Ukrainian Mathematical Journal, Vol. 58, No. 1, 2006

SOME PROPERTIES OF A CAUCHY-TYPE INTEGRAL FOR THE MOISIL–THEODORESCO SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

B. Schneider

UDC 517.5

Our main interest is an analog of a Cauchy-type integral for the theory of the Moisil–Theodoresco system of differential equations in the case of a piecewise-Lyapunov surface of integration. The topics of the paper concern theorems that cover basic properties of this Cauchy-type integral: the Sokhotskii–Plemelj theorem for it as well as a necessary and sufficient condition for the possibility of extending a given Hölder function from such a surface up to a solution of the Moisil–Theodoresco system of partial differential equations in a domain. A formula for the square of a singular Cauchy-type integral is given. The proofs of all these facts are based on intimate relations between the theory of the Moisil–Theodoresco system of partial differential equations and some versions of quaternionic analysis.

1. Introduction

As is well known, the role of Cauchy-type integrals in the holomorphic function theory of one complex variable is very important. In this article, we investigate the properties of Cauchy-type integrals for a first-order elliptic system in $\mathbb{R}^3$. Let Ω be a domain in $\mathbb{R}^3$. Suppose that $f = f_0 + \vec{f} \in C^1(\Omega, \mathbb{R}^4)$. The homogeneous system

$$\mathrm{div} f = 0,$$

$$\mathrm{grad} f_0 + \mathrm{rot} \vec{f} = 0$$

is called the Moisil–Theodoresco system and is the simplest analog of the Cauchy–Riemann system in the three-dimensional case. Thus, the theory of solutions of the Moisil–Theodoresco system of differential equations reduces, in some degenerate cases, to that of complex holomorphic functions. Hence, one may consider the former to be a generalization of the latter.

Note that if $f_0 = 0$, then we have

$$\begin{aligned} \mathrm{div} f &= 0, \\ \mathrm{rot} f &= 0. \end{aligned} \tag{1}$$

Solutions to system (1) are called solenoidal and irrotational vector fields (cf. [1], where some applications to geophysics are given). It is known that solutions of (1) satisfy the Laplace equation and are sometimes called Laplacian or harmonic vector fields. In [2], we studied some properties of a Cauchy-type integral for the theory of Laplace vector fields as well.

In the present paper, we follow the approach presented in [3], where we studied an analog of a Cauchy-type integral for the theory of time-harmonic solutions of the relativistic Dirac equation in the case of a piecewise-Lyapunov surface of integration. The paper is organized as follows: In Sec. 2, we formulate a series of theorems

Izmir University of Economics, Izmir, Turkey.

Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 58, No. 1, pp. 105–112, January, 2006. Original article submitted May 16, 2005.

 0041–5995/06/5801–0118

that cover basic properties of a Cauchy-type integral for the theory of the Moisil–Theodoresco system of differential equations in the case of a piecewise-Lyapunov surface of integration. The proofs of all of them can be found in Sec. 4 in the form of more or less direct corollaries of the corresponding facts valid for hyperholomorphic function theory, which is developed in Sec. 3 and [4].

2. Moisil–Theodoresco System of Partial Differential Equations and Cauchy–Moisil–Theodoresco Integral

2.1. Let Ω denote a domain in $\mathbb{R}^3$ and let $\Gamma := \partial\Omega$ be its boundary. For $\Omega \subset \mathbb{R}^3$, consider an $\mathbb{R}^4$-valued function $f = (f_0, f_1, f_2, f_3)$ that satisfies the following system of partial differential equations:

$$0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = 0,$$

$$\frac{\partial f_0}{\partial x_1} + 0 - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0,$$

$$\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + 0 - \frac{\partial f_3}{\partial x_1} = 0,$$

