|\triangle a_2||2z_2 +2z_3|$ is defined as follows:

$$v = -[1 + \hat{p}(z)]sign(s) \tag{46}$$

After the finite time $t_{reach} < |s(t = 0)|/\eta$ [42], the system in (45) reaches the sliding manifold and s becomes zero and so $z_4 = G_c(z_1, z_2, z_3)$. Then the system in (45) with its first 3 states matches the reference chaos model in (3) as

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= G_c(z_1, z_2, z_3) \\
\dot{z}_4 &= \dot{G}_c(z_1, z_2, z_3)
\end{aligned} \tag{47}$$

and the system in (40) becomes topologically conjugate to the reference chaotic system [32].

In Figure 1, the simulation result for $\triangle a_0 = -0.4596, \triangle a_1 = 0.8103$, and $\triangle a_2 = -0.8812$ is presented. The chaotic attractor of the reference chaotic system given in (9) is shown in Figure 1a. The chaotic attractor of the chaotified system with the model-based method in [32, 33] and the chaotic attractor of the chaotified system with the proposed sliding mode control method are shown in Figure 1b,1c. In Figure 1b, the chaotified system with the model-based method in [32, 33] is observed to exhibit limit cycle behavior due to the parameter uncertainty with Lyapunov exponents $\lambda_1 \cong 0(-0.0027), \lambda_2 = -2.428, \lambda_3 = -2.4893$ calculated by [45]. As

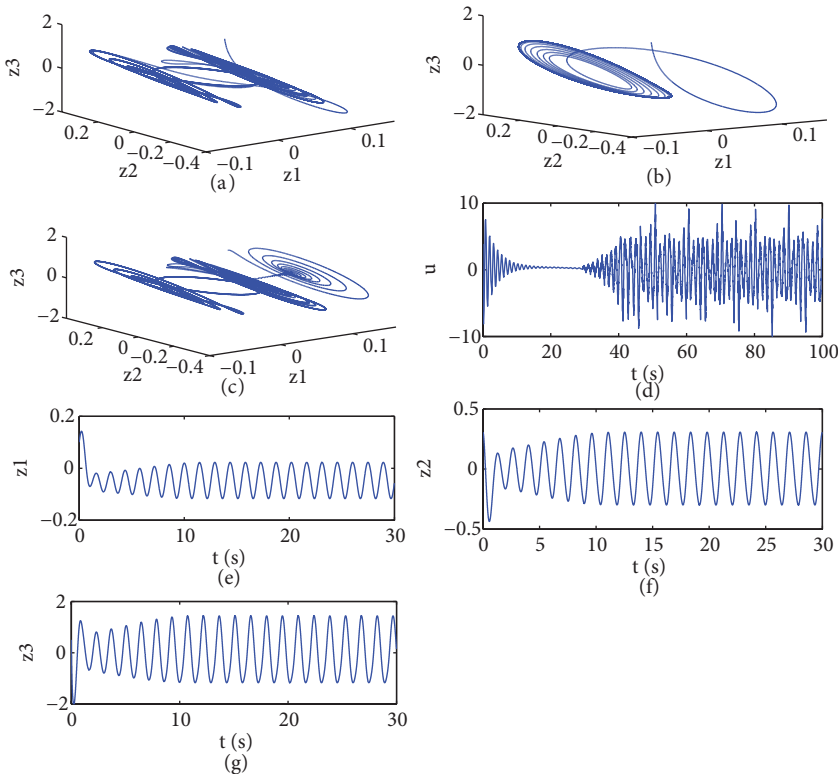**Figure 1**. Chaotic attractor of (a) reference chaotic system, i.e. the cubic Chua's circuit (9), (b) the chaotified system with the model-based method in [32, 33], which causes limit cycle behavior with Lyapunov exponents $\lambda_1 \cong 0(-0.0027), \lambda_2 = -2.428, \lambda_3 = -2.4893$, (c) the chaotified system with the proposed sliding mode control method, (d) chaotifying control input in (44) for the proposed method, (e) z_1 versus time, (f) z_2 versus time, (g) z_3 versus time for $t \leq 30s$ of the chaotified system with the model-based methods in [32, 33].

