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On the Hilbert formulas and of change of integration order for some singular integrals in the unit circle

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Abstract: We obtain some analogues of the Hilbert formulas on the unit circle for α-hyperholomorphic function theory when α is a complex number. Such formulas relate a pair of components of the boundary value of an α-hyperholomorphic function in the unit circle with the other one. Furthermore, the corresponding Poincaré–Bertrand formula for the α-hyperholomorphic singular integrals in the unit circle is presented.

Key words: Hilbert formulas, Poincaré–Bertrand formulas

1. Introduction

The classical Hilbert formulas describing the relation between the boundary values in the unit circle of a pair of conjugate harmonic functions are a well-known result in one-dimensional complex analysis. For exhaustive information on the topic, we refer the reader to [7].

Let us denote by $\mathbb{S}$ the unit circle in the complex plane $\mathbb{C}$ and set f a limit function in $\mathbb{S}$. Denote $g(\theta) := f(e^{i\theta})$, $0 \leq \theta < 2\pi$, and $g = g_1 + ig_2$. Then the real components g_1 and g_2 of g are related by the following formulas, known as the Hilbert formulas:

$$\begin{aligned} M[g_1] + \mathcal{H}[g_2] &= g_1, \\ M[g_2] - \mathcal{H}[g_1] &= g_2, \end{aligned} \tag{1}$$

where M and $\mathcal{H}$ are given by

$$\mathcal{H}[g](\theta) := \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - \theta}{2} g(\tau) d\tau, \quad \theta \in [0, 2\pi), \tag{2}$$

$$M[g] := \frac{1}{2\pi} \int_0^{2\pi} g(\tau) d\tau, \tag{3}$$

which are both defined on the linear space of real valued Hölder continuous functions $C^{0,\mu}(\mathbb{S}, \mathbb{R})$, $\mu \in (0, 1]$.

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The integral $\mathcal{H}[g]$ (well defined on $C^{0,\mu}(\mathbb{S},\mathbb{R})$, $\mu \in (0,1]$) is understood in the sense of Cauchy's principal value, generating the so-called Hilbert operator with (real) kernel $\frac{1}{2\pi}\cot\frac{\tau-\theta}{2}$. Meanwhile, $M[g]$ is a functional that can be viewed as the average value of g.

Assume now that $g \in \ker M$, and then the Hilbert formulas take the form

$$\begin{aligned} \mathcal{H}[g_2] &= g_1, \\ -\mathcal{H}[g_1] &= g_2. \end{aligned} \tag{4}$$

The Hilbert operator (2) is a well-known transformation in mathematics and in signal processing; for example, in geophysics and astrophysics it deals with input signals. Examples of this type of signals are seismic, satellite, and gravitational data, and the Hilbert operator proves to be useful for a local analysis of them, providing a set of rotation-invariant local properties: the local amplitude, local orientation, and local phase. See, for example, [23].

Various analogues of the Hilbert formulas on the unit sphere have kept interest until our days. For instance, in [11] there are introduced analogues for the case of solenoidal and irrotational vector fields, in [15] are presented analogues for time-harmonic electromagnetic fields, and in [14] are presented analogues for the time-harmonic relativistic Dirac bispinors theory. For the case of a half space there already exist works in that direction; see for example, [8, 17].

The names of Bertrand [1] and Poincaré [16] are attached to the formula of change of integration order in iterated singular integrals of Cauchy principal value type. It was previously obtained by Hardy [6] under certain conditions. This formula, which has proved useful to the study of one-dimensional singular integral equations and physical applications, takes the following form:

$$\frac{1}{\pi i}\int_{\gamma_\tau}\frac{d\tau}{\tau-t}\cdot\frac{1}{\pi i}\int_{\gamma_\varsigma}\frac{f(\tau,\varsigma)}{\varsigma-\tau}d\varsigma = f(t,t)+\frac{1}{\pi i}\int_{\gamma_\varsigma}d\varsigma\cdot\frac{1}{\pi i}\int_{\gamma_\tau}\frac{f(\tau,\varsigma)}{(\tau-t)(\varsigma-\tau)}d\tau,$$

or

$$\int_{\gamma_\tau}\frac{d\tau}{\tau-t}\cdot\int_{\gamma_\varsigma}\frac{f(\tau,\varsigma)}{\varsigma-\tau}d\varsigma = -\pi^2 f(t,t)+\int_{\gamma_\varsigma}d\varsigma\cdot\int_{\gamma_\tau}\frac{f(\tau,\varsigma)}{(\tau-t)(\varsigma-\tau)}d\tau, \tag{5}$$

where γ is a simple closed smooth curve in $\mathbb{R}^2$, t is a fixed point on γ, and f lies on some appropriate function space.

The Poincaré–Bertrand formula for the unit circle $\mathbb{S}$ is simplified by assuming $\tau = e^{i\theta}$, $t = e^{i\theta_0}$, and thus

$$\frac{d\tau}{\tau-t} = \frac{1}{2}\cot\frac{\theta-\theta_0}{2}\,d\theta+\frac{i}{2}\,d\theta.$$

Consequently, we can rewrite (5) as

$$\int_0^{2\pi}\cot\frac{\theta_0-\theta}{2}\,d\theta\int_0^{2\pi}\cot\frac{\theta-\theta_1}{2}g(\theta_1)\,d\theta_1 = -4\pi^2 g(\theta_0)+2\pi\int_0^{2\pi}g(\theta)\,d\theta. \tag{6}$$

Here the following have been used:

$$\cot\frac{\theta-\theta_0}{2}\,\cot\frac{\theta_1-\theta}{2} = -\cot\frac{\theta_0-\theta_1}{2}\left(\cot\frac{\theta_1-\theta}{2}+\cot\frac{\theta-\theta_0}{2}\right)+1,$$

$$\int_0^{2\pi} \cot\frac{\theta-\theta_0}{2}\,\cot\frac{\theta_1-\theta}{2}\,d\theta = 2\pi,$$

for $\theta_0 \neq \theta_1$.