$$\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + 0 = 0.$$

It is usually called the *Moisil–Theodoresco system*. Let $V_{st} := \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ with $a = (\delta_j^k)_{j,k=1}^3$ (δ_j^k is the Kronecker symbol), $x = (0, x_1, x_2, x_3)^T$, and $d\hat{x} = (0, dx_{[1]}, -dx_{[2]}, dx_{[3]})^T$, where $dx_{[k]}$ denotes, as usual, the differential form $dx_1 \wedge dx_2 \wedge dx_3$ with the factor dx_k omitted. The integral

$${}^{V_{st}}K_\Gamma[f](x) := \frac{1}{4\pi} \int\limits_\Gamma \frac{1}{|\tau - x|^3} B_l(V_{st}^T \cdot (\tau - x)) B_l(V_{st}^T \cdot d\hat{\tau}) f(\tau), \quad x \notin \Gamma,$$

plays the role of an analog of a Cauchy-type integral in the theory of the Moisil–Theodoresco system of partial differential equations with $f\colon \Gamma \to \mathbb{R}^4$ (see [5]). We call it the *Cauchy–Moisil–Theodoresco integral.*

2.2. For reader's convenience, we collect here some definitions which we use in the sequel. Let $H_\mu(\Gamma, \mathbb{R}^4)$ denote the class of functions that satisfy the Hölder condition $\{f \in \mathbb{R}^4 \,\big|\, |f(t_1) - f(t_2)| \le L_f |t_1 - t_2|^\mu \ \forall \{t_1, t_2\} \subset \Gamma, \ L_f = \text{const}\}$ with exponent $0 < \mu \le 1$. Here, $|f|$ means the Euclidean norm in $\mathbb{R}^4$, while $|t|$ is the Euclidean norm in $\mathbb{R}^3$. We say (see, e.g., [6]) that the surface Γ in $\mathbb{R}^3$ is a Lyapunov surface if the following conditions are satisfied:

1. At every point $t \in \Gamma$, there is the tangential hyperplane.
2. There exists a constant number $R > 0$ such that, for any point $t \in \Gamma$, the set $\Gamma \cap \mathbb{B}^3(t, R)$ is connected and lines that are parallel to the normal $\vec{n}(t)$ to the surface Γ at the point t intersect $\Gamma \cap \mathbb{B}^3(t, R)$ at at most one point. Here, $\mathbb{B}^3(t, R)$ is an open ball of radius R centered at the point t in $\mathbb{R}^3$.
3. The normal vector field $\vec{n}\colon \Gamma \to \mathbb{R}^3$ satisfies the Hölder condition.

A conical surface in $\mathbb{R}^3$ is the surface generated by a straight line (the generator) that passes through a fixed point (the vertex or conical point) and moves along a fixed curve (the directing curve). A solid angle in $\mathbb{R}^3$ is a part of the space $\mathbb{R}^3$ bounded by some conical surface. A tangential conical surface to Γ at a point t_0 is the conical surface generated by straight tangent lines to the surface Γ at the point t_0 (the conical point of the tangential conical surface). In particular, for a smooth point, the tangential conical surface is its tangential plane. The measure of a solid angle in $\mathbb{R}^3$ is the surface area cut out by the solid angle from the unit sphere centered at the vertex; the value of the measure is defined in accordance with the orientation of the conical surface.

Let $\mathbf{l}$ be a smooth, closed, simple curve on the surface $\Gamma \subset \mathbb{R}^3$ such that $\Gamma \setminus \mathbf{l}$ is a Lyapunov surface. Then the curve $\mathbf{l}$ is called an edge of the surface Γ, and Γ is called a Lyapunov surface with edge.

For $\mathbf{l}$ as above, let $t_0 \in \mathbf{l}$. Then the normal plane to the curve at the point t_0 intersects the surface Γ by the curve l_{t_0}. The curve l_{t_0} is a smooth curve except, possibly, t_0. Assume that the curve l_{t_0} has both one-sided tangents P_1 and P_2 at t_0. Let $\mathbf{p}$ be a tangent line to the curve $\mathbf{l}$ itself at the point t_0. Then the plane T_1 that passes through P_1 and $\mathbf{p}$ and the plane T_2 that passes through P_2 and $\mathbf{p}$ generate a dihedral angle, which is called the tangential dihedral angle.

