



The Legendre Matrix-Collocation Approach for Some Nonlinear Differential Equations Arising in Physics and Mechanics

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Abstract

In this study, the Legendre operational matrix method based on collocation points is introduced to solve high order ordinary differential equations with some nonlinear terms arising in physics and mechanics. This technique transforms the nonlinear differential equation into a matrix equation with unknown Legendre coefficients via mixed conditions. This solution of this matrix equation yields the Legendre coefficients of the solution function. Thus, the approximate solution is obtained in terms of Legendre polynomials. Some test problems together with residual error estimation are given to show the usefulness and applicability of the method and the numerical results are compared.

Keywords: Legendre Polynomials and Series, Nonlinear Ordinary Differential Equation, Matrix Method, Residual Error.

Fizik ve Mekanikte Ortaya Çıkan Bazı Lineer Olmayan Diferansiyel Denklemler için Legendre Matris-Kollokasyon Yaklaşımı

Özet

Bu çalışmada, fizik ve mekanikte ortaya çıkan lineer olmayan bazı terimlere sahip yüksek mertebeden adi diferansiyel denklemlerin çözümü için, kollokasyon noktalarına dayanan operasyonel Legendre matris metodu takdim edilmiştir. Bu teknik, karışık koşullar sayesinde, bilinmeyen Legendre katsayıları ile lineer olmayan bir diferansiyel denkleme bir matris denkleme dönüştürür. Bu matris denkleminin çözümü, çözüm fonksiyonunun Legendre katsayılarını verir. Böylece, yaklaşık çözüm Legendre polinomları cinsinden elde edilir. Yöntemin faydasını ve uygulanabilirliğini göstermek için, rezidüel hata tahmini ile birlikte bazı test problemleri verilir ve nümerik sonuçlar kıyaslanır.

Anahtar Kelimeler: Legendre Polinomları ve Serisi, Lineer Olmayan Adi Diferansiyel Denklemler, Matris Metodu, Rezidüel Hata.

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1. Introduction

Orthogonal polynomials are extensively considered in many area of mathematics, sciences and engineering. One of these polynomials is the Legendre polynomials which are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1$. The mentioned polynomials play an important role in many branches such as mathematics, statics and other scientific (Kreyszig, 2013; El-Mikkawy et. al. 2005; Everitt et. al. 2002; Sezer and Gülsu, 2009; Gülsu et. al. 2009).

The solutions of nonlinear ordinary differential equations are frequently investigated by many researches (Yüksel et. al. 2011; Akyüz Daşcıoğlu and Çerdik Yaslan, 2011; Gürbüz and Sezer 2016). These equations are characterized by the presence of the nonlinear terms and have a great importance in explaining many different phenomena. Generally, the nonlinear differential equations have no analytical solution. Therefore, we need to numerical methods to obtain approximate solutions.

In this study, we develop a numerical method based on the matrix relations of Legendre polynomials and their derivatives by means of the matrix methods based on collocation points given by Sezer and co-workers (Sezer and Gülsu, 2009; Gülsu et. al. 2009; Yüksel et. al., 2011; Kürkçü et. al., 2017) and apply to the m-th order ordinary differential equation with first and second order nonlinear terms in the general form (Yüksel et. al., 2011; Akyüz Daşcıoğlu and Çerdik Yaslan, 2011)

$$\sum_{k=0}^m F_k(t)y^{(k)}(t) + \sum_{p=0}^2 \sum_{q=0}^p Q_{pq}(t)y^{(p)}(t)y^{(q)}(t) = g(t); \quad -1 \leq t \leq 1 \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^m (a_{kj}y^{(k)}(-1) + b_{kj}y^{(k)}(0) + c_{kj}y^{(k)}(1)) = \lambda_i; \quad i=0,1,\dots,m-1 \tag{2}$$

where $F_k(t), Q_{pq}(t)$ and $g(t)$ are analytic functions on interval $-1 \leq t \leq 1$, a_{kj}, b_{kj}, c_{kj} and λ_i are convenient real constants. The aim of this study is to get the solution of the problem (1)-(2) as the truncated Legendre series defined by

$$y(t) \cong y_N(t) = \sum_{n=0}^N a_n P_n(t); \quad -1 \leq t \leq 1 \tag{3}$$

where $P_n(t)$, $n = 0, 1, \dots, N$ denotes the Legendre polynomials given in (Kreyszig, 2013; Everitt et. al. 2002).

and a_n ($0 \leq n \leq N$) are unknown Legendre coefficients ($N \geq m$).

