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Legendre wavelet solution of high order nonlinear ordinary delay differential equations

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Abstract: The purpose of this paper is to illustrate the use of the Legendre wavelet method in the solution of high-order nonlinear ordinary differential equations with variable and proportional delays. The main advantage of using Legendre polynomials lies in the orthonormality property, which enables a decrease in the computational cost and runtime. The method is applied to five differential equations up to sixth order, and the results are compared with the exact solutions and other numerical solutions when available. The accuracy of the method is presented in terms of absolute errors. The numerical results demonstrate that the method is accurate, effectual and simple to apply.

Key words: Legendre wavelets, nonlinear ordinary differential equations, variable delay, proportional delay

1. Introduction

Many physical phenomena are modeled by using both the present and past states of the model. Hence, differential equations with time delays are needed in the modeling of real life situations. Applications of these equations can be seen in many areas such as human body control and multibody control systems, electric circuits, dynamical behavior of a system in fluid mechanics, chemical engineering [18], spread of bacteriophage infection [35], population dynamics, epidemiology, physiology, immunology, neural networks, and cell kinetics [9].

Several numerical techniques are introduced to find approximate solutions of nonlinear differential equations with proportional and constant delays. These methods can be listed as: Aboodh transformation method [2], Adomian decomposition method [8, 14, 30], power series method [7], decomposition method [43], differential transform method [26], Hermite wavelet-based method [34], variational iteration method [22], power and Padé series-based method [21], spectral method [3], variable multistep methods [25], quasilinearization technique [31], the Runge–Kutta–Fehlberg methods [29], polynomial least squares method [12], homotopy perturbation method [33], variational approach [1], Bessel functions of first kind [36, 37], shifted Legendre polynomials [38, 42], and first Boubaker polynomial approach [13]. There are also studies on the solutions of nonlinear ordinary differential equations solved by collocation methods which are based on Bessel polynomials [39–41].

On the other hand, there are few studies about numerical solutions of nonlinear ordinary differential equations with variable delays. A new multistep technique with differential transform method was studied by Benhammouda and Vazquez-Leal [6]. The Legendre–Gauss collocation method for the solution of DDEs with

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variable delays was introduced by Wang and Wang [44]. Ismail et al. [20] used the Runge–Kutta method with Hermite interpolation. The Lucas polynomial method is used by Gümgüm et al. [17]. Another method, power and Padé series is used by Vanani and Aminataei [21]. In literature, Legendre wavelets are used to solve system of first order ordinary differential equations with constant delays by Kumar et al. [23] and first order differential equations with proportional delays by Hafshejani et al. [19]. To the best of our knowledge, this is the first application to high-order derivatives with variable and proportional delays.

In the present study, our main goal is to use the Legendre wavelet method to solve the initial value problem in the form

$$\sum_{i=0}^6 \sum_{j=0}^J P_{ij}(t) y^{(i)}(t - \tau_{ij}(t)) + \sum_{p=0}^2 \sum_{q=0}^p R_{pq}(t) y^{(p)}(\alpha_{pq}t) y^{(q)}(\beta_{pq}t) = g(t) \quad , J \leq 6 \quad (1.1)$$

subject to the initial conditions

$$y^{(i)}(0) = \lambda_i, \quad i = 0, 1, \dots, 5 \quad (1.2)$$

where $P_{ij}(t)$, $R_{pq}(t)$, $g(t)$, and the variable delays $\tau_{ij}(t)$ are given continuous functions on the interval $0 \leq t \leq 1$, α_{pq} and β_{pq} are given constants to express proportional delays. We note that the delays are nonnegative for $t \geq 0$.

This method has the main advantage of not requiring any discretization of the domain, and it can be simply implemented by conversion of the differential equation to a system of nonlinear algebraic equations, which then can be solved by MATLAB tools.

In the next section, we describe the fundamentals of the wavelets and Legendre wavelets which is required for the representation of the differential equation and the approximated solution.