The linear measure of the tangential dihedral angle is the value of the angle formed by the one-sided tangents P_1 and P_2. Denote it by $\eta(t)$. In the sequel, we take $\eta(t_0) = \mathrm{const}$ on $\mathbf{l}$, where the constant is different from 0 and 2π. If $\eta(t) = \pi$ on $\mathbf{l}$, then Γ is a smooth surface. In particular, for a smooth surface, any closed, smooth, simple curve is an edge.

The solid measure of the tangential dihedral angle is the surface area cut out by the planes T_1 and T_2 from the unit sphere centered at the point $t_0 \in \mathbf{l}$; the value of the measure is defined in accordance with the orientation of the surface with edge.

Let Γ be a surface in $\mathbb{R}^3$ that contains a finite number of conical points and a finite number of nonintersecting edges such that none of the edges contains any conical point. If the complement (in Γ) of the union of conical points and edges is a Lyapunov surface, then we call Γ a piecewise-Lyapunov surface in $\mathbb{R}^3$.

***2.3.* Theorem** (Sokhotskii–Plemelj formulas for the Cauchy–Moisil–Theodoresco integral with piecewise-Lyapunov surface of integration)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$ with piecewise-Lyapunov boundary. Let $f \in H_\mu(\Gamma, \mathbb{R}^4)$. Then the following limits exist:*

$$\lim_{\Omega^{\pm} \ni x \to t \in \Gamma} {}^{V_{st}}K_\Gamma[f](x) =: {}^{V_{st}}K_\Gamma[f]^{\pm}(t).$$

Moreover, the following identities hold for all $t \in \Gamma$:

$${}^{V_{st}}K_\Gamma[f]^{+}(t) = \left(1 - \frac{\gamma(t)}{4\pi}\right) f(t) + {}^{V_{st}}K_\Gamma[f](t) := \left(1 - \frac{\gamma(t)}{4\pi}\right) f(t) + \frac{1}{2}\,{}^{V_{st}}S_\Gamma[f](t),$$

$${}^{V_{st}}K_\Gamma[f]^{-}(t) = -\frac{\gamma(t)}{4\pi} f(t) + {}^{V_{st}}K_\Gamma[f](t) := -\frac{\gamma(t)}{4\pi} f(t) + \frac{1}{2}\,{}^{V_{st}}S_\Gamma[f](t),$$

where

$${}^{V_{st}}S_\Gamma[f](t) := 2\,{}^{V_{st}}K_\Gamma[f](t),$$

the integrals are understood in the sense of the Cauchy principal value, and $\gamma(t)$ is the measure of the solid angle of the tangential conical surface at the point t or is the solid measure of the tangential dihedral angle at the point t.

2.4. We call the operator ${}^{V_{st}}S_\Gamma$ the *singular Cauchy–Moisil–Theodoresco integral operator*. It turns out that many properties that are of interest for us can better be expressed in terms of another operator, namely

$$ {}^{V_{st}}\breve{S}_\Gamma[f](t) := \frac{2\pi - \gamma(t)}{2\pi} f(t) + {}^{V_{st}}S_\Gamma[f](t) $$

for any $t \in \Gamma$. We call ${}^{V_{st}}\breve{S}_\Gamma$ the *modified singular Cauchy–Moisil–Theodoresco integral operator*.

2.5. **Theorem** (Plemelj–Privalov-type theorem for the theory of the Moisil–Theodoresco system of partial differential equations)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with piecewise-Lyapunov boundary. Then*

$$ f \in H_\mu(\Gamma, \mathbb{R}^4) \Rightarrow {}^{V_{st}}\breve{S}_\Gamma[f](t) \in H_\mu(\Gamma, \mathbb{R}^4) \tag{2} $$

for $0 < \mu < 1$.