Combining (2) and (3) in (6), we have

$$\mathcal{H}[\mathcal{H}[g](\theta)](\theta_0) = g(\theta_0) + M[g].$$

If $g \in ker\, M$, then

$$\mathcal{H}[\mathcal{H}[g](\theta)](\theta_0) = g(\theta_0).$$

There have already been considered extensions of the the Poincaré–Bertrand formula for problems with different backgrounds. For example, Mitelman and Shapiro [12] established a Poincaré–Bertrand formula for quaternion singular integrals of Cauchy type over a smooth Lyapunov surface, while Kytmanov [9] established an extension for the Bochner–Martinelli integral over a smooth manifold. Another important extension was achieved by Hang and Jiang [5] on smooth hypersurfaces in higher dimensions. For more recent references in different contexts, see, for instance, [2, 10, 13, 18–20, 24].

The outline of this paper is as follows. We collect in Section 2 basic facts and results of the α-hyperholomorphic function theory in $\mathbb{R}^2$, and Section 3 is devoted to the study of some analogues of the Hilbert formulas on the unit circle for α-hyperholomorphic function theory, α being a complex number. Finally, the corresponding Poincaré–Bertrand formula is derived.

2. Preliminaries

We start by giving a brief summary of some facts and results from α-hyperholomorphic function theory in $\mathbb{R}^2$ to be used in this paper. For more details, we refer the reader to [8] and the references therein.

Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary γ and introduce the temporary notations $\Omega := \Omega^+$ and $\Omega^- := \mathbb{R}^2 \setminus (\Omega^+ \cup \gamma)$.

Let $\mathbb{H}(\mathbb{C})$ be the set of complex quaternions, i.e. each quaternion a is represented in the form $a = \sum_{k=0}^{3} a_k i_k$, with $\{a_k\} \subset \mathbb{C}$; $i_0 = 1$ stands for the unit and i_1, i_2, i_3 stand for the quaternionic imaginary units. Denote the complex imaginary unit in $\mathbb{C}$ by i as usual. By definition, i commutes with all the quaternionic imaginary units i_1, i_2, i_3.

The set $\mathbb{H}(\mathbb{C})$ is a complex noncommutative, associative algebra with zero divisors. The involution $a \to \bar{a}$, called quaternionic conjugation, is defined by

$$\bar{a} := \sum_{k=0}^{3} a_k \cdot \bar{i}_k = a_0 - \sum_{k=1}^{3} a_k \cdot i_k.$$

It satisfies $\overline{ab} = \bar{b}\bar{a}$. We use the Euclidean norm $|a|$ in $\mathbb{H}(\mathbb{C})$, defined by $|a| := \sqrt{\sum_{k=0}^{3}|a_k|^2}$. Writing for $a = \sum_{k=0}^{3} a_k i_k \in \mathbb{H}(\mathbb{C}), a_0 =: Sc\,(a); \vec{a} := \sum_{k=1}^{3} a_k i_k =: Vect(a)$, we have $a = a_0 + \vec{a}$. We call a_0 the scalar part of the complex quaternion a and $\vec{a}$ the vector part of a. Then $\{Vect(a) : a \in \mathbb{H}(\mathbb{C})\}$ is identified with $\mathbb{C}^3$. This enables us to write $\bar{a} = Sc\,(a) - \vec{a}$.

For any $a, b \in \mathbb{H}(\mathbb{C})$:

$$a\,b := a_0\,b_0 - \langle \vec{a}, \vec{b}\rangle + a_0\,\vec{b} + b_0\,\vec{a} + [\vec{a}, \vec{b}],$$

where

$$\langle \vec{a}, \vec{b} \rangle := \sum_{k=1}^{3} a_k\, b_k,\ [\vec{a}, \vec{b}] := \begin{vmatrix} i_1 & i_2 & i_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

In particular, if $a_0 = b_0 = 0$ then $a\, b := -\langle \vec{a}, \vec{b} \rangle + [\vec{a}, \vec{b}]$.

Denote by $\mathfrak{S}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$ and by $G\mathbb{H}(\mathbb{C})$ the subset of invertible elements from $\mathbb{H}(\mathbb{C})$. If $a \notin \mathfrak{S} \cup \{0\}$ then $a^{-1} := \frac{\bar{a}}{(a\bar{a})}$ is the inverse of the complex quaternion a. Note that $G\mathbb{H}(\mathbb{C}) = \mathbb{H}(\mathbb{C}) \setminus (\mathfrak{S} \cup \{0\})$.

Typical points of the Euclidean space $\mathbb{R}^2$ will denoted by $z := xi_1 + yi_2$, $\zeta := \xi i_1 + \eta i_2$, $\tau := \tau_1 i_1 + \tau_2 i_2$, etc.

We shall consider functions $f : \Omega \longrightarrow \mathbb{C}$ of the form $f = f_0 i_0 + f_1 i_1 + f_2 i_2 + f_3 i_3$, where the component functions f_k are $\mathbb{C}$-valued functions. As usual, we denote by $C^n(\Omega, \mathbb{C})$, $n \in \mathbb{N} \cup \{0\}$ the complex linear spaces of n times continuously differentiable functions. For $n \in \mathbb{N} \cup \{0\}$ set $C^n(\Omega, \mathbb{H}(\mathbb{C})) := \{f : \Omega \longrightarrow \mathbb{H}(\mathbb{C}) \mid f_k \in C^n(\Omega, \mathbb{C})\}$.

On $C^2(\Omega, \mathbb{H}(\mathbb{C}))$ the two-dimensional Helmholtz operator with wave number $\lambda \in \mathbb{C} \setminus \{0\}$ is defined as

$$\Delta_\lambda := \Delta_{\mathbb{R}^2} + \lambda,$$

where $\Delta_{\mathbb{R}^2} := \partial_1^2 + \partial_2^2$ is the Laplacian with $\partial_k := \dfrac{\partial}{\partial x_k}$. Let α be the square root of λ, i.e. $\alpha^2 = \lambda$.