2. Wavelets and Legendre wavelets

Wavelets are defined as the functions generated from a single function, called the mother wavelet $\psi(t)$. A family of continuous functions of the form

$$\psi_{r,s}(t) = |r|^{-1/2} \psi\left(\frac{t-s}{r}\right)$$

is constituted by dilation and translation of the mother wavelet by the factors r and s , respectively. Here, the scaling parameter $r \neq 0$ and translation parameter s may vary in R , but if we restrict these parameters to $r = r_0^{-k}$ and $s = ns_0 r_0^{-k}$ for $r_0 > 1$, $s_0 > 0$ and for positive integers k, n , then we obtain the discrete wavelets

$$\psi_{k,n}(t) = r_0^{k/2} \psi(r_0^k t - ns_0)$$

which form a wavelet basis in $L^2(R)$ and if $r_0 = 2$, $s_0 = 1$ this basis is then orthonormal [10, 15].

Different types of discrete wavelets have drawn much attention from scientists and engineers due to the advantages of the wavelets. One of the main advantages is that unlike cosine and sine as Fourier bases, these wavelet basis are not periodic and do not continue to infinity; additionally, wavelets with compact support are good at modeling localized features in applications [15]. Haar wavelet [16], Legendre wavelet [27, 28, 32], and Chebyshev wavelet [5] are some of discrete wavelets used to solve varieties of differential and integral equations.

2.1. Formulation of Legendre wavelets

Legendre wavelets are derived from Legendre polynomials with four arguments $\psi_{nm} = \psi_{nm}(k, n, m, t)$ which are defined in the form

$$\psi_{nm}(t) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{(k+1)/2} P_m^*(2^{k+1}t - (2n + 1)), & \text{if } \frac{n}{2^k} < t < \frac{n+1}{2^k}, \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

Here, $m = 0, 1, 2, \dots, M$ and $n = 0, 1, 2, \dots, 2^k - 1$ for any nonnegative integer k . $P_m^*(t)$ is the Legendre polynomial of order m which is orthogonal with respect to weight function $w(t) = 1$ and satisfy the recurrence relation, [11, 24]

$$P_{m+1}^*(t) = \left(\frac{2m+1}{m+1}\right) t P_m^*(t) - \left(\frac{m}{m+1}\right) P_{m-1}^*(t), \quad m = 1, 2, 3, \dots$$

where $P_0^*(t) = 1$ and $P_1^*(t) = t$. $\sqrt{m + \frac{1}{2}}$ in Eq. (2.1) is used to satisfy orthonormality property, and t is the normalized time.

When the Legendre polynomials are shifted by $t = 2x - 1$, the shifted Legendre polynomials $P_m(x) = P_m^*(2x - 1)$, which are orthogonal in $[0, 1]$, are obtained and can be expressed in the analytical form as follows [11, 24, 27]:

$$P_m(x) = \sum_{k=0}^m (-1)^{m+k} \frac{(m+k)! x^k}{(m-k)! (k!)^2}$$

2.2. Function approximation

We can expand any function $f(t)$ with the domain $[0, 1]$ by the infinite series as [27]

$$f(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} A_{nm} \psi_{nm}(t)$$

where $A_{nm} = (f(t), \psi_{nm}(t)) = \int_0^1 f(t) \psi_{nm}(t) dt$. Truncating this series yields

$$f(t) \approx \sum_{k=0}^{2^k-1} \sum_{m=0}^M A_{nm} \psi_{nm}(t) = \mathbf{A}^T \boldsymbol{\Psi}(t) \tag{2.2}$$

where \mathbf{A} and $\boldsymbol{\Psi}(t)$ are $2^k(M + 1) \times 1$ matrices in the form

$$\mathbf{A} = [A_{00}, A_{01}, \dots, A_{0M}, A_{10}, \dots, A_{1M}, A_{(2^k-1)0}, \dots, A_{(2^k-1)M}]^T,$$

$$\boldsymbol{\Psi}(t) = [\psi_{00}(t), \psi_{01}(t), \dots, \psi_{0M}(t), \psi_{10}(t), \dots, \psi_{1M}(t), \dots, \psi_{(2^k-1)0}(t), \dots, \psi_{(2^k-1)M}(t)]^T.$$