2.6. **Theorem** (extension of a Hölder function given on Γ up to a solution of the Moisil–Theodoresco system of partial differential equations)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with piecewise-Lyapunov boundary.*

1. *In order that a function* $f \in H_\mu(\Gamma, \mathbb{R}^4)$ *be a boundary value of a function* $\tilde{f}$ *that satisfies a Moisil–Theodoresco system of partial differential equations in* Ω^+ *and is continuous in* $\overline{\Omega^+}$, *it is necessary and sufficient that*

$$ f(t) = {}^{V_{st}}\breve{S}_\Gamma[f](t) \quad \forall t \in \Gamma. $$

2. *In order that a function* $f \in H_\mu(\Gamma, \mathbb{R}^4)$ *be a boundary value of a function* $\tilde{f}$ *that satisfies a Moisil–Theodoresco system of partial differential equations in* Ω^-, *is continuous in* $\overline{\Omega^-}$, *and vanishes at infinity, it is necessary and sufficient that*

$$ f(t) = -{}^{V_{st}}\breve{S}_\Gamma[f](t) \quad \forall t \in \Gamma. $$

2.7. **Theorem** (on the square of the operators ${}^{V_{st}}S_\Gamma$ and ${}^{V_{st}}\breve{S}_\Gamma$)**.** *If* Γ *is a piecewise Lyapunov surface, then we have the following formulas for* $f \in H_\mu(\Gamma, \mathbb{R}^4)$, $0 < \mu < 1$:

$$ {}^{V_{st}}S^2_\Gamma[f](t) = a_1(t)f(t) + a_2(t)^{V_{st}}S_\Gamma[f](t) + {}^{V_{st}}S_\Gamma[a_3 f](t), \tag{3} $$

$$ {}^{V_{st}}\breve{S}^2_\Gamma[f](t) = f(t) \tag{4} $$

for all $t \in \Gamma$, *i.e., the modified singular Cauchy–Moisil–Theodoresco integral operator* ${}^{V_{st}}\breve{S}_\Gamma$ *is an involution on* $H_\mu(\Gamma, \mathbb{R}^4)$, $0 < \mu < 1$,

$$ {}^{V_{st}}\breve{S}^2_\Gamma = I, $$

where

$$ a_1(t) := \frac{\gamma(t)}{\pi} - \frac{\gamma^2(t)}{4\pi^2}, \qquad a_2(t) := \frac{\gamma(t)}{2\pi} - 2, \qquad a_3(t) := \frac{\gamma(t)}{2\pi}. $$

The proofs of these theorems can be found in Sec. 4.

3. Hyperholomorphic Function Theory: General Information

In this section, we provide some background on quaternionic analysis needed in this paper. For more information, we refer the reader to [7–9].

3.1. Consider the skew-field of real quaternions $\mathbb{H}$:

$$\mathbb{H} := \{x = x_0 i_0 + x_1 i_1 + x_2 i_2 + x_3 i_3;\ (x_0, x_1, x_2, x_3)^T \in \mathbb{R}^4\},$$

where i_0 is the unit, and $i_1,\ i_2,$ and i_3 are the quaternionic imaginary units with the properties:

$$i_0^2 = i_0 = -i_k^2, \quad i_0 i_k = i_k i_0 = i_k, \quad k \in \mathbb{N}_3,$$

$$i_1 i_2 = -i_2 i_1 = i_3, \quad i_2 i_3 = -i_3 i_2 = i_1, \quad i_3 i_1 = -i_1 i_3 = i_2.$$

Let $x = \sum_{k=0}^{3} x_k i_k \in \mathbb{H}.$ Then

$$x_0 =: \mathrm{Sc}(x) \quad \text{and} \quad \vec{x} := \sum_{k=1}^{3} x_k \cdot i_k =: \mathrm{Vect}(x)$$

are called, respectively, the scalar and the vector part of a quaternion. We can write

$$x = x_0 + \vec{x}.$$

In vector terms, the multiplication of two arbitrary real quaternions x and y can be rewritten as follows:

$$x \cdot y = \big(x_0 + \vec{x}\big) \cdot \big(y_0 + \vec{y}\big) = x_0 \cdot y_0 - \langle \vec{x}, \vec{y} \rangle + x_0 \vec{y} + y_0 \vec{x} + [\vec{x}, \vec{y}],$$