2. Material and Method

2.1. Matrix Relations associated with Legendre Polynomials and Series

In this section, we constitute the matrix forms of each term in Eq. (1). Firstly, we can write the approximate solution $y_N(t)$ defined by truncated Legendre series (3) in the matrix form

$$y(t) \cong y_N(t) = \mathbf{P}(t)\mathbf{A} \tag{4}$$

where $\mathbf{P}(t) = [P_0(t) \ P_1(t) \ \dots \ P_N(t)]$ and $\mathbf{A} = [a_0 \ a_1 \ \dots \ a_N]^T$. Using the recurrence formulas

$(2n + 1)P_n(t) = P'_{n+1}(t) - P'_{n-1}(t)$; $n \geq 1$ where $P'_0(t) = 0$ and $P'_1(t) = P_0(t)$ for $n = 0, 1, \dots, N$, we have recurrence relation between the matrix $\mathbf{P}(t)$ and its derivate $\mathbf{P}^{(k)}(t)$ in the form

$$\mathbf{P}^{(k)}(t) = \mathbf{P}(t)\mathbf{\Pi}^k; \quad k = 0, 1, \dots, m \tag{5}$$

$$\mathbf{\Pi} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 3 & 0 & 3 & \dots & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 & \dots & 0 & 5 \\ 0 & 0 & 0 & 0 & 7 & \dots & 7 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2N-3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2N-1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{(N+1) \times (N+1)}$$

$$\mathbf{\Pi}^0 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{(N+1) \times (N+1)}$$

From (3) and (5), we obtain the matrix relation as follows:

$$y^{(k)}(t) \cong y_N^{(k)}(t) = \mathbf{P}^{(k)}(t)\mathbf{A} = \mathbf{P}(t)\mathbf{\Pi}^k \mathbf{A}; \quad k = 0, 1, \dots, m \tag{6}$$

Using the Legendre explicit form and taking the matrix relation (6) for $n = 0, 1, \dots, N$, the matrix relations in the following are obtained (Gülsu et. al., 2009; Yüksel et. al., 2011):

$$\mathbf{P}(t) = \mathbf{X}(t)\mathbf{D}; \quad \mathbf{P}(x) = [P_0(t) \quad P_1(t) \quad \dots \quad P_N(t)], \quad \mathbf{X}(t) = [1 \quad t \quad \dots \quad t^N] \tag{7}$$

$$\begin{aligned} (y^{(0)}(t))^2 &= \mathbf{P}(t)\bar{\mathbf{P}}(t)\bar{\mathbf{A}} & y^{(1)}(t)y^{(0)}(t) &= \mathbf{P}(t)\mathbf{\Pi}\bar{\mathbf{P}}(t)\bar{\mathbf{A}} & \bar{\mathbf{A}} &= [a_0A \quad a_1A \quad \dots \quad a_NA]^T \\ (y^{(1)}(t))^2 &= \mathbf{P}(t)\mathbf{\Pi}\bar{\mathbf{P}}(t)\bar{\mathbf{\Pi}}\bar{\mathbf{A}} & y^{(2)}(t)y^{(1)}(t) &= \mathbf{P}(t)\mathbf{\Pi}^2\bar{\mathbf{P}}(t)\bar{\mathbf{\Pi}}\bar{\mathbf{A}} & y^{(2)}(t)y^{(0)}(t) &= \mathbf{P}(t)\mathbf{\Pi}^2\bar{\mathbf{P}}(t)\bar{\mathbf{A}} \\ (y^{(2)}(t))^2 &= \mathbf{P}(t)\mathbf{\Pi}^2\bar{\mathbf{P}}(t)\bar{\mathbf{\Pi}}^2\bar{\mathbf{A}} \end{aligned}$$

$$\bar{\mathbf{P}}(t) = \text{diag} [P(t) \quad P(t) \quad \dots \quad P(t)]_{(N+1)^2 \times (N+1)^2} \quad \bar{\mathbf{\Pi}}^2 = \text{diag} [\mathbf{\Pi}^2 \quad \mathbf{\Pi}^2 \quad \dots \quad \mathbf{\Pi}^2]_{(N+1)^2 \times (N+1)^2}$$