seen in Figure 1c, the chaotified system with the proposed method matches the chaotic system despite the uncertainty and after a finite transient time slides on it. In Figure 1d, the chaotifying control input in (44) for the proposed method is presented. In order to observe limit cycle behavior of the chaotified system with the model-based method in [32, 33], the states z_1, z_2, and z_3 versus time for $t \leq 30s$ are shown in Figure 1e–1g.

In Figure 2, the simulation result for $\triangle a_0 = -0.8359, \triangle a_1 = -0.9793$, and $\triangle a_2 = -0.9844$ is presented. The chaotic attractor of the reference chaotic system given in (9) is shown in Figure 2a. The chaotic attractor of the chaotified system with the model-based method in [32, 33] and the chaotic attractor of the chaotified system with the proposed sliding mode control method are shown in Figure 2b, 2c. In Figure 2b, the chaotified system with the model-based method in [32, 33] is observed to tend towards an equilibrium point due to the parameter uncertainty with Lyapunov exponents $\lambda_1 = -0.0432, \lambda_2 = -0.0446, \lambda_3 = -6.3534$ calculated by [45]. As seen in Figure 2c, the chaotified system with the proposed method matches the chaotic system despite the uncertainty and after a finite transient time slides on it. In Figure 2d, the chaotifying control input in (44) for the proposed method is presented. In order to observe the asymptotically stability of the chaotified system with the model-based method in [32, 33], the states z_1, z_2 and z_3 versus time are shown in Figure 2e–2g.

Furthermore, in order to show the effectiveness of the proposed method, the system in (40) is subjected to randomly chosen Δa_i's in the range [-1,1] for 100 trials. For the model-based method [32, 33] just 32 of
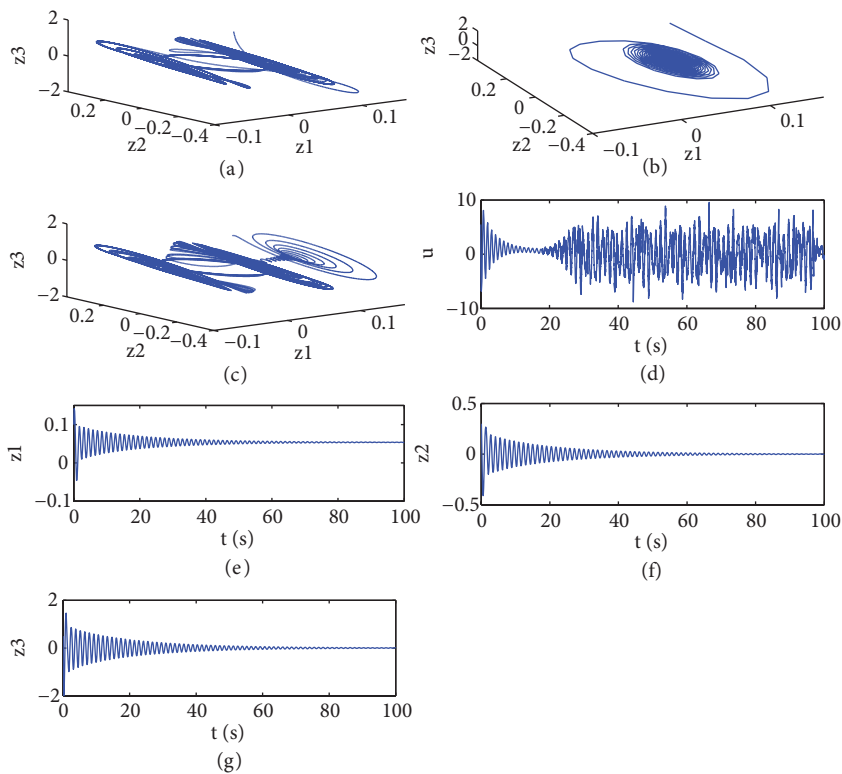