On $C^1(\Omega, \mathbb{H}(\mathbb{C}))$ the left- and right-operators ∂_l and ∂_r are defined according to the following rules:

$$\begin{aligned} \partial_l &:= i_1\partial_1 + i_2\partial_2; & \bar{\partial}_l &:= \bar{i}_1\partial_1 + \bar{i}_2\partial_2; \\ \partial_r &:= \partial_1 M^{i_1} + \partial_2 M^{i_2}; & \bar{\partial}_r &:= \partial_1 M^{\bar{i}_1} + \partial_2 M^{\bar{i}_2}, \end{aligned}$$

where $M^a[f] := f\, a$, ${}^aM[f] := af$, for any $a \in \mathbb{H}(\mathbb{C})$.

One can readily see that

$$\partial_l \cdot \bar{\partial}_l = \bar{\partial}_l \cdot \partial_l = \Delta_{\mathbb{R}^2} = \partial_r \cdot \bar{\partial}_r = \bar{\partial}_r \cdot \partial_r,$$

or

$$\partial_r^2 = \partial_l^2 = -\Delta_{\mathbb{R}^2}.$$

Set

$$\partial_\alpha := \partial_l + \alpha.$$

Therefore,

$$\Delta_\lambda = -\partial_\alpha \cdot \partial_{-\alpha} = -\partial_{-\alpha} \cdot \partial_\alpha.$$

In analogy with the usual notion of a holomorphic function, consider the following definition of an α-hyperholomorphic function in $\mathbb{R}^2$.

A function $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ is said to be left α-hyperholomorphic if $\partial_\alpha f \equiv 0$ in Ω. Let $\mathfrak{M}_\alpha(\Omega, \mathbb{H}(\mathbb{C})) := \{f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \mid \partial_\alpha f \equiv 0 \text{ in } \Omega\}$. Similar definitions can be written for the right-operator but we confine the discussion to the left one.

It is known (see, e.g., [22]) that a fundamental solution θ_α of Δ_λ is given by

$$\theta_\alpha(z) = \begin{cases} \frac{-i}{4} H_0^{(1)}(\alpha|z|), & \alpha \neq 0, \\ \frac{1}{2\pi} \ln|z|, & \alpha = 0, \end{cases} \tag{7}$$

where $H_\iota^{(1)}$ is the Hankel function of the first kind of order $\iota = 0, 1, 2$.

Let us show some properties of the Hankel functions $H_\iota^{(1)}(t)$ and $H_\iota^{(2)}(t)$, i.e. $H_\iota^{(1),(2)}(t)$, $t \in \mathbb{C}$:

$$H_0^{(1),(2)}(t) - H_2^{(1),(2)}(t) = 2\frac{d}{dt} H_1^{(1),(2)}(t),$$

$$-H_1^{(1),(2)}(t) = \frac{d}{dt} H_0^{(1),(2)}(t).$$

About the series expansion (at the origin) of $H_0^{(1)}$, $H_1^{(1)}$, and H_2^1, only the first term is of our interest; it is given by:

$$H_0^{(1)}(t) \underset{t\to 0}{=} i\frac{2}{\pi} \ln t + 1 + i\frac{2}{\pi}(\kappa - \ln 2) + O(t), \tag{8}$$

$$H_1^{(1)}(t) \underset{t\to 0}{=} -i\frac{1}{\pi}\frac{2}{t} + O(t), \tag{9}$$

$$H_2^{(1)}(t) \underset{t\to 0}{=} -i\frac{1}{\pi}\frac{4}{t^2} + O(t), \tag{10}$$

with $t \in \mathbb{C}$, $t \neq 0$ and where κ is the Euler number. The asymptotic expansion (at infinity) is given by:

$$H_\iota^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z-(\iota+1/2)(\pi/2))}(P_\iota(z) + iQ_\iota(z)), \quad -\pi < \arg(z) < \pi,$$

where

$$P_\iota(z) \approx 1 - \frac{(r-1)(r-9)}{2!(8z)^2} + \frac{(r-1)(r-9)(r-25)(r-49)}{4!(8z)^4} - \cdots,$$

$$Q_\iota(z) \approx \frac{(r-1)}{1!(8z)} - \frac{(r-1)(r-9)(r-25)}{3!(8z)^3} + \cdots,$$

and

$$r = 4\iota^2.$$

Recall that

$$H_\iota^1 := I_\iota + iN_\iota,$$

for each $\iota = 0, 1, 2$, where

$$I_\iota(r) = \frac{r^\iota}{2^\iota \Gamma(1/2)\Gamma(\iota + 1/2)} \int_0^\pi e^{ir\cos\theta} \sin^{2\iota}\theta d\theta$$

is the Bessel function of the first kind with Γ denoting the gamma function and

$$N_\iota(r) = \frac{I_\iota(r)\cos\iota\pi - I_{-\iota}(r)}{\sin\iota\pi}$$

is the Bessel function of the second kind. For more information about properties of Hankel functions, see [4, 22].

Let $C^{0,\mu}(\gamma,\mathbb{H}(\mathbb{C}))$ denote the class of $\mathbb{H}(\mathbb{C})$-functions satisfying the μ-Hölder condition in γ,

$$\{f\in C^{0,\mu}(\gamma,\mathbb{H}(\mathbb{C})):\ |f(t_1)-f(t_2)|\leq L_f\cdot|t_1-t_2|^{\mu};\ \forall\{t_1,t_2\}\subset\gamma,\ L_f=const\},$$

with the exponent $0<\mu\leq 1$. Here $|f|$ means the Euclidean norm in $\mathbb{C}^4\approx\mathbb{R}^8$ while $|t|$ is the Euclidean norm in $\mathbb{R}^2$.