Here, the transient matrix \mathbf{D} is given in (Sezer and Gülsu, 2010; Gülsu et. al., 2009). Now, we define the collocation points for interval $[-1, 1]$ as

$$t_i = -1 + \frac{2}{N}i; \quad i = 0, 1, \dots, N \tag{8}$$

Substituting collocation points (8) into Eq. (1) and the matrix relations (7) yields

$$\sum_{k=0}^2 \mathbf{F}_k \mathbf{Y}^{(k)} + \sum_{p=0}^2 \sum_{q=0}^p \mathbf{Q}_{pq} \mathbf{Y}^{(p,q)} = \mathbf{G} \tag{9}$$

where $\mathbf{F}_k = \text{diag} [F_k(t_0) \quad F_k(t_1) \quad \dots \quad F_k(t_N)]$ and

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(t_0) \\ y^{(k)}(t_1) \\ \vdots \\ y^{(k)}(t_N) \end{bmatrix} \quad \mathbf{Y}^{(p,q)} = \begin{bmatrix} y^{(p)}(t_0)y^{(q)}(t_0) \\ y^{(p)}(t_1)y^{(q)}(t_1) \\ \vdots \\ y^{(p)}(t_N)y^{(q)}(t_N) \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_N) \end{bmatrix}$$

Putting the collocation points (8) into the matrix relations (6) and (7), we have the system in the following

$$y^{(k)}(t_i) \cong y_N^{(k)}(t_i) = \mathbf{P}(t_i)\mathbf{\Pi}^k \mathbf{A}; \quad i = 0, 1, \dots, N \text{ or } \mathbf{Y}^{(k)} = \mathbf{P}\mathbf{\Pi}^k \mathbf{A} \tag{10}$$

Thus, the nonlinear part of Eq. (9) can be written as

$$\sum_{p=0}^2 \sum_{q=0}^p \mathbf{Q}_{pq} \mathbf{Y}^{(p,q)} = \mathbf{Q}_{00} \mathbf{Y}^{(0,0)} + \mathbf{Q}_{10} \mathbf{Y}^{(1,0)} + \mathbf{Q}_{11} \mathbf{Y}^{(1,1)} + \mathbf{Q}_{20} \mathbf{Y}^{(2,0)} + \mathbf{Q}_{21} \mathbf{Y}^{(2,1)} + \mathbf{Q}_{22} \mathbf{Y}^{(2,2)}$$

Substituting the collocation points (8) into the mentioned matrix relations, then the following matrices are obtained:

$$\mathbf{Y}^{(0,0)} = \mathbf{P}_{0,0}^* \bar{\mathbf{A}}, \quad \mathbf{Y}^{(1,0)} = \mathbf{P}_{1,0}^* \bar{\mathbf{A}}, \quad \mathbf{Y}^{(1,1)} = \mathbf{P}_{1,1}^* \bar{\mathbf{A}}, \quad \mathbf{Y}^{(2,0)} = \mathbf{P}_{2,0}^* \bar{\mathbf{A}}, \quad \mathbf{Y}^{(2,1)} = \mathbf{P}_{2,1}^* \bar{\mathbf{A}}, \quad \mathbf{Y}^{(2,2)} = \mathbf{P}_{2,2}^* \bar{\mathbf{A}}$$

