Cauchy's Theorem for Orthogonal Polyhedra of Genus 0

Therese Biedl[1,⋆] and Burkay Genc[2,⋆⋆]

[1] David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
biedl@uwaterloo.ca
[2] Izmir University of Economics, Faculty of Computer Science, Sakarya Cad. No:156, Balcova, Izmir, Turkey
burkaygenc@gmail.com

Abstract. A famous theorem by Cauchy states that the dihedral angles of a convex polyhedron are determined by the incidence structure and face-polygons alone. In this paper, we prove the same for orthogonal polyhedra of genus 0 as long as no face has a hole. Our proof yields a linear-time algorithm to find the dihedral angles.

1 Introduction

A famous theorem by Cauchy states that for a convex polyhedron, the incidence structure and the face-polygons determine the polyhedron uniquely. Put differently, if we are given a graph with a fixed order of edges around each vertex, and we are given the angles at every vertex-face incidence and edge lengths, then there can be at most one set of dihedral angles such that graph, facial angles, edge lengths and dihedral angles are those of a convex polyhedron. Many books and monographs give this theorem, proofs, and related results [1,10].

Cauchy's theorem does not hold for polyhedra that are not convex. An easy example is a polyhedron where one face has a rectangular "hole" where a small box can be popped to the "inside" or "outside". But in fact, there are even so-called *flexible* polyhedra where the dihedral angles change continuously (see e.g. [10].)

We show in this paper that Cauchy's theorem *does* hold for orthogonal polyhedra of genus 0, as long as we exclude holes in faces. (Rather than defining holes, we will express this by saying that the graph of the polyhedron must be connected; see Section 2 for precise definitions.) Thus, while a big cube with a small cube attached on one face has two possible realizations, this is in essence the only way in which multiple realizations are possible.

Our proof is algorithmic and yields a linear-time algorithm to find the only possible set of dihedral angles of a realizing orthogonal polyhedron. This is in

⋆ Supported by NSERC.
⋆⋆ This work was done while the author was a student at University of Waterloo and supported by NSERC.

A. Fiat and P. Sanders (Eds.): ESA 2009, LNCS 5757, pp. 71–82, 2009.

contrast to convex polyhedra, where determining whether a working set of dihedral angles exist is not even known to be in NP; see [4,9] and the references therein for some recent progress on this tantalizing problem.

1.1 Roadmap

We first briefly outline the approach of this paper. Rather than proving uniqueness and then deriving an algorithm from the proof, we provide an algorithm that reconstructs an orthogonal polyhedron. There will never be any choice in the assignment of dihedral angles, except at one moment when we can choose one dihedral angle. Hence we obtain two sets of dihedral angles, and can argue that only one of them could possibly do; this then proves uniqueness.

Our algorithm proceeds in three steps. In the first step in Section 3, we only identify which dihedral angles must be flat, i.e., have value 180°. We do this by determining the orientation of each face; the algorithm to do so is simple, but proving its correctness is not.[1] Two adjacent faces with the same orientation must have a flat dihedral angle between them, so this determines all flat dihedral angles.

The problem hence reduces to reconstructing an orthogonal polyhedron where all dihedral angles are non-flat. In Section 4, we show that there are only 7 possible configurations of vertices for such a polyhedron. Moreover, if we fix one dihedral angle and know all facial angles, this determines all other dihedral angles at a vertex, and hence with a simple propagation scheme, all dihedral angles can be computed as long as the graph is connected.

Finally, we study in Section 5 which of the two resulting sets of dihedral angles can possibly be the correct set of dihedral angles. This is the only part of the algorithm that uses edge lengths. We conclude with remarks in Section 6.

2 Definitions

A *polygonal curve* is a simple closed curve in the plane that consists of a finite number of line segments. A *polygon* is a set in a plane whose boundary is one polygonal curve. A *polygonal region* is an interior connected set in a plane that is a finite union of polygons. A *polyhedral surface* is a connected 2-manifold that is a finite union of polygonal regions. A *polyhedron* is a set in 3D whose boundary is a polyhedral surface. Its *genus* is the genus of the surface that bounds it.

A *face* of a polyhedron is a maximal polygonal region on the boundary of the polyhedron. Note that a face need not be a polygon, because its boundary may be disconnected and/or touch itself and hence not be simple. A *vertex* is a point that belongs to at least three faces. An *edge* is a maximal line segment that belongs to two faces and contains no vertex other than its endpoints. A *facial angle* is the interior angle of a face at a vertex. A *dihedral angle* is the interior angle at an edge between two adjacent faces.

[1] A preliminary version of this algorithm appeared in 2004 [2], but its correctness was shown only for orthogonally convex polyhedra for which all faces are rectangles.