where $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$ denote the usual scalar and vector products of three-dimensional vectors. In particular, if $x_0 = y_0 = 0,$ then

$$x \cdot y = -\langle \vec{x}, \vec{y} \rangle + [\vec{x}, \vec{y}\,].$$

The quaternionic conjugation of $x = x_0 i_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ is given by

$$\bar{x} := x_0 i_0 - x_1 i_1 - x_2 i_2 - x_3 i_3.$$

We use the Euclidean norm $|x|$ in $\mathbb{H},$ defined by

$$|x| := \sqrt{x \bar{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

An important property is that

$$|xy| = |x| \cdot |y|.$$

3.2. Let the matrix

$$B_l(b) := \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} \tag{5}$$

be the left regular representation of a real quaternion b, and, correspondingly, let the matrix

$$B_r(b) := \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{pmatrix}$$

be the right regular representation of the real quaternion b. Then $\mathbb{H}$ can be identified as a skew-field with $\mathcal{B}_l := \{B_l(b) \mid b \in \mathbb{H}\}$. The same holds for $\mathcal{B}_r := \{B_r(b) \mid b \in \mathbb{H}\}$ and $\mathbb{H}$. Moreover, the left multiplication by the real quaternion b corresponds to the multiplication by the matrix $B_l(b)$, i.e.,

$$b \cdot x \leftrightarrow B_l(b) \cdot (x_0, x_1, x_2, x_3)^T,$$

where $(x_0, x_1, x_2, x_3)^T := \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

3.3. We consider functions ranged in $\mathbb{H}$ and defined in a domain $\Omega \subset \mathbb{R}^3$. The notation $C^p(\Omega, \mathbb{H})$, $p \in \mathbb{N} \cup \{0\}$, has the usual componentwise meaning. A function f is called left-hyperholomorphic if

$$D[f] := \sum_{k=1}^{3} i_k \frac{\partial f}{\partial x_k} =: \sum_{k=1}^{3} i_k \partial_k[f] = 0$$

in Ω. Let $\theta = -\frac{1}{4\pi}\frac{1}{|x|}$ be a fundamental solution of the Laplace operator. Then a fundamental solution $\mathcal{K}$ of the operator D is given by the formula (see [9])

$$\mathcal{K}(x) := -D[\theta](x) = \frac{1}{4\pi} \sum_{k=1}^{3} \bar{i}_k \frac{x_k}{|x|^3} = \frac{1}{4\pi}\frac{1}{|x|^3} B_l(V_{st}^T \cdot x), \tag{6}$$

where $st := \{i_1, i_2, i_3\}$. We set

$$\sigma_x := i_1 dx_{[1]} - i_2 dx_{[2]} + i_3 dx_{[3]},$$

where $dx_{[k]}$ denotes, as usual, the differential form $dx_1 \wedge dx_2 \wedge dx_3$ with the factor dx_k omitted. Note that if Γ is a piecewise-smooth surface in $\mathbb{R}^3$ and if $\vec{n}(\tau) = (n_1(\tau), n_2(\tau), n_3(\tau))$ is the outward unit normal to the surface Γ at τ, then

$$\sigma\,|_\Gamma = \vec{n}(\tau) ds_\tau =: \sum_{k=1}^{3} n_k(\tau) i_k ds_\tau,$$

where ds is the differential form of the two-dimensional surface Γ in $\mathbb{R}^3$. Let $\Omega = \Omega^+$ be a domain in $\mathbb{R}^3$ with boundary Γ that is assumed to be a piecewise-Lyapunov surface; denote $\Omega^- := \mathbb{R}^3 \setminus (\Omega^+ \cup \Gamma)$. If f is a Hölder function, then its left-hyperholomorphic Cauchy-type integral is defined as follows:

$$K_\Gamma[f](x) := \int_\Gamma \mathcal{K}(\tau - x) \cdot \sigma_\tau \cdot f(\tau), \quad x \in \Omega^\pm.$$

For more information about hyperholomorphic functions, we refer the reader to [7–10] (see also [11]).

