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Laguerre wavelet method for solving Troesch equation

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Abstract

The purpose of this paper is to illustrate the use of the Laguerre wavelet method in the solution of Troesch's equation, which is a stiff nonlinear equation. The unknown function is approximated by Laguerre wavelets and the equation is transformed into a system of algebraic equations. One of the advantages of the method is that it does not require the linearization of the nonlinear term. The problem is solved for different values of Troesch's parameter (μ) and the results are compared with both the analytical and other numerical results to validate the accuracy of the method.

***Keywords**: Laguerre Wavelet method, Troesch equation, Laguerre polynomial, Nonlinear differential equation.*

Troesch denkleminin çözümü için Laguerre dalgacık yöntemi

Özet

Bu makalenin amacı lineer olmayan Troesch denklemini Laguerre dalgacık yöntemini kullanarak çözmektir. Bilinmeyen fonksiyon Laguerre dalgacıkları ile yaklaştırılarak denklem bir cebirsel denklem sistemine dönüştürülür. Bu yöntemin avantajlarından biri, lineer olmayan terimin lineer hale dönüştürülmesine gerek kalmamasıdır. Denklem Troesch parametresinin farklı değerleri için çözülmüştür. Yöntemin etkin olduğunu göstermek için elde edilen sonuçlar gerek gerçek gerekse literatürdeki diğer sayısal sonuçlar ile karşılaştırılmıştır.

***Anahtar Kelimeler:** Laguerre dalgacık yöntemi, Troesch denklemi, Laguerre polinomu, Lineer olmayan diferensiyel denklem.*

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1. Introduction

Boundary value problems are used in several fields such as chemical physics, chemistry, biology, nanotechnology, natural science, and engineering. One of the important problem is the well known Troesch's problem which comes from the theory of gas porous electrodes. This problem arises in some chemical reaction-diffusion and heat transfer processes as well as a plasma column under radiation pressure.

In literature, several numerical methods have been employed to solve this nonlinear problem. We can list these methods as: Finite difference method [1], Chebyshev wavelet method [2], Chebysev collocation method [3], A finite-element approach based on cubic B-spline collocation [4], An accurate asymptotic approximation [5], Adomian decomposition method and the reproducing kernel method [6], Christov rational functions [7], Decomposition method [8], Differential transform method [9], High-Order Difference Schemes [10], Homotopy perturbation method [11], Hybrid heuristic computing [12], Jacobi-Gauss collocation method [13], Laplace transform and a modified decomposition technique [14], Modified Homotopy perturbation method [15], Newton-Raphson-Kantorovich approximation method [16], Optimal Homotopy asymptotic method [17], Perturbation Method and Laplace-Padé Approximation [18], Scott and the Kagiwada-Kalaba algorithms [19], Modified nonlinear Shooting method [20], Sinc-Collocation Method [21], sinc–Galerkin method [22], Variational iteration method [23, 24].

Laguerre series are used in the solution of delayed single degree-of-Freedom oscillator problem [25], high- order linear Fredholm integro-differential equations [26], and pantograph-type Volterra integro-differential equations [27].

In this study, Laguerre wavelets is used in the solution of the Troesch's problem. The unknown function and its derivatives are approximated by the Laguerre wavelets and the nonlinear differential equation is transformed into a system of nonlinear system of equations. The paper is organized as follows. In Section 2, we introduce wavelets, the Laguerre wavelets and their properties. In Section 3, we introduce the method of solving Troesch's problem by Laguerre wavelets. In Section 4, numerical results are presented. Some conclusions are drawn in Section 5.

2. Laguerre wavelets

2.1. Wavelets

A single function $\varphi(t)$ which is called mother wavelet is dilated (scaled) and translated by the parameters a and b, respectively in order to generate a family of functions of the form [28]

$$\varphi_{a,b}(t) = |a|^{-\frac{1}{2}}\varphi\left(\frac{t-b}{a}\right), a, b \in R, a \neq 0. \quad (1)$$

If the dilation parameter a and translation paramater b is restricted to $a = 2^{-k}$ and $b = n2^{-k}$, then the wavelets

$$\varphi_{k,n}(t) = 2^{k/2}\varphi(2^k t - n)$$

form an orthonormal basis in Hilbert space $L^2(R)$.

Due to the main advantages, different wavelet types such as Haar wavelet [29], Chebyshev wavelet [30], Legendre wavelet [31-34] now attract great attention of researchers. One of these advantages is that the wavelets are not periodic and do not continue to the infinity. Additionally, the wavelets with compact support are compatible to model localized features in applications.