Figure 2. Chaotic attractor of (a) reference chaotic system, i.e. the cubic Chua's circuit (9), (b) the chaotified system with the model-based method in [32, 33], which tends toward an equilibrium point with Lyapunov exponents $\lambda_1 = -0.0432, \lambda_2 = -0.0446, \lambda_3 = -6.3534$, (c) the chaotified system with the proposed sliding mode control method, (d) chaotifying control input in (44) for the proposed method, (e) z_1 versus time, (f) z_2 versus time, (g) z_3 versus time of the chaotified system with the model-based method in [32, 33].

the 100 trials have Lyapunov exponents $\lambda_1 > 0$ in the interval of $[0.0468,0.5529]$, $\lambda_2 \cong 0$ in the interval of $[-0.0071,0.0015]$, and $\lambda_3 < 0$ in the interval of $[-6.8093,-3,4982]$, which is the sign of chaotic behavior for 3-dimensional systems [46], whereas the proposed method copes with uncertainties and after a finite transient it exhibits chaotic behavior for all trials.

4.2. Nonlinear system application

A link driven by a motor through a torsional spring (a single-link flexible-joint robot arm) with unit values of coefficients is considered. The details of the system may be found in [42]. A state space form of the system is given as

$$\dot{x} = \begin{pmatrix} x_2 \\ -sinx_1 - (x_1 - x_3) \\ x_4 \\ x_1 - x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} (u + \delta(t)) \tag{48}$$

where $\delta(t)$ is a uniformly distributed random noise added to see the effect of noise and it is in the interval $[d_0, d_1]$. $\delta(t)$ is bounded by a known scalar: $|\delta(t)| \leq p = max(|d_0|, |d_1|)$.

By the change of the variables, the feedback linearizable system in (48) can be transformed with the global diffeomorphism $z = (x_1 \ x_2 \ -sinx_1 - (x_1 - x_3) \ -x_2cosx_1 - (x_2 - x_4))^T$ into the normal form

$$\dot{z} = \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ sinz_1(z_2^2 + cosz_1 + 1) - (z_3 + sinz_1)(2 + cosz_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} (u + \delta(t)) \tag{49}$$

In order to chaotify the system in (49), $G_c(z_1, z_2, z_3, z_4)$ is taken to be the jerk function of the 4-dimensional chaotic system defined in (15) with quadratic nonlinearity as given in (16) and the dynamical feedback law is chosen as

$$\begin{aligned} u &= \dot{z}_4 - \ddot{z}_4 - sinz_1(z_2^2 + cosz_1 + 1) + \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + \frac{\partial G_c}{\partial z_4} \dot{z}_4 + v \\ &= \dot{z}_4 - \ddot{z}_4 - sinz_1(z_2^2 + cosz_1 + 1) + 2z_2^2 + 2z_1(z_2 + z_3) \\ &\quad + \beta z_3 + (\alpha + \beta)z_4 + (\alpha - 1)\dot{z}_4 + v \end{aligned} \tag{50}$$

By inserting the control input in (50) into the system in (49) and by defining a new state variable $\dot{z}_4 = z_5$ with $\ddot{z}_4 = \dot{z}_5 = \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + \frac{\partial G_c}{\partial z_4} z_5 + v + \delta(t)$, the 5-dimensional state space form of the system can be obtained as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= z_5 \\ \dot{z}_5 &= \frac{\partial G_c}{\partial z_1} z_2 + \frac{\partial G_c}{\partial z_2} z_3 + \frac{\partial G_c}{\partial z_3} z_4 + \frac{\partial G_c}{\partial z_4} z_5 + v + \delta(t) \end{aligned} \tag{51}$$