where

$$\mathbf{P}_{0,0}^* = \begin{bmatrix} P(t_0)\bar{P}(t_0) \\ P(t_1)\bar{P}(t_1) \\ \vdots \\ P(t_N)\bar{P}(t_N) \end{bmatrix}, \quad \mathbf{P}_{1,0}^* = \begin{bmatrix} P(t_0)\Pi\bar{P}(t_0) \\ P(t_1)\Pi\bar{P}(t_1) \\ \vdots \\ P(t_N)\Pi\bar{P}(t_N) \end{bmatrix}, \quad \mathbf{P}_{1,1}^* = \begin{bmatrix} P(t_0)\Pi\bar{P}(t_0)\bar{\Pi} \\ P(t_1)\Pi\bar{P}(t_1)\bar{\Pi} \\ \vdots \\ P(t_N)\Pi\bar{P}(t_N)\bar{\Pi} \end{bmatrix}, \quad \mathbf{P}_{2,0}^* = \begin{bmatrix} P(t_0)\Pi^2\bar{P}(t_0) \\ P(t_1)\Pi^2\bar{P}(t_1) \\ \vdots \\ P(t_N)\Pi^2\bar{P}(t_N) \end{bmatrix}$$

$$\mathbf{P}_{2,1}^* = \begin{bmatrix} P(t_0)\Pi^2\bar{P}(t_0)\bar{\Pi} \\ P(t_1)\Pi^2\bar{P}(t_1)\bar{\Pi} \\ \vdots \\ P(t_N)\Pi^2\bar{P}(t_N)\bar{\Pi} \end{bmatrix}, \quad \mathbf{P}_{2,2}^* = \begin{bmatrix} P(t_0)\Pi^2\bar{P}(t_0)\bar{\Pi}^2 \\ P(t_1)\Pi^2\bar{P}(t_1)\bar{\Pi}^2 \\ \vdots \\ P(t_N)\Pi^2\bar{P}(t_N)\bar{\Pi}^2 \end{bmatrix}$$

2.2. Legendre Operational Matrix Method

Substituting the matrix relations (10) and (11) into Eq. (9), we obtain the fundamental matrix equation

$$\sum_{k=0}^m \mathbf{F}_k \mathbf{P} \mathbf{\Pi}^k \mathbf{A} + \sum_{p=0}^2 \sum_{q=0}^p \mathbf{Q}_{pq} \mathbf{P}_{p,q}^* \bar{\mathbf{A}} = \mathbf{G} \quad \text{or} \quad \mathbf{W} \mathbf{A} + \mathbf{V} \bar{\mathbf{A}} = \mathbf{G} \tag{12}$$

where $\mathbf{W} = [w_{ij}] = \sum_{k=0}^m \mathbf{F}_k \mathbf{P} \mathbf{\Pi}^k$; $i, j = 0, 1, \dots, N$ and

$$\mathbf{V} = [v_{m^*,(n+1)^2}] = \sum_{p=0}^2 \sum_{q=0}^p \mathbf{Q}_{pq} \mathbf{P}_{p,q}^* \quad ; \quad m^* = 0, 1, \dots, N; \quad n = 0, 1, \dots, (N+1)^2 - 1$$

Then, the matrix equation (12) can be written in the augmented form $[\mathbf{W}; \mathbf{V} : \mathbf{G}]$ or clearly

$$[\mathbf{W}; \mathbf{V} : \mathbf{G}] = \begin{bmatrix} W_{00} & W_{01} & \dots & W_{0N} & : & V_{00} & V_{01} & \dots & V_{0,(N+1)^2-1} & : & g(t_0) \\ W_{10} & W_{11} & \dots & W_{1N} & : & V_{10} & V_{11} & \dots & V_{1,(N+1)^2-1} & : & g(t_1) \\ \vdots & \vdots & \ddots & \vdots & : & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ W_{N0} & W_{N1} & \dots & W_{NN} & : & V_{N0} & V_{N1} & \dots & V_{N,(N+1)^2-1} & : & g(t_N) \end{bmatrix} \tag{13}$$

Using the matrix relation (6), the fundamental matrix equation corresponding to the mixed conditions (2) is obtained as

$$\sum_{k=0}^{m-1} (a_{kj} \mathbf{P}(-1) + b_{kj} \mathbf{P}(0) + c_{kj} \mathbf{P}(1)) \mathbf{\Pi}^k \mathbf{A} = \lambda_j ; \quad j = 0, 1, \dots, N \quad \text{or} \quad \mathbf{U} \mathbf{A} + \mathbf{O}^* \bar{\mathbf{A}} = \lambda = [\mathbf{U}; \mathbf{O}^* : \lambda] \tag{14}$$