2.2. Laguerre wavelets and properties

If the dilation and translation parameters in Eq. (1) are chosen respectively as $a = 2^{-(k+1)}$ and $b = (2n+1)2^{-(k+1)}$, then the Laguerre wavelets $\varphi_{nm}(t) = \varphi_{nm}(t; k; n; m)$ can be defined on [0, 1) for integers $k \geq 0;\ n = 0,1,2, \dots, 2^k - 1;$

$$\varphi_{nm}(t) = \begin{cases} 2^{(k+1)/2} L_m(2^{k+1}t - 2n - 1)\,, & \text{if } \dfrac{n}{2^k} \leq t < \dfrac{n+1}{2^k} \\ 0\,, & \text{otherwise} \end{cases}$$

where t is the normalized time and $m = 0, 1, 2, \dots, M$ is the order of very well known Laguerre polynomials $L_m(t)$; the dilation and translation parameters in Eq. (1) are respectively $a = 2^{-(k+1)}$ and $b = (2n+1)2^{-(k+1)}$.

The Laguerre polynomials are m-th degree polynomials which satisfy the differential equation

$$xy''(x) + (1-x)y'(x) + my(x) = 0,\ \ x \in (0, \infty)$$

and can be explicitly determined by the recurrence relation

$$(m+2)L_{(m+2)}(x) = (2m+3-x)L_{(m+1)}(x) - (m+1)L_m(x)$$

with $L_0(x) = 1$ and $L_1(x) = 1 - x$, [35]. The first few Laguerre polynomials are listed as

$$L_0(x) = 1,$$
$$L_1(x) = 1 - x,$$
$$L_2(x) = \frac{1}{2!}(x^2 - 4x + 2),$$
$$L_3(x) = \frac{1}{3!}(-x^3 + 9x^2 - 18x + 6),$$
$$L_4(x) = \frac{1}{4!}(x^4 - 16x^3 + 72x^2 - 96x + 24),$$
$$L_5(x) = \frac{1}{5!}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120),$$

$$L_6(x) = \frac{1}{6!}(x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720).$$

As a natural result of the orthogonality of the Laguerre polynomials over the interval $(0,\infty)$, the Laguerre wavelets $\varphi_{nm}(t)$ are orthogonal with respect to the dilated and translated weight function $w_n(t) = w(2^{k+1}t - 2n - 1) = e^{-(2^{k+1}t-2n-1)}$. This is an essential property to expand a function $f(t)$, defined on $[0,1)$ as the infinite series of Laguerre wavelets

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\varphi_{nm}(t),$$

where A_{nm} are coefficients obtained by the inner product $A_{nm} = < f(t), \varphi_{nm}(t) > = \int_0^{\infty} w_n(t) f(t)\varphi_{nm}(t)dt$. If the series is truncated then it can be written as

$$f(t) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi_{nm}(t).$$

3. Application to Troesch's problem

In this section, we consider Troesch's problem and discuss the implementation of the Laguerre wavelet method.

A boundary value problem (BVP) of Troesch's equation is introduced by the second order nonlinear differential equation and boundary conditions

$$u''(t) = \mu\sinh(\mu u(t)),\ t \in [0,1], \tag{2}$$
$$u(0) = 0, u(1) = 1. \tag{3}$$

Here, positive constant μ is called Troesch's parameter. The closed form solution of Eq. (2) and Eq. (3) is given by means of Jacobian elliptic function $sc(\mu|r)$ as

$$u(t) = \frac{2}{\mu}\sinh^{-1}\left[\frac{u'(0)}{2}\ sc(\mu t|r)\right]$$

where $r = 1 - \frac{1}{4}\left(u'(0)\right)^2$ and $sc(\mu|r)(1-r)^{\frac{1}{2}} = \sinh(\frac{\mu}{2})$ [36]. In order to solve Troesch's problem, we expand the solution of Eq. (2) in terms of Laguerre wavelets in the form

$$u(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi_{nm}(t) \tag{4}$$

where A_{nm} are unknown coefficients to be determined. In order to find these coefficients we express the boundary conditions in Eq. (3) by using Eq. (4) as

$$u(0) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi_{nm}(0) = 0 \tag{5}$$

$$u(1) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi_{nm}(0) = 1 \tag{6}$$

These boundary conditions provide two algebraic equations to be solved for $2^k(M+1)$ unknown coefficients A_{nm}. For other $2^k(M+1)-2$ equations, we rewrite the differential equation in Eq.(2) in the form

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi''_{nm}(t) - \mu \sinh\left[\mu \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi_{nm}(t)\right] = 0 \tag{7}$$

and we use the roots, t_i, of shifted Chebyshev polynomials $U_{2^k(M+1)}$ as collocation points in Eq. (7)

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi''_{nm}(t_i) - \mu \sinh\left[\mu \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi_{nm}(t_i)\right] = 0 \tag{8}$$

for $i = 1,2,\dots,2^k(M+1)-2$. The system of algebraic equations in Eqs. (5), (6) and (8) can be solved for the same number of unknown coefficients for A_{nm}, $n = 0,1,2,\dots, 2^k-1$; $m = 0,1,2,\dots,M$ by using MATLAB tools. The approximate solution of BVP in Eqs. (2)-(3) is determined with the obtained values of A_{nm}, by $u(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\varphi_{nm}(t)$.