2.3. Operational matrix of differentiation

The i -th derivative of the vector $\boldsymbol{\Psi}(t)$, defined above, can be obtained by

$$\frac{d^i}{dt^i} \boldsymbol{\Psi}(t) = \mathbf{D}^i \boldsymbol{\Psi}(t), \quad i = 1, \dots, 6 \tag{2.3}$$

where \mathbf{D}^i is the i -th power of the $2^k(M + 1) \times 2^k(M + 1)$ operational matrix \mathbf{D} , defined in [27] as

$$\mathbf{D} = \begin{pmatrix} \mathbf{F} & 0 & \dots & 0 \\ 0 & \mathbf{F} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \mathbf{F} \end{pmatrix} \tag{2.4}$$

and \mathbf{F} is an $(M + 1) \times (M + 1)$ matrix of the form

$$\mathbf{F} = 2^{k+1} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \sqrt{5}\sqrt{3} & 0 & 0 & \dots & 0 & 0 \\ \sqrt{7} & 0 & \sqrt{7}\sqrt{5} & 0 & \dots & 0 & 0 \\ 0 & \sqrt{9}\sqrt{3} & 0 & \sqrt{9}\sqrt{7} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \sqrt{2M+1} & 0 & \sqrt{2M+1}\sqrt{5} & 0 & \dots & \sqrt{2M+1}\sqrt{2M-1} & 0 \end{pmatrix}$$

if M is odd and

$$\mathbf{F} = 2^{k+1} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \sqrt{5}\sqrt{3} & 0 & 0 & 0 & \dots & 0 & 0 \\ \sqrt{7} & 0 & \sqrt{7}\sqrt{5} & 0 & 0 & \dots & 0 & 0 \\ 0 & \sqrt{9}\sqrt{3} & 0 & \sqrt{9}\sqrt{7} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \sqrt{2M+1}\sqrt{3} & 0 & \sqrt{2M+1}\sqrt{7} & 0 & \dots & \sqrt{2M+1}\sqrt{2M-1} & 0 \end{pmatrix}$$

if M is even, related to the definition of components

$$F_{ab} = \begin{cases} 2^{k+1} \sqrt{(2a-1)(2b-1)}, & a=2, \dots, (M+1), b=1, \dots, (a-1), \text{ and } a+b \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

2.4. Application of the operational matrix of differentiation

In order to solve the problem given in Eqs. (1.1) and (1.2), firstly we approximate the solution with the truncated series in Eq. (2.2), using Legendre wavelets as

$$y(t) = \sum_{k=0}^{2^k-1} \sum_{m=0}^M A_{nm} \psi_{nm}(t) = \mathbf{A}^T \Psi(t) \tag{2.5}$$

where A_{nm} are unknown coefficients to be determined.

Then, we use Eq. (2.3) and the operational matrix for differentiation in Eq. (2.4) to approximate the derivatives of $y(t)$ as follows, [27]

$$y^{(i)}(t) = \mathbf{A}^T \mathbf{D}^i \Psi(t), \text{ for } i = 1, \dots, 6. \tag{2.6}$$

The differential equation in Eq. (1.1) can be written by substituting $t - \tau_{ij}(t)$, $\alpha_{pq}t$ and $\beta_{pq}t$ in place of t in Eq. (2.6) as

$$\sum_{i=0}^6 \sum_{j=0}^J P_{ij}(t) \mathbf{A}^T \mathbf{D}^{(i)} \Psi(t - \tau_{ij}(t)) + \sum_{p=0}^2 \sum_{q=0}^p R_{pq}(t) \left(\mathbf{A}^T \mathbf{D}^{(p)} \Psi(\alpha_{pq}t) \right) \left(\mathbf{A}^T \mathbf{D}^{(q)} \Psi(\beta_{pq}t) \right) = g(t), \quad J \leq 6 \tag{2.7}$$