The fundamental solution of the operator ∂_α, the Cauchy kernel denoted by $\mathcal{K}_\alpha$, is given by formula

$$\mathcal{K}_\alpha(z):=-\partial_{-\alpha}\,\theta_\alpha(z),\ z\in\mathbb{R}^2\setminus\{0\},$$

and its explicit form can be seen, e.g., in [8]:

$$\mathcal{K}_\alpha(z)=\begin{cases}\frac{-i\alpha}{4}\left[H_1^{(1)}(\alpha|z|)\frac{z_{st}}{|z|}+H_0^{(1)}(\alpha|z|)\right], & \alpha\neq 0,\\ \frac{-z_{st}}{2\pi|z|^2}, & \alpha=0,\end{cases}\tag{11}$$

where $z_{st}:=xi_1+yi_2$.

Notice that from expressions (8) and (9) one can conclude that

$$\lim_{\alpha\to 0}\frac{-i\alpha}{4}\left[H_1^{(1)}(\alpha|z|)\frac{z_{st}}{|z|}+H_0^{(1)}(\alpha|z|)\right]=\frac{-z_{st}}{2\pi|z|^2}.$$

The assumption $\alpha\in\mathbb{C}(\neq 0)$ will be needed throughout the paper. The general case requires further analysis, but we will not develop this point here.

According to the identity

$$\sigma_\tau=\vec{n}(\tau)\,ds_\tau:=n_1(\tau)\,i_1\,ds_\tau+n_2(\tau)\,i_2\,ds_\tau,$$

where $\vec{n}(\tau)$ and ds_τ denote respectively the outward unit normal to γ at τ and the element of arc-length measure, we are in a position to specify the appearance of the unit normal vector to the boundary in the denition of the Cauchy-type integrals of the α-hyperholomorphic function theory. To this end, one has

$$K_\alpha[f](z):=-\int_\gamma\mathcal{K}_\alpha(z-\tau)\sigma_\tau f(\tau)=-\int_\gamma\mathcal{K}_\alpha(z-\tau)\vec{n}(\tau)f(\tau)ds_\tau,\quad z\in\mathbb{R}^2\setminus\gamma,\tag{12}$$

while

$$S_\alpha[f](t):=-2\int_\gamma\mathcal{K}_\alpha(t-\tau)\sigma_\tau f(\tau)=-2\int_\gamma\mathcal{K}_\alpha(t-\tau)\vec{n}(\tau)f(\tau)ds_\tau,\quad t\in\gamma.\tag{13}$$

The central formula establishing the relation between the boundary value $K_\alpha[f]^{\pm}$ (if it existed) of the Cauchy-type integral and it singular version $S_\alpha[f]$ is the so-called Plemelj–Sokhotski formula:

$$K_\alpha[f]^{\pm}(t)=\pm\frac{1}{2}f(t)+\frac{1}{2}S_\alpha[f](t).\tag{14}$$

Moreover, S_α is an involution operator on $C^{0,\mu}(\gamma,\mathbb{H}(\mathbb{C}))$, and $L_p(\gamma,\mathbb{H}(\mathbb{C}))$, $p>1$:

$$S_\alpha^2=I.\tag{15}$$

In order for f to be a boundary value (i.e. a trace on γ) of a function $\tilde{f}$ from $\mathfrak{M}_\alpha(\Omega^+,\mathbb{H}(\mathbb{C}))\cap C^{0,\mu}(\Omega^+\cup\gamma,\mathbb{H}(\mathbb{C}))$, the following condition is necessary and sufficient:

$$f(t)=S_\alpha[f](t),\ \forall t\in\gamma. \tag{16}$$

The interested reader is referred to [3, 8, 21] for further information.

3. Hilbert and Poincaré–Bertrand formulas on the unit circle

In this section, we suppose that γ is merely the unit circle in $\mathbb{R}^2$, often denoted by $\mathbb{S}=\mathbb{S}(0;1)$, which is the boundary of the unit ball $\mathbb{B}^2=\mathbb{B}^2(0;1)$. We wish to illustrate as quickly and easily as possible that analogues of the Hilbert and Poincaré–Bertrand formulas on the unit circle in $\mathbb{R}^2$ for α-hyperholomorphic function theory with α being a complex number can be obtained.

One more structure of complex quaternions proved to be useful for our purposes. Let $f\in C(\mathbb{S},\mathbb{H}(\mathbb{C})$, and then:

$$f=\sum_{k=0}^{3}f_ki_k=(f_0+f_3i_3)+(f_1+f_2i_3)i_1=:F_1+F_2i_1.$$

The functions F_1, F_2 are of the form $a+bi_3$ with a, b usual complex numbers, and thus belong to the (commutative) algebra of bicomplex numbers generated by the imaginary units i and i_3, which shall be denoted by $\mathbb{C}(i)\otimes\mathbb{C}(i_3)$. We shall call the functions F_1 and F_2 the bicomplex components (or coordinates) of f. Conjugation with respect to i_3 will be denoted as follows:

$$\hat{F}_1:=f_0-f_3i_3.$$

The corresponding operator will be denoted as $\hat{Z}\ :\ \hat{Z}[F_1]:=\hat{F}_1$. It is clear that $F_1i_1=i_1\hat{F}_1$; i.e. $M^{i_1}[F_1]=i_1\hat{Z}[F_1]$.

Define the following operators for $f\in C^{0,\mu}(\mathbb{S},\mathbb{C}(i)\otimes\mathbb{C}(i_3))$:

$$\mathcal{H}_\alpha[f](t):=-\int_{\mathbb{S}}\frac{i\alpha}{2}H_0^{(1)}(\alpha|t-\tau|)(\tau_1+\tau_2i_3)\hat{Z}[f](\tau)ds_\tau, \tag{17}$$

$$M_\alpha[f](t):=\int_{\mathbb{S}}\frac{i\alpha}{4}H_1^{(1)}(\alpha|t-\tau|)|t-\tau|\left(1+\cot\left(\frac{\Xi}{2}\right)i_3\right)f(\tau)ds_\tau, \tag{18}$$

which have to be understood in the sense of Cauchy's principal value.