4. Results and discussion

In this Section, the problem given in Eqs. (2)-(3) is solved for two values of Troesch's parameter. Table 1 presents the absolute errors obtained by taking M = 4, 5 and 6. We can see that the maximum absolute error for M = 4 is 10^{-4}, and for M = 6 maximum absolute error is obtained as 10^{-7}. One can say that the method yields high accuracy even with low degree polynomials.

Table 1. Absolute errors for different values of M.

t_i	Error for M = 6	Error for M = 5	Error for M = 4
0.1	1.02705262211567e-07	6.50748263789081e-07	1.50597029199839e-05
0.2	2.04455795155267e-07	1.31084622134736e-06	2.99308713823387e-05
0.3	3.08525196668352e-07	1.97144941621596e-06	4.45469093270368e-05
0.4	4.13512608266053e-07	2.62164521075414e-06	5.97260537494315e-05
0.5	5.16097657887737e-07	3.28527395815348e-06	7.59601747001293e-05
0.6	6.20768004666594e-07	3.99985017496274e-06	9.21456752845939e-05
0.7	7.37419484031499e-07	4.71848371341732e-06	1.04237391663986e-04
0.8	8.40628852527559e-07	5.10860093361210e-06	1.03799293054374e-04
0.9	7.64399135100291e-07	4.22126587584781e-06	7.64257817268410e-05

Table 2 presents the comparison of the numerical results of the proposed method taking M=6 with the Homotopy perturbation method (HPM) [11], Perturbation method with Pade approximation [18], Modified nonlinear shooting method (MNLSM) [20], and Variational iteration method (VIM) [24], as well as the analytical solution. One can see that even with a low degree polynomial, the proposed method has a better accuracy than these methods.

Table 2. Comparison of the present method with exact and other numerical solutions for $\mu = 0.5$.

t_i	Exact Sln.	Present Method	HPM[11]	PM-Pade [18]	MNLSM[20]	VIM [24]
0.1	0.095944	0.095944	0.095948	0.095941	0.095972	0.100042
0.2	0.192128	0.192128	0.192135	0.192123	0.192185	0.200334
0.3	0.288794	0.288794	0.288804	0.288786	0.288879	0.301128
0.4	0.386184	0.386185	0.386196	0.386174	0.386298	0.402677
0.5	0.484547	0.484547	0.484559	0.484534	0.484416	0.505241
0.6	0.584133	0.584133	0.584145	0.584117	0.584281	0.609082
0.7	0.685201	0.685201	0.685212	0.685182	0.685256	0.714470
0.8	0.788016	0.788017	0.788025	0.787994	0.788079	0.821682
0.9	0.892854	0.892854	0.892859	0.892829	0.892926	0.931008

Table 3 presents the numerical results of the present method and the same numerical methods used in the previous comparison for $\mu = 1.$ We again observe that the present method has a better accuracy for increasing value of the Troesch's parameter.

Table 3. Comparison of the present method with exact and other numerical solutions for $\mu = 1.$

t_i	Exact Sln.	Present Method	HPM[11]	PM-Pade[18]	MNLSM[20]	VIM [24]
0.1	0.084661	0.084668	0.084934	0.871733	0.084730	0.100167
0.2	0.170171	0.170186	0.170697	0.170260	0.170310	0.201339
0.3	0.257393	0.257417	0.258133	0.257531	0.257603	0.304541
0.4	0.347222	0.347254	0.348116	0.347413	0.347506	0.410841
0.5	0.440599	0.440639	0.44157	0.440849	0.439937	0.521373
0.6	0.538534	0.538582	0.539498	0.538848	0.538905	0.637362
0.7	0.642128	0.642187	0.642987	0.642508	0.642093	0.760162
0.8	0.752608	0.752676	0.753267	0.753043	0.752558	0.891287
0.9	0.871362	0.871426	0.871733	0.871811	0.871310	1.032460

5. Conclusion

In this study, Laguerre wavelets are used to solve the nonlinear Troesh problem. The results are presented for several values of M and μ, and we observed that accurate numerical results are obtained by using quite small values of M. Compared with other numerical results, it has seen that the present method has a better accuracy. Furthermore, the application of the method does not require the approximation of the nonlinear terms, it is efficient and easy to implement.

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