In order to find the unknown coefficients A_{nm} in the vector \mathbf{A} , we need $2^k(M + 1)$ equations. The first six equations are obtained from the initial conditions

$$\begin{aligned} y(0) &= \mathbf{A}^T \Psi(0), \\ y^{(i)}(0) &= \mathbf{A}^T \mathbf{D}^i \Psi(0), \quad i = 1, \dots, 5 \end{aligned}$$

and $(2^k(M + 1)) - 6$ equations are obtained by substituting the first $(2^k(M + 1)) - 6$ roots of shifted Legendre polynomial $P_{(2^k(M+1))}(t)$ in Eq. (2.7). The resulting system of nonlinear equations are solved for the coefficients A_{nm} by MATLAB tools; then, the approximate solution in Eq. (2.5) is computed.

3. Convergence analysis

Let $L^2(R)$ denote the Hilbert space with the inner product on the interval $[0, 1]$, $(u, v) = \int_0^1 u(t)v(t)dt$ for any arbitrary functions $u(t), v(t) \in L^2(R)$. By the properties, $\psi_{k,n}(t) = r_0^{k/2} \psi(r_0^k t - ns_0)$ form a wavelet basis in $L^2(R)$ and if $r_0 = 2, s_0 = 1$, this basis is orthonormal [10, 15].

For $k = 1$, let $y(t) = \sum_{i=1}^M A_{1i} \psi_{1i}(t)$ be the solution of Eq. (1.1), where $A_{1i} = (y(t), \psi_{1i}(t))$. In order to show that this series converges to the solution $y(t)$ of Eq. (1.1), we define a partial sum S_n and our aim is to show that S_n is a Cauchy sequence in Hilbert space which implies the convergence. For this, firstly we denote $\psi_{1i}(t) = \psi(t)$ and let $\gamma_j = (y(t), \psi(t))$.

Let $S_n = \sum_{j=1}^n \gamma_j \psi(t_j)$ and $S_m = \sum_{j=1}^m \gamma_j \psi(t_j)$ be the partial sums with $n \geq m$.

$$(y(t), S_n) = (y(t), \sum_{j=1}^n \gamma_j \psi(t_j)) = \sum_{j=1}^n \bar{\gamma}_j (y(t), \psi(t_j)) = \sum_{j=1}^n \bar{\gamma}_j \gamma_j = \sum_{j=1}^n |\gamma_j|^2.$$

It is clear that $S_n - S_m = \sum_{j=m+1}^n \gamma_j \psi(t_j)$, then we consider

$$\begin{aligned} \|S_n - S_m\|^2 &= \left\| \sum_{j=m+1}^n \gamma_j \psi(t_j) \right\|^2 = \left(\sum_{i=m+1}^n \gamma_i \psi(t_i), \sum_{j=m+1}^n \gamma_j \psi(t_j) \right) \\ &= \sum_{i=m+1}^n \sum_{j=m+1}^n \gamma_i \bar{\gamma}_j (\psi(t_i), \psi(t_j)) = \sum_{j=m+1}^n |\gamma_j|^2. \end{aligned}$$

By Bessel's inequality, $\sum_{j=m+1}^n |\gamma_j|^2$ converges, as $n \rightarrow \infty$, [4]. Hence, S_n is a Cauchy sequence in Hilbert space and it converges to a sum S , then

$$\begin{aligned} (S - y(t), \psi(t_i)) = (S, \psi(t_i)) - (y(t), \psi(t_i)) &= (\lim_{n \rightarrow \infty} S_n, \psi(t_i)) - \gamma_i \\ &= \lim_{n \rightarrow \infty} (S_n, \psi(t_i)) - \gamma_i \\ &= \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \gamma_j \psi(t_j), \psi(t_i) \right) - \gamma_i \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \gamma_j (\psi(t_j), \psi(t_i)) - \gamma_i \\ &= \lim_{n \rightarrow \infty} (\gamma_i - \gamma_i) \\ &= 0. \end{aligned}$$

As a result, $(S - y(t), \psi(t_i)) = 0$ which implies $S = y(t)$, and thus $y(t) = \sum_{j=1}^{\infty} \gamma_j \psi(t_j)$.