4. Proofs of the Theorems from Section 2

In this section, we prove all theorems from Sec. 2 using the relations between the theory of the Moisil–Theodoresco system of partial differential equations and the theory of hyperholomorphic functions.

4.1. We begin this section with a brief description of relations between the theory of the Moisil–Theodoresco system of partial differential equations and the theory of hyperholomorphic functions.

On the set $C^1(\Omega, \mathbb{H})$, the well-known Moisil–Theodoresco operator is defined by the formula

$$D := \sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}.$$

Using matrix (5), we can rewrite the equality $D[f] = 0$ (the Moisil–Theodoresco system) as follows:

$$B_l\left(\sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}\right) f^T = 0,$$

where

$$B_l\left(\sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k}\right) = \begin{pmatrix} 0 & -\dfrac{\partial}{\partial x_1} & -\dfrac{\partial}{\partial x_2} & -\dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & 0 & -\dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 & -\dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0 \end{pmatrix}.$$

Thus,

$$D[f]=0 \Longleftrightarrow \begin{pmatrix} 0 & -\dfrac{\partial}{\partial x_1} & -\dfrac{\partial}{\partial x_2} & -\dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & 0 & -\dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 & -\dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = 0,$$

i.e., one can identify the class of solutions of the elliptic system of partial differential equations with constant coefficients with the set of hyperholomorphic functions. By equality (6) for the $\mathbb{R}^4$-valued function $f,$ we have

$$K_\Gamma[f](x) := \frac{1}{4\pi}\int\limits_\Gamma \frac{1}{|\tau-x|^3} B_l(V_{st}^T\cdot(\tau-x))B_l(V_{st}^T\cdot d\hat{\tau})f(\tau), \quad x\notin\Gamma.$$

So, the integral $K_\Gamma[f](x)$ coincides with ${}^{V_{st}}K_\Gamma[f](x).$ In the same way,

$$S_\Gamma[f](t) := 2K_\Gamma[f](t) = \frac{1}{2\pi}\int\limits_\Gamma \frac{1}{|\tau-t|^3} B_l(V_{st}^T\cdot(\tau-t))B_l(V_{st}^T\cdot d\hat{\tau})f(\tau) \quad \forall t\in\Gamma,$$

and, hence, the integral S_Γ for $f\in H_\mu(\Gamma,\mathbb{R}^4)$ coincides with ${}^{V_{st}}S_\Gamma[f].$

4.2. Proof of Theorem 2.3. Let $f\in H_\mu(\Gamma,\mathbb{R}^4).$ Consider ${}^{V_{st}}K_\Gamma[f](x).$ It was proved that

$${}^{V_{st}}K_\Gamma[f](x) = K_\Gamma[f](x).$$

By [4] (Theorem 2.1 for $\alpha=0$) (see also [7, 8]), there exists $K_\Gamma[f]^\pm(t)$ and

$$K_\Gamma[f]^+(t) = \left(1-\frac{\gamma(t)}{4\pi}\right)f(t) + K_\Gamma[f](t) =: \left(1-\frac{\gamma(t)}{4\pi}\right)f(t) + \frac{1}{2}S_\Gamma[f](t),$$

$$K_\Gamma[f]^-(t) = -\frac{\gamma(t)}{4\pi}f(t) + K_\Gamma[f](t) =: -\frac{\gamma(t)}{4\pi}f(t) + \frac{1}{2}S_\Gamma[f](t).$$