The sliding manifold is specified as $s = z_5 - G_c(z_1, z_2, z_3, z_4)$ and the switching control input v described in (34) with setting the parameters of it as k=0, $\eta = 1$, and $\hat{p}(t) = p$ is defined as follows:

$$v = -(1 + p)sign(s) \tag{52}$$

After the finite time $t_{reach} < |s(t = 0)|/\eta$ [42], the system reaches the sliding manifold and s becomes zero and

so $z_5 = G_c(z_1, z_2, z_3, z_4)$ and the system in (51) with its 4 states matches the reference chaos model in (3) as

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= G_c(z_1, z_2, z_3, z_4) \\
\dot{z}_5 &= \dot{G}_c(z_1, z_2, z_3, z_4)
\end{aligned}
\tag{53}
$$

and the system in (48) becomes topologically conjugate to the reference chaotic system [32].

In Figure 3, the simulation result for $d_0 = -0.18$ and $d_1 = 1.93$ is presented. In Figure 3a–3c, z_1 versus z_2, z_3, and z_4 of the reference chaotic system in (15) are shown. In Figure 3d–3f, z_1 versus z_2, z_3, and z_4 of
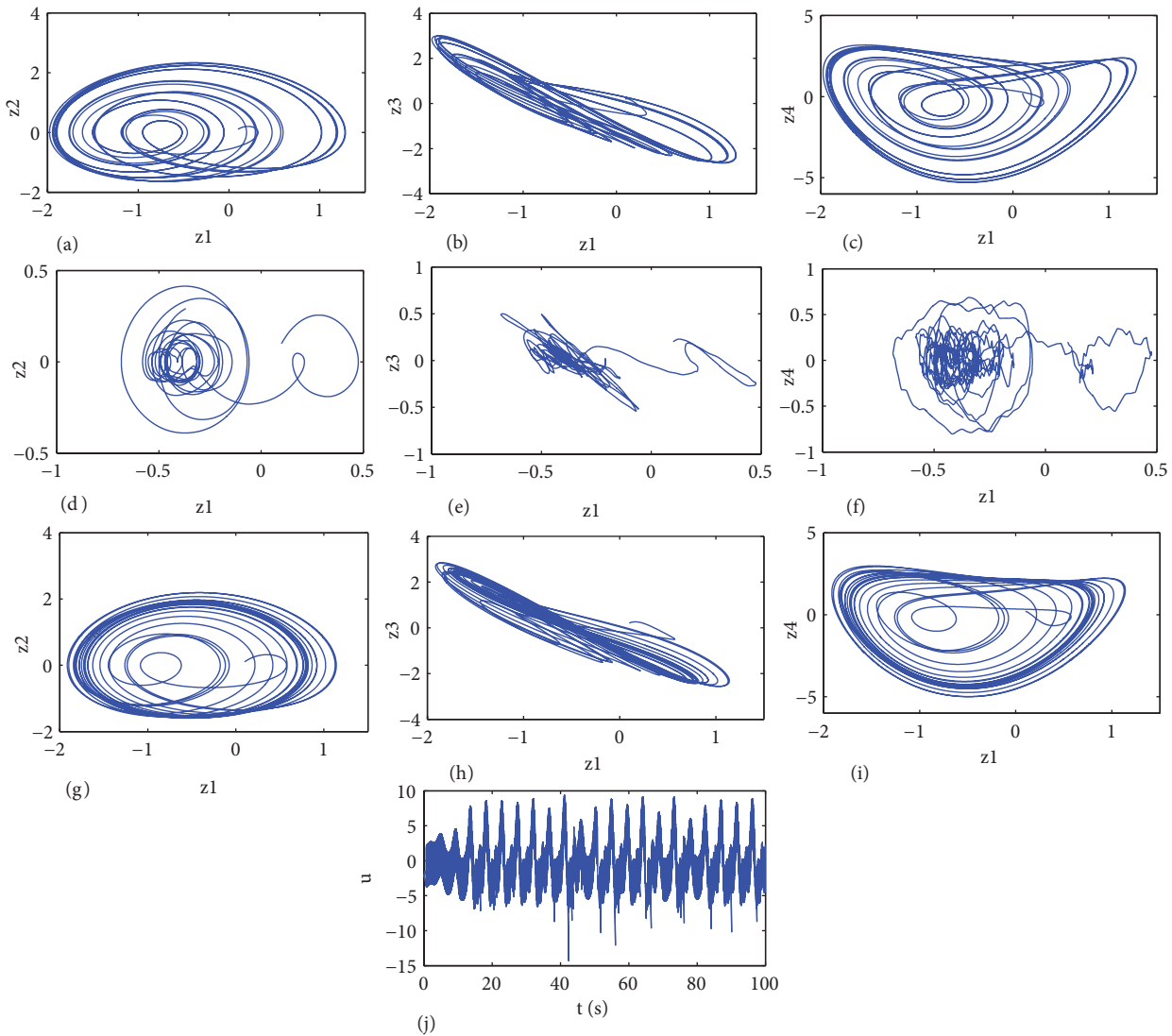