3.1. Error bound

If we assume that the function $y(t)$, defined in $[0, 1]$, is m times continuously differentiable function, then there exists a mean error bound for the approximation of $\sum_{n=0}^{2^k-1} \sum_{m=0}^M A_{nm} \psi_{nm}(t) = \mathbf{A}^T \Psi(t)$ to $y(t)$ as follows [45]

$$\|y - \mathbf{A}^T(t)\| \leq \frac{1}{m!2^{mk}} \sup_{t \in [0,1]} |y^{(m)}(t)|.$$

To show this inequality, the interval $[0, 1]$ is partitioned into subintervals $[\frac{n}{2^k}, \frac{n+1}{2^k}]$. Here $y(t)$ is approximated by an m -th degree polynomial $\mathbf{A}^T \Psi(t)$ with a minimum mean error in these subintervals. Then, the maximum error estimation is used for this polynomial which interpolates $y(t)$, that is

$$\begin{aligned} \|y - \mathbf{A}^T \Psi(t)\|^2 &= \int_0^1 [y(t) - \mathbf{A}^T \Psi(t)]^2 dt \\ &= \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} [y(t) - \mathbf{A}^T \Psi(t)]^2 dt \\ &\leq \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} [y(t) - y^*(t)]^2 dt \\ &\leq \sum_{n=0}^{2^k-1} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \left[\frac{1}{m!2^{mk}} \sup_{t \in [0,1]} |y^{(m)}(t)| \right]^2 dt \\ &\leq \int_0^1 \left[\frac{1}{m!2^{mk}} \sup_{t \in [0,1]} |y^{(m)}(t)| \right]^2 dt \\ &= \left[\frac{1}{m!2^{mk}} \sup_{t \in [0,1]} |y^{(m)}(t)| \right]^2 \end{aligned}$$

where $y^*(t)$ is the m -th order interpolation of $y(t)$. Taking square roots of both sides gives the desired result, [45].

4. Results and discussion

In this section, we apply the method to five examples. In order to show that the method is capable of finding the exact solution, we start with a first-order nonlinear differential equation with variable delay. The next two problems are second-order equations with variable delays, the fourth problem is a third-order equation with proportional delay, and finally the last problem is a sixth-order equation again with proportional delay. We solve these problems using quite small values of M and compare the solutions with the exact and other numerical solutions where available. Comparisons are given in terms of absolute error tables.

4.1. Example 1

Consider the first order nonlinear differential equation with variable delay t^2

$$\begin{cases} y'(t) + ty(t - t^2) + ty^2(t) = 1 + t^2, & 0 \leq t \leq 1 \\ y(0) = 0 \end{cases} \tag{4.1}$$

The analytical solution of the above problem is $y(t) = t$.

The unknown function can be approximated using Eq. (2.5) and taking $k = 0$ and $M = 1$ as

$$y(t) = \mathbf{A}^T \mathbf{\Psi}(t) = [A_{00} \quad A_{01}] \begin{bmatrix} \psi_{00}(t) \\ \psi_{01}(t) \end{bmatrix}$$

where A_{00} and A_{01} are unknown coefficients, $\psi_{00}(t) = 1$, and $\psi_{01}(t) = \sqrt{3}(2t - 1)$. Hence, $y(t)$ and $y(t - t^2)$ can be written as

$$\begin{aligned} y(t) &= A_{00} + A_{01}\sqrt{3}(2t - 1) \\ y(t - t^2) &= A_{00} + A_{01}\sqrt{3}(2(t - t^2) - 1). \end{aligned} \tag{4.2}$$