Definition 3.1 *$\mathfrak{U}_\alpha(\mathbb{B}^2(0;1);C^{0,\mu}(\mathbb{S},\mathbb{H}(\mathbb{C})))$, $0<\mu\leq 1$, denotes the class of functions $\tilde{f}$ such that*

1) $\tilde{f}\in\mathfrak{M}_\alpha(\mathbb{B}^2(0;1),\mathbb{H}(\mathbb{C}))$;

2) there exists everywhere on $\mathbb{S}$ the limit $\lim\limits_{\mathbb{B}^2(0;1)\ni x\to t\in\mathbb{S}}\tilde{f}(x)=:f(t)$ generating the function f in $C^{0,\mu}(\mathbb{S},\mathbb{H}(\mathbb{C}))$.

Theorem 3.2 *(Analogue of the Hilbert formulas for α-hyperholomorphic functions). A function $f = F_1 + F_2 i_1$ is the limit function of $\tilde{f} \in \mathfrak{U}_\alpha(\mathbb{B}^2(0;1); C^{0,\mu}(\mathbb{S}, \mathbb{H}(\mathbb{C}))$, if, and only if, F_1 and F_2 are related by the following formulas:*

$$\begin{aligned} M_\alpha[F_1] + \mathcal{H}_\alpha[F_2] &= F_1, \\ M_\alpha[F_2] - \mathcal{H}_\alpha[F_1] &= F_2. \end{aligned} \tag{19}$$

Furthermore, if $f \in \ker M_\alpha$ then

$$\begin{aligned} \mathcal{H}_\alpha[F_2] &= F_1, \\ -\mathcal{H}_\alpha[F_1] &= F_2. \end{aligned} \tag{20}$$

Proof The Cauchy singular operator $S_\alpha[f]$ on the unit circle can easily be shown to have the following form:

$$S_\alpha[f](t) = \frac{i\alpha}{2} \int_{\mathbb{S}} \left[H_1^{(1)}(\alpha|t-\tau|) \frac{(t-\tau)_{st}}{|t-\tau|} + H_0^{(1)}(\alpha|t-\tau|) \right] \tau_{st} f(\tau) ds_\tau,$$

where $(t-\tau)_{st} := (t_1 - \tau_1) i_1 + (t_2 - \tau_2) i_2$.

Observe that if Ξ is the angle between t and τ, then

$$(t-\tau)_{st} \tau_{st} = 1 - \cos\Xi + \sin\Xi i_3 \quad \text{and} \quad |t-\tau|^2 = 2 - 2\cos\Xi,$$

and we can write

$$\begin{aligned} S_\alpha[f] &= \frac{i\alpha}{2} \int_{\mathbb{S}} \left[H_1^{(1)}(\alpha|t-\tau|)|t-\tau| \frac{1 - \cos\Xi + \sin\Xi i_3}{2 - 2\cos\Xi} + H_0^{(1)}(\alpha|t-\tau|)\tau_{st} \right] f(\tau) ds_\tau \\ &= \frac{i\alpha}{2} \int_{\mathbb{S}} \left[H_1^{(1)}(\alpha|t-\tau|) \frac{|t-\tau|}{2} \left(1 + \cot\left(\frac{\Xi}{2}\right) i_3 \right) + H_0^{(1)}(\alpha|t-\tau|)\tau_{st} \right] f(\tau) ds_\tau \\ &= \int_{\mathbb{S}} \left[\mathcal{A}_\alpha(t,\tau) + \mathcal{B}_\alpha(t,\tau) i_1 \right] f(\tau) ds_\tau, \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_\alpha(t,\tau) &:= \frac{i\alpha}{4} H_1^{(1)}(\alpha|t-\tau|)|t-\tau| \left(1 + \cot\left(\frac{\Xi}{2}\right) i_3 \right), \\ \mathcal{B}_\alpha(t,\tau) &:= \frac{i\alpha}{2} H_0^{(1)}(\alpha|t-\tau|)(\tau_1 + \tau_2 i_3). \end{aligned}$$

In what follows, we regard f as being interpreted in terms of its bicomplex components F_1 and F_2 and the next identities can be derived:

$$\begin{aligned} S_\alpha[f] &= \int_{\mathbb{S}} \left[\mathcal{A}_\alpha(t,\tau) + \mathcal{B}_\alpha(t,\tau) i_1 \right] f(\tau) ds_\tau \\ &= \int_{\mathbb{S}} \left[\mathcal{A}_\alpha(t,\tau) + \mathcal{B}_\alpha(t,\tau) i_1 \right] (F_1 + F_2 i_1)(\tau) ds_\tau \\ &= \int_{\mathbb{S}} \left[\mathcal{A}_\alpha(t,\tau) F_1(\tau) - \mathcal{B}_\alpha(t,\tau) \hat{F}_2(\tau) \right] ds_\tau \\ &\quad + \int_{\mathbb{S}} \left[\mathcal{A}_\alpha(t,\tau) F_2(\tau) + \mathcal{B}_\alpha(t,\tau) \hat{F}_1(\tau) \right] i_1 ds_\tau. \end{aligned}$$

Then

$$S_\alpha[f] = (M_\alpha[F_1] + \mathcal{H}_\alpha[F_2]) + (M_\alpha[F_2] - \mathcal{H}_\alpha[F_1])\, i_1.$$

Recall that (16) is a necessary and sufficient condition for f to be a boundary value of a function $\tilde{f}$ from $\mathfrak{M}_\alpha(\mathbb{B}^2(0;1), \mathbb{H}(\mathbb{C})) \cap C^{0,\mu}(\mathbb{B}^2(0;1) \cup \mathbb{S}, \mathbb{H}(\mathbb{C}))$. Hence,

$$f = F_1 + F_2 i_1 = (M_\alpha[F_1] + \mathcal{H}_\alpha[F_2]) + (M_\alpha[F_2] - \mathcal{H}_\alpha[F_1])\, i_1,$$

From this equality, one gets the following relations:

$$M_\alpha[F_1] + \mathcal{H}_\alpha[F_2] = F_1,$$
$$M_\alpha[F_2] - \mathcal{H}_\alpha[F_1] = F_2.$$

If $f \in \ker M_\alpha$, then

$$\mathcal{H}_\alpha[F_2] = F_1,$$
$$-\mathcal{H}_\alpha[F_1] = F_2.$$

□

Below we will see that the relation with the usual complex operators is more sophisticated. Notice that formulae (19) have the same structure as those of the complex case (1).