Figure 3. (a) z_1 versus z_2, (b) z_1 versus z_3, (c) z_1 versus z_4 of the reference chaotic system in (15), (d) z_1 versus z_2, (e) z_1 versus z_3, (f) z_1 versus z_4 of the chaotified system with the model based method in [32, 33], (g) z_1 versus z_2, (h) z_1 versus z_3, (i) z_1 versus z_4 of the chaotified system with the proposed sliding mode control method, and (j) chaotifying control input in (50) for the proposed method.

the chaotified system with the model-based method in [32, 33] are shown. In Figure 3g–3i, z_1 versus z_2, z_3, and z_4 of the chaotified system with the proposed sliding mode control method are shown. In Figure 3j, chaotifying control input in (50) for the proposed method is presented. In Figure 3d–3f, it is observed that the chaotified system with the model-based methods in [32, 33] exhibits different behavior than the reference chaotic system due to the effect of the uniformly distributed noise. As seen in Figure 3g–3i, the chaotified system with the proposed method reaches the chaotic manifold despite the noise and slides on it thereafter.

Furthermore, in order to show the effectiveness of the proposed method, the system in (48) is subjected to uniformly distributed random noise $\delta(t)$ in the interval $[d_0, d_1]$, where d_0 and d_1 are chosen randomly between [-2,2] for 100 trials. For the model-based method [32, 33] just 7 of the 100 trials have Lyapunov exponents $\lambda_1 > 0$ in the interval of [0.0145,0.1543], $\lambda_2 \cong 0$ in the interval of [-0.0049,0.0061], $\lambda_3 < 0$ in the interval of [-0.6464,-0.5121], and $\lambda_4 < 0$ in the interval of [-1.014,-1.0018], which is the sign of chaotic behavior for 4-dimensional systems [46], whereas the proposed method copes with noises and after a finite transient time it exhibits chaotic behavior for all trials.

5. Results and conclusion

A sliding mode control-based robust chaotification scheme has been introduced for model-based chaotification. The scheme can be applied to any continuous time single input controllable linear and input state linearizable nonlinear systems subject to parameter uncertainties, nonlinearities, noises, and disturbances that are all additive to the input and can be modeled as an unknown function but bounded by a known function. It is assumed that a reference chaotic system exists in the normal form and the designed dynamical state feedback control law forces the system to match the reference chaotic system in finite time irrespective of the mentioned uncertainties, noises, and disturbances, as provided by the developed sliding mode control scheme. The matching of the considered system to the reference chaotic system is always achieved in finite time, which can be made arbitrarily small by modifying a parameter changing the control input. Several simulations have demonstrated the robustness and effectiveness of the chaotification scheme.

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