The first derivative $y'(t)$ is approximated as

$$y'(t) = \mathbf{A}^T \mathbf{D}\mathbf{\Psi}(t) = [A_{00} \quad A_{01}] \begin{bmatrix} 0 & 0 \\ 2\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{3}(2t - 1) \end{bmatrix} = A_{01}2\sqrt{3}.$$

Substituting the approximations for $y(t)$, $y(t - t^2)$ and $y'(t)$ in Eq. (4.1), yields

$$A_{01}2\sqrt{3} + t(A_{00} + A_{01}\sqrt{3}(2(t - t^2) - 1)) + t(A_{00} + A_{01}\sqrt{3}(2t - 1))^2 = 1 + t^2 \tag{4.3}$$

Since there are two unknowns, we need two equations. The first equation is obtained by inserting the initial condition $y(0) = 0$ into Eq. (4.2)

$$y(0) = A_{00} + A_{01}(-\sqrt{3}) = 0.$$

The second equation is obtained by inserting $t = 0.211$, which is the root of the second-order shifted Legendre polynomial, into Eq. (4.3).

$$A_{01}2\sqrt{3} + 0.211(A_{00} + A_{01}\sqrt{3}(2(0.211 - (0.211)^2) - 1)) + 0.211(A_{00} + A_{01}\sqrt{3}(2 * 0.211 - 1))^2 = 1 + (0.211)^2$$

Solving this system of 2×2 nonlinear equations results in

$$\mathbf{A}^T = [A_{00}, A_{01}] = [0.5, 0.2887]$$

and hence,

$$y(t) = \mathbf{A}^T \Psi(t) = [0.5, 0.2887] \left[\begin{matrix} 1 \\ \sqrt{3}(2t - 1) \end{matrix} \right] = t,$$

which is the exact solution. This shows that the proposed method is efficient and simple to implement.

4.2. Example 2

The next example is a second-order nonlinear differential equation with several delays, t^2 and $-\frac{t}{2}$

$$\begin{cases} y''(t) + y'(t - t^2) - t^2 y(t + \frac{t}{2}) + (y'(t))^2 - y'(t)y(t) = e^t + e^{t-t^2} - t^2 e^{3t/2} \\ y(0) = y'(0) = 1, t \in [0, 1] \end{cases}$$

The analytical solution of this problem is $y(t) = e^t$. We solved the problem for $k = 0$ and several values of M . Recall that M refers to the degree of the polynomial approximating the solution. Absolute errors for $M = 2, 3$, and 5 can be seen in Table 1. We observe that the quadratic polynomial does not match the exact solution sufficiently when t gets close to 1. Thus, we increase the degree to 3 and then to 5. It can be seen that the error increases as t gets closer to 1 in all cases.

Table 1. Comparison of the absolute errors for $M = 2$, $M = 3$, and $M = 5$.

t_i	E_2	E_3	E_5
0.1	0.0003700607	2.681857e-05	3.759234e-08
0.2	0.0007611571	1.770104e-05	1.250407e-07
0.3	1.000193e-05	7.705389e-05	8.144094e-08
0.4	0.0031690362	0.0001848035	3.440118e-07
0.5	0.0101967998	9.769431e-05	4.761915e-07
0.6	0.0226435623	0.0005415589	4.774446e-07
0.7	0.0422447445	0.0022553881	1.197968e-06
0.8	0.0709182830	0.0057487414	5.626756e-06
0.9	0.1107838255	0.0119282777	3.901048e-05
1	0.1641839450	0.0219235813	0.000137395

Figure 1 illustrates the comparison of the exact and numerical solution. One can see that the numerical solution approximates the exact solution very well even for $M = 5$.