Corollary 3.3 *If $f = F_1 + F_2 i_1$ is the limit function of $\tilde{f}$, then f is determined by*

$$\begin{aligned} f &= F_1 + (M_\alpha[F_2] - \mathcal{H}_\alpha[F_1])\, i_1 \\ &= (M_\alpha[F_1] + \mathcal{H}_\alpha[F_2]) + F_2 i_1. \end{aligned}$$

In particular, if $f \in \ker M_\alpha$, then

$$\begin{aligned} f &= F_1 - \mathcal{H}_\alpha[F_1] i_1 \\ &= \mathcal{H}_\alpha[F_2] + F_2 i_1. \end{aligned}$$

That is, if $f \in \ker M_\alpha$, then f is completely determined by any one of its bicomplex components.

Corollary 3.4 *(Analogue of the Schwarz formulas for α-hyperholomorphic functions). If $f = F_1 + F_2 i_1$ is the limit function of $\tilde{f}$, then $\tilde{f}$ is determined by*

$$\begin{aligned} \tilde{f} &= K_\alpha \left[F_1 + (M_\alpha[F_2] - \mathcal{H}_\alpha[F_1])\, i_1\right] \\ &= K_\alpha \left[(M_\alpha[F_1] + \mathcal{H}_\alpha[F_2]) + F_2 i_1\right]. \end{aligned}$$

In particular, if $f \in \ker M_\alpha$, then

$$\begin{aligned} \tilde{f} &= K_\alpha \left[F_1 - \mathcal{H}_\alpha[F_1] i_1\right] \\ &= K_\alpha \left[\mathcal{H}_\alpha[F_2] + F_2 i_1\right]. \end{aligned}$$

That is, if $f \in \ker M_\alpha$, then $\tilde{f}$ is determined by any one of the limit functions of its bicomplex components. From expressions (8) and (9), one can conclude that

$$\begin{aligned}\lim_{\alpha\to 0} M_\alpha[F] &= \frac{1}{2\pi}\int_{\mathbb{S}} \left(1+\cot\left(\frac{\Xi}{2}\right) i_3\right) F(\tau) ds_\tau, \\ &= M[F] + \mathcal{H}[F] i_3 \\ \lim_{\alpha\to 0} \mathcal{H}_\alpha[F] &= 0,\end{aligned}$$

where M and $\mathcal{H}$ are the usual operators from one-dimensional complex analysis defined in (2) and (3).

Applying (19) and (20) when $\alpha \to 0$, one obtains

$$\begin{aligned}\lim_{\alpha\to 0} M_\alpha[f_0 + f_3 i_3] &= f_0 + f_3 i_3, \\ \lim_{\alpha\to 0} M_\alpha[f_1 + f_2 i_3] &= f_1 + f_2 i_3,\end{aligned}$$

i.e.

$$\begin{aligned}M[f_0 + f_3 i_3] + \mathcal{H}[f_0 + f_3 i_3] i_3 &= f_0 + f_3 i_3, \\ M[f_1 + f_2 i_3] + \mathcal{H}[f_1 + f_2 i_3] i_3 &= f_1 + f_2 i_3.\end{aligned}$$

Therefore, we arrive at the following relations:

$$\begin{aligned}M[f_0] - \mathcal{H}[f_3] &= f_0, \\ M[f_3] + \mathcal{H}[f_0] &= f_3,\end{aligned} \tag{21}$$

and

$$\begin{aligned}M[f_1] - \mathcal{H}[f_2] &= f_1, \\ M[f_2] + \mathcal{H}[f_1] &= f_2.\end{aligned} \tag{22}$$

In order to understand how the usual case is embedded here we need to note that in the case $\alpha = 0$ the operator that defines the class of hyperholomorphic functions is the following:

$$i_1 \frac{\partial}{\partial x} + i_2 \frac{\partial}{\partial y} = i_1 \left(\frac{\partial}{\partial x} - i_3 \frac{\partial}{\partial y}\right).$$

Then, in order for a function $f = \sum_{k=0}^{3} f_k i_k = (f_0 + f_3 i_3) + (f_1 + f_2 i_3) i_1 = F_1 + F_2 i_1$ to be 0-hyperholomorphic, it has to satisfy

$$\begin{aligned}i_1 \left(\frac{\partial}{\partial x} - i_3 \frac{\partial}{\partial y}\right) f &= i_1 \left(\frac{\partial}{\partial x} - i_3 \frac{\partial}{\partial y}\right)(F_1 + F_2 i_1) \\ &= i_1 \left(\frac{\partial F_1}{\partial x} - i_3 \frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x} i_1 - i_3 \frac{\partial F_1}{\partial y} i_1\right) \\ &= i_1 \left(\frac{\partial F_1}{\partial z} + \frac{\partial F_2}{\partial z} i_1\right) = 0,\end{aligned} \tag{23}$$

i.e. f is a couple of antiholomorphic functions of the complex variable $z = x+i_3y$ but with values in $\mathbb{C}(i)\otimes\mathbb{C}(i_3)$.