Hence, there exists ${}^{V_{st}}K_\Gamma[f]^\pm(t),$ and, after noncomplicated computation, we obtain the required result. We set

$$\breve{S}_\Gamma[f](t) := \frac{2\pi-\gamma(t)}{2\pi}f(t) + S_\Gamma[f](t)$$

for any $t\in\Gamma.$

4.3. Proof of Theorem 2.5. Let $f \in H_\mu(\Gamma, \mathbb{R}^4)$. Consider ${}^{V_{st}}K_\Gamma[f](x)$. By Theorem 2.3, there exists ${}^{V_{st}}K_\Gamma[f]^\pm(t)$ and

$$ {}^{V_{st}}K_\Gamma[f]^+(t) = \frac{1}{2}[f(t) + {}^{V_{st}}\breve{S}_\Gamma[f](t)], $$

$$ {}^{V_{st}}K_\Gamma[f]^-(t) = \frac{1}{2}[-f(t) + {}^{V_{st}}\breve{S}_\Gamma[f](t)], $$

where ${}^{V_{st}}\breve{S}_\Gamma$ has been defined in Sec. 2.4. According to Sec. 4.1, we have $f \in H_\mu(\Gamma, \mathbb{R}^4)$, and, hence, ${}^{V_{st}}\breve{S}_\Gamma[f] = \breve{S}_\Gamma[f]$ on Γ. In [4] (Sec. 2.2 for $\alpha = 0$), it was proved that $\breve{S}_\Gamma$ satisfy the Hölder condition. Therefore, recalling the relationship between the operators $\breve{S}_\Gamma$ and ${}^{V_{st}}\breve{S}_\Gamma$, we conclude that ${}^{V_{st}}\breve{S}_\Gamma[f] \in H_\mu(\Gamma, \mathbb{R}^4)$.

4.4. Proof of Theorem 2.6. This proof follows from [4] (Theorem 2.3 for $\alpha = 0$) with regard for the above relation between the class of solutions of the Moisil–Theodoresco system of partial differential equations and the set of hyperholomorphic functions.

4.5. Proof of Theorem 2.7. Let $f \in H_\mu(\Gamma, \mathbb{R}^3)$. Consider ${}^{V_{st}}K_\Gamma[f]$. In Sec. 4.1, it has been proved that

$$ f \in H_\mu(\Gamma, \mathbb{R}^4) \Longrightarrow f \in H_\mu(\Gamma, \mathbb{R}^4). $$

So, we obtain (3) after taking into account [4] (Theorem 2.4 for $\alpha = 0$) (see also [8]) combined with straightforward calculation. Using the definition of modified singular operator ${}^{V_{st}}S_\Gamma$, we obtain (4).

REFERENCES

1. M. Zhdanov, *Integral Transforms in Geophysics*, Springer, Heidelberg (1998).
2. B. Schneider and M. Shapiro, "Some properties of the Cauchy-type integral for the Laplace vector fields theory," *Global Anal. Appl. Math.*, 274–280 (2004).
3. B. Schneider and M. Shapiro, "Some properties of the Cauchy-type integral for the time-harmonic relativistic Dirac equation," *Math. Meth. Appl. Sci.*, **25**, No. 16–18, 1441–1463 (2002).
4. B. Schneider and M. Shapiro, "Some properties of the quaternionic Cauchy-type integral for a piecewise Lyapunov surface of integration," *Contemp. Math.*, **364**, 243–260 (2004).
5. I. Mitelman and M. Shapiro, "Formulae of changing of integration order and of inversion for some multidimensional singular integrals and hypercomplex analysis," *J. Natural Geom.*, **5**, 11–27 (1994).
6. S. Mikhlin, *Multidimensional Singular Integrals and Integral Equations* [in Russian], Fizmatgiz, Moscow (1962).
7. K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, Wiley (1997).
8. K. Gürlebeck and W. Sprössig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Akademie, Berlin (1989).
9. M. Shapiro and N. Vasilevski, "Quaternionic ψ-hyperholomorphic functions, singular integral operators and boundary value problems. I. ψ-Hyperholomorphic function theory," *Complex Variables, Theory Appl.*, **27**, 17–46 (1995).
10. V. Kravchenko and M. Shapiro, *Integral Representations for Spatial Models of Mathematical Physics*, Chapman & Hall (1996).
11. M. Mitrea, *Clifford Wavelets, Singular Integrals, and Hardy Spaces*, Springer (1994).