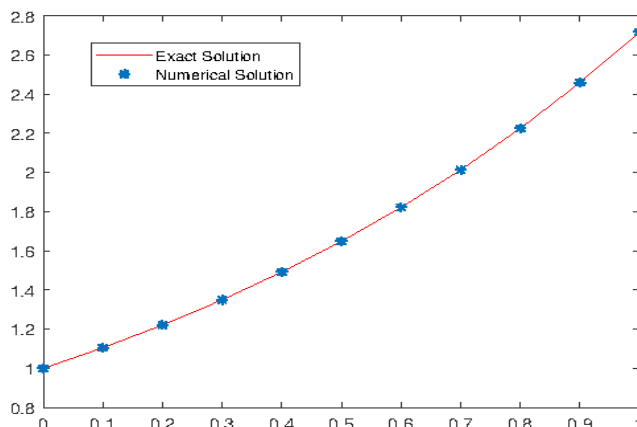


Figure 1. Exact solution and approximate solution for $k = 0$, $M = 5$

4.3. Example 3

For the third example, we consider a second-order nonlinear differential equation with the variable delay $t - t^3/8$, [6]

$$\begin{cases} y''(t) + 2y(t) - y^2(t) + y(t^3/8) = \sin t - \sin^2 t + \sin(t^3/8), & 0 \leq t \leq 1 \\ y(0) = 0, y'(0) = 1 \end{cases}$$

The analytical solution of this problem is $y(t) = \sin t$. The problem is solved by taking $M = 5$ and 6 . Absolute errors for these values are presented in Table 2. One can see that even $M = 5$ results in accuracy up to five decimal places. It is also observed that the numerical results match better with the exact solution when t is closer to zero, and the error increases as t gets closer to one.

Table 2. Absolute errors for $M = 5$ and $M = 6$

t_i	E_5	E_6
0.1	8.963065387e-09	3.389353381e-10
0.2	2.720358344e-08	3.618011279e-09
0.3	2.394278514e-08	3.060617093e-09
0.4	6.937304025e-08	7.998320783e-09
0.5	1.053035117e-07	7.465058682e-09
0.6	1.310158346e-07	1.884611267e-08
0.7	4.027837364e-07	2.564863577e-08
0.8	9.757849775e-07	3.424606509e-08
0.9	8.700010081e-06	6.393175213e-07
1	3.267790566e-05	3.926144182e-06

Figure 2 presents the analytical and numerical solutions calculated with a fifth degree polynomial. We can see that the numerical results agree very well with the analytical results.

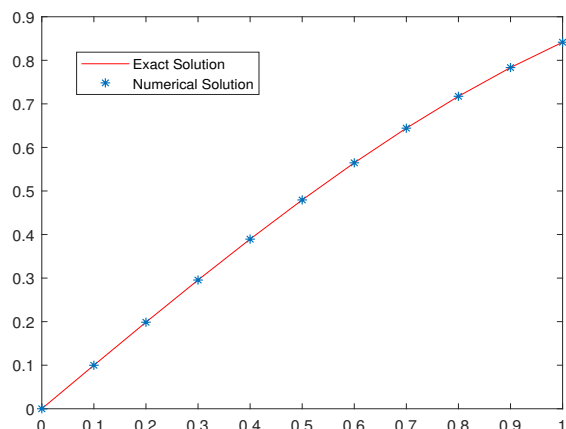


Figure 2. Exact solution and approximate solution for $k = 0$, $M = 5$

4.4. Problem 4:

The fourth example is a third-order nonlinear differential equation with proportional delay, [14, 43]

$$y'''(t) + 1 - 2y^2(\frac{t}{2}) = 0 \quad 0 \leq t \leq 1$$

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0$$

The exact solution of the above problem is $y(t) = \sin(t)$. Comparison of the present method using 5^{th} and 6^{th} -order polynomials with decomposition method with 13^{th} -order polynomial [43], and Adomian decomposition method with 9^{th} -order polynomial [14] are given in Table 3. The results in [14] were not given explicitly, so we derived this polynomial for comparison purposes. When the results are compared, one can see that approximately the same accuracy is obtained by using smaller degree polynomials with the present method. This shows that the present method is more efficient than the other two methods.