Furthermore, we have

$$\begin{aligned} F_1 &= f_0 + f_3 i_3 = \mathrm{Re}f_0 + i\mathrm{Im}f_0 + (\mathrm{Re}f_3 + i\mathrm{Im}f_3)i_3 \\ &= (\mathrm{Re}f_0 + i_3\mathrm{Re}f_3) + i(\mathrm{Im}f_0 + i_3\mathrm{Im}f_3) \\ &=: p_1 + iq_1, \end{aligned}$$

and

$$\begin{aligned} F_2 &= f_1 + f_2 i_3 = \mathrm{Re}f_1 + i\mathrm{Im}f_1 + (\mathrm{Re}f_2 + i\mathrm{Im}f_2)i_3 \\ &= (\mathrm{Re}f_1 + i_3\mathrm{Re}f_2) + i(\mathrm{Im}f_1 + i_3\mathrm{Im}f_2) \\ &=: p_2 + iq_2. \end{aligned}$$

From (23) one obtains

$$\frac{\partial F_1}{\partial z} = \frac{\partial p_1}{\partial z} + i\frac{\partial q_1}{\partial z} = 0 \Leftrightarrow \frac{\partial p_1}{\partial z} = \frac{\partial q_1}{\partial z} = 0,$$

and

$$\frac{\partial F_2}{\partial z} = \frac{\partial p_2}{\partial z} + i\frac{\partial q_2}{\partial z} = 0 \Leftrightarrow \frac{\partial p_2}{\partial z} = \frac{\partial q_2}{\partial z} = 0.$$

Hence, p_1, q_1, p_2, and q_2 are also "usual" antiholomorphic complex functions.

Now, applying (21) and (22), we obtain

$$\begin{aligned} M[\mathrm{Re}f_0 + i\mathrm{Im}f_0] - \mathcal{H}[\mathrm{Re}f_3 + i\mathrm{Im}f_3] &= \mathrm{Re}f_0 + i\mathrm{Im}f_0, \\ M[\mathrm{Re}f_3 + i\mathrm{Im}f_3] + \mathcal{H}[\mathrm{Re}f_0 + i\mathrm{Im}f_0] &= \mathrm{Re}f_3 + i\mathrm{Im}f_3, \end{aligned}$$

and

$$\begin{aligned} M[\mathrm{Re}f_1 + i\mathrm{Im}f_1] - \mathcal{H}[\mathrm{Re}f_2 + i\mathrm{Im}f_2] &= \mathrm{Re}f_1 + i\mathrm{Im}f_1, \\ M[\mathrm{Re}f_2 + i\mathrm{Im}f_2] + \mathcal{H}[\mathrm{Re}f_1 + i\mathrm{Im}f_1] &= \mathrm{Re}f_2 + i\mathrm{Im}f_2, \end{aligned}$$

and we arrive at the following relations between the real components:

$$\begin{aligned} M[\mathrm{Re}f_0] - \mathcal{H}[\mathrm{Re}f_3] &= \mathrm{Re}f_0, \\ M[\mathrm{Re}f_3] + \mathcal{H}[\mathrm{Re}f_0] &= \mathrm{Re}f_3, \end{aligned} \tag{24}$$

$$\begin{aligned} M[\mathrm{Im}f_0] - \mathcal{H}[\mathrm{Im}f_3] &= \mathrm{Im}f_0, \\ M[\mathrm{Im}f_3] + \mathcal{H}[\mathrm{Im}f_0] &= \mathrm{Im}f_3, \end{aligned} \tag{25}$$

$$\begin{aligned} M[\mathrm{Re}f_1] - \mathcal{H}[\mathrm{Re}f_2] &= \mathrm{Re}f_1, \\ M[\mathrm{Re}f_2] + \mathcal{H}[\mathrm{Re}f_1] &= \mathrm{Re}f_2, \end{aligned} \tag{26}$$

$$\begin{aligned} M[\mathrm{Im}f_1] - \mathcal{H}[\mathrm{Im}f_2] &= \mathrm{Im}f_1, \\ M[\mathrm{Im}f_2] + \mathcal{H}[\mathrm{Im}f_1] &= \mathrm{Im}f_2. \end{aligned} \tag{27}$$

Additionally, if $f \in \ker M$, then

$$\begin{aligned} M[f] &= M[f_0 + f_1 i_1 + f_2 i_2 + f_3 i_3] \\ &= M[f_0] + M[f_1] i_1 + M[f_2] i_2 + M[f_3] i_3 \\ &= 0, \end{aligned}$$

and thus

$$M[f_k] = 0 \; \forall k \in \mathbb{N}_3 \cup \{0\} \Rightarrow M[\mathrm{Re}f_k + i\mathrm{Im}f_k] = 0 \; \forall k \in \mathbb{N}_3 \cup \{0\},$$
$$\Rightarrow M[\mathrm{Re}f_k] = 0 \text{ and } M[\mathrm{Im}f_k] = 0 \; \forall k \in \mathbb{N}_3 \cup \{0\}.$$

Therefore, we obtain

$$\begin{aligned} -\mathcal{H}[\mathrm{Re}f_3] &= \mathrm{Re}f_0, \\ \mathcal{H}[\mathrm{Re}f_0] &= \mathrm{Re}f_3, \end{aligned} \tag{28}$$

$$\begin{aligned} -\mathcal{H}[\mathrm{Im}f_3] &= \mathrm{Im}f_0, \\ \mathcal{H}[\mathrm{Im}f_0] &= \mathrm{Im}f_3, \end{aligned} \tag{29}$$

$$\begin{aligned} -\mathcal{H}[\mathrm{Re}f_2] &= \mathrm{Re}f_1, \\ \mathcal{H}[\mathrm{Re}f_1] &= \mathrm{Re}f_2, \end{aligned} \tag{30}$$

$$\begin{aligned} -\mathcal{H}[\mathrm{Im}f_2] &= \mathrm{Im}f_1, \\ \mathcal{H}[\mathrm{Im}f_1] &= \mathrm{Im}f_2. \end{aligned} \tag{31}$$

Finally we obtained a family of Hilbert formulas for the parameter $\alpha \neq 0$, but we can extend to the case $\alpha = 0$ and for this case the bicomplex Hilbert formulas have become four pairs of complex Hilbert formulas for antiholomorphic functions. Compare these results with (1).

The Poincaré–Bertrand formula on the unit circle $\mathbb{S}$ in $\mathbb{R}^2$ and $f \in C^{0,\mu}(\mathbb{S}, \mathbb{H}(\mathbb{C}))$ can be adapted from [20, Theorem 3.5] as follows:

$$\int_{\mathbb{S}_\tau} \mathcal{K}_\alpha(t-\tau)\sigma_\tau \int_{\mathbb{S}_\zeta} \mathcal{K}_\alpha(\tau-\zeta)\sigma_\zeta f(\zeta,\tau) = \tag{32}$$

$$= \int_{\mathbb{S}_\zeta} \int_{\mathbb{S}_\tau} \mathcal{K}_\alpha(t-\tau)\,\sigma_\tau\,\mathcal{K}_\alpha(\tau-\zeta)\,\sigma_\zeta\,f(\zeta,\tau) + \frac{1}{4}\,f(t,t).$$

Suppose that $f(\zeta,\tau)=f(\zeta)\in C^{0,\mu}(\mathbb{S},\mathbb{H}(\mathbb{C}))$; then, from (32), we have:

$$\int_{\mathbb{S}_\tau}\mathcal{K}_\alpha(t-\tau)\sigma_\tau\int_{\mathbb{S}_\zeta}\mathcal{K}_\alpha(\tau-\zeta)\sigma_\zeta f(\zeta)=f(t_1,t_2).$$

Therefore, we have

$$\int_{\mathbb{S}_\tau}[\mathcal{A}_\alpha(t,\tau)+\mathcal{B}_\alpha(t,\tau)i_1]\,ds_\tau\int_{\mathbb{S}_\zeta}[\mathcal{A}_\alpha(\zeta,\tau)+\mathcal{B}_\alpha(\zeta,\tau)i_1]f(\zeta,\tau)\,ds_\zeta=$$

$$\int_{\mathbb{S}_\zeta}\int_{\mathbb{S}_\tau}[\mathcal{A}_\alpha(t,\tau)+\mathcal{B}_\alpha(t,\tau)i_1]\,d\Gamma_\tau\,[\mathcal{A}_\alpha(\zeta,\tau)+\mathcal{B}_\alpha(\zeta,\tau)i_1]f(\zeta,\tau)\,d\Gamma_\zeta+\frac{1}{4}f(t,t).$$

As before, let $f=F_1+F_2i_1$, and then we have

$$\int_{\mathbb{S}_\tau}\int_{\mathbb{S}_\zeta}[\mathcal{A}_\alpha(t,\tau)\mathcal{A}_\alpha(\zeta,\tau)F_1(\zeta,\tau)-\mathcal{B}_\alpha(t,\tau)\hat{\mathcal{B}}_\alpha(\zeta,\tau)F_1(\zeta,\tau)-\mathcal{A}_\alpha(t,\tau)\mathcal{B}_\alpha(\zeta,\tau)\hat{F}_2(\zeta,\tau)-$$

$$-\mathcal{B}_\alpha(t,\tau)\hat{\mathcal{A}}_\alpha(\zeta,\tau)\hat{F}_2(\zeta,\tau)]\,ds_\zeta\,ds_\tau=\frac{1}{4}F_1(t,t)+$$

$$+\int_{\mathbb{S}_\zeta}\int_{\mathbb{S}_\tau}[\mathcal{A}_\alpha(t,\tau)\mathcal{A}_\alpha(\zeta,\tau)F_1(\zeta,\tau)-\mathcal{B}_\alpha(t,\tau)\hat{\mathcal{B}}_\alpha(\zeta,\tau)F_1(\zeta,\tau)-\mathcal{A}_\alpha(t,\tau)\mathcal{B}_\alpha(\zeta,\tau)\hat{F}_2(\zeta,\tau)-$$

$$-\mathcal{B}_\alpha(t,\tau)\hat{\mathcal{A}}_\alpha(\zeta,\tau)\hat{F}_2(\zeta,\tau)]\,ds_\tau\,ds_\zeta,$$

$$\int_{\mathbb{S}_\tau}\int_{\mathbb{S}_\zeta}[\mathcal{A}_\alpha(t,\tau)\mathcal{B}_\alpha(\zeta,\tau)F_1(\zeta,\tau)+\mathcal{B}_\alpha(t,\tau)\hat{\mathcal{A}}_\alpha(\zeta,\tau)\hat{F}_1(\zeta,\tau)+\mathcal{A}_\alpha(t,\tau)\mathcal{A}_\alpha(\zeta,\tau)F_2(\zeta,\tau)-$$

$$-\mathcal{B}_\alpha(t,\tau)\hat{\mathcal{B}}_\alpha(\zeta,\tau)F_2(\zeta,\tau)]\,ds_\zeta\,ds_\tau=\frac{1}{4}F_2(t,t)+$$

$$+\int_{\mathbb{S}_\zeta}\int_{\mathbb{S}_\tau}[\mathcal{A}_\alpha(t,\tau)\mathcal{B}_\alpha(\zeta,\tau)F_1(\zeta,\tau)+\mathcal{B}_\alpha(t,\tau)\hat{\mathcal{A}}_\alpha(\zeta,\tau)\hat{F}_1(\zeta,\tau)+\mathcal{A}_\alpha(t,\tau)\mathcal{A}_\alpha(\zeta,\tau)F_2(\zeta,\tau)-$$

$$-\mathcal{B}_\alpha(t,\tau)\hat{\mathcal{B}}_\alpha(\zeta,\tau)F_2(\zeta,\tau)]\,ds_\tau\,ds_\zeta.$$

Remark 3.5 *It is possible, and indeed desirable, to consider the analogous formulas in other spaces than Hölder space, for example, the Banach space $L_p(\mathbb{S},\mathbb{H}(\mathbb{C}))$, $p>1$. If $f\in L_p(\mathbb{S},\mathbb{H}(\mathbb{C}))$, $p>1$ then the Sokhotski–Plemelj formulas, the Poincaré–Bertrand formula, and the Hilbert formulas are valid almost everywhere on $\mathbb{S}$.*

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