Table 3. Comparison of the absolute errors with other numerical methods.

t_i	Present method, E_5	Present method, E_6	Decomposition M. E_{13} [43]	Adomian decomposition M. E_9 [14]
0.1	2.54e-09	5.37e-10	0.0	1.02e-15
0.2	3.24e-09	1.39e-09	0.0	5.28e-13
0.3	2.11e-08	1.59e-09	0.0	2.02e-11
0.4	1.44e-08	7.06e-09	0.0	2.69e-10
0.5	1.21e-07	3.52e-09	2.61e-09	2.00e-09
0.6	1.42e-07	3.27e-08	1.04e-08	1.03e-08
0.7	2.48e-06	6.05e-08	4.07e-08	4.12e-08
0.8	1.11e-05	1.05e-06	1.38e-07	1.36e-07
0.9	3.47e-05	5.08e-06	4.00e-07	3.92e-07
1	8.87e-05	1.67e-05	1.03e-06	1.00e-06

4.5. Problem 5:

The last example is a sixth-order equation with proportional delay

$$\begin{aligned}
 y^{(6)}(t) - 1 + 2y^2\left(\frac{t}{2}\right) &= 0 \quad 0 \leq t \leq 1 \\
 y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1 \\
 y'''(0) = 0, \quad y^{(4)}(0) = 1, \quad y^{(5)}(0) = 0.
 \end{aligned}$$

The exact solution of the above problem is $y(t) = \cos(t)$. We solve the problem by taking $M = 6$. Table 4 presents the absolute errors. It is seen that even with a 6th-order polynomial, we can obtain an accuracy of at least five decimal places. This shows that the method is very efficient in solving high order nonlinear differential equations.

Table 4. Comparison of the approximate solution with the exact solution.

t_i	Exact solution for M = 6	Numerical solution for M = 6	Error of solution for M = 6
0.1	0.995004165278026	0.995004165278227	2.01616501271928e-13
0.2	0.980066577841242	0.980066577806563	3.46787043525865e-11
0.3	0.955336489125606	0.955336487827882	1.29772448342891e-09
0.4	0.921060994002885	0.921060979620032	1.43828530196899e-08
0.5	0.877582561890373	0.877582472305420	8.95849524562564e-08
0.6	0.825335614909678	0.825335220984428	3.93925250197213e-07
0.7	0.764842187284489	0.764840818192666	1.36909182280043e-06
0.8	0.696706709347165	0.696702695682056	4.01366510993650e-06
0.9	0.621609968270664	0.621599626525754	1.03417449105470e-05
1	0.540302305868140	0.540278227546903	2.40783212364093e-05

Figure 3 presents the graphical interpretation of the numerical and analytical solution for $M = 6$.

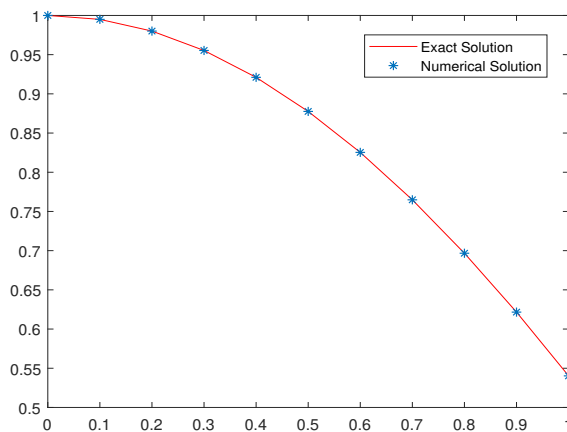


Figure 3. Exact solution and approximate solution for $M = 6$

5. Conclusion

In this study, the application of the Legendre wavelet method is extended to solve high order nonlinear differential equations with variable and proportional delays. In order to show the efficiency of the proposed method, we considered five nonlinear equations up to 6 order. We observed that the method is capable of finding the analytical solution when the solution is a polynomial function. It is also observed that accurate numerical results can be obtained by using quite small values of M . Furthermore, the application of the method does not require the approximation of the nonlinear terms, it is efficient and easy to implement.

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