(Here, zero matrix indicated with \mathbf{O}^* is a $(N+1) \times (N+1)^2$ matrix) or more clearly

$$[\mathbf{U}; \mathbf{O}^* : \lambda] = \begin{bmatrix} u_{00} & u_{01} & \dots & u_{0N} & : & 0 & 0 & \dots & 0 & : & \lambda_0 \\ u_{10} & u_{11} & \dots & u_{1N} & : & 0 & 0 & \dots & 0 & : & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & : & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ u_{N0} & u_{N1} & \dots & u_{NN} & : & 0 & 0 & \dots & 0 & : & \lambda_{m-1} \end{bmatrix} \tag{15}$$

To calculate Legendre coefficients a_n ($n=0,1,\dots,N$) related to the approximate solution (3) of the problem (1)-(2), by replacing the m row matrices (14) by the last m rows (or any m rows) of the augmented matrix (16), we obtain the resulting matrix as follows;

$$[\tilde{W}; \tilde{V}; \tilde{G}] = \begin{bmatrix} W_{00} & W_{01} & \dots & W_{0N} & : & V_{00} & V_{01} & \dots & V_{0,(N+1)^2-1} & : & g(t_0) \\ W_{10} & W_{11} & \dots & W_{1N} & : & V_{10} & V_{11} & \dots & V_{1,(N+1)^2-1} & : & g(t_1) \\ \vdots & \vdots & \ddots & \vdots & : & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ W_{N-m,0} & W_{N-m,1} & \dots & W_{N-m,N} & : & V_{N-m,0} & V_{N-m,1} & \dots & V_{N-m,(N+1)^2-1} & : & g(t_{N-m}) \\ U_{00} & \vdots & \dots & \vdots & : & 0 & 0 & \dots & 0 & : & \lambda_0 \\ U_{10} & \vdots & \dots & \vdots & : & 0 & 0 & \dots & 0 & : & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & : & \vdots & \vdots & \ddots & \vdots & : & \vdots \\ U_{m-1,0} & U_{m-1,1} & \dots & U_{m-1,N} & : & 0 & 0 & \dots & 0 & : & \lambda_{m-1} \end{bmatrix} \quad (16)$$

From this nonlinear algebraic system, that is, from the matrix equation $\tilde{W}\mathbf{A} + \tilde{V}(\bar{\mathbf{A}}) = \tilde{G}$, the unknown Legendre coefficients a_n ($n=0,1,\dots,N$) are determined. Thus, the truncated Legendre series solution (3) is obtained from Eqs. (4) and (7).

3. Accuracy of Solutions and Residual Error Estimation

We consider the residual error estimation to check accuracy of the obtained solutions. Since the truncated Legendre series (1) is an approximate solution of Eq. (1), when the solution $y_N(x)$ and its derivatives are substituted in Eq. (1), then resulting equation must be satisfied approximately as follows:

$$R_N(t_l) = \sum_{k=0}^m F_k(t_l) y^{(k)}(t_l) + \sum_{p=0}^2 \sum_{q=0}^p Q_{pq}(t_l) y^{(p)}(t_l) y^{(q)}(t_l) - g(t_l) \cong 0 \quad (17)$$

for $t_l \in [a, b]$ ($l=0,1,2,\dots$) or $R_N(t_l) \leq 10^{-k_l}$ where k_l is any positive numbers. Predetermining $\max 10^{-k_l} = 10^{-k}$, then the truncation limit N is increased until the difference $R_N(t_l)$ at each of the points becomes smaller than predetermined 10^{-k} . Therefore, if $R_N(t_l) \rightarrow 0$ when N is sufficiently large enough, then the error decreases (Everitt et.al. 2002; Sezer and Gülsu, 2010).

The accuracy of the solution can be checked and the error can be estimated via the residual function $R_N(t)$ (Balci and Sezer, 2016; Oğuz and Sezer, 2015, Kürkçü et. al., 2016, Gürbüz and Sezer, 2017) and the mean value of the function $|R_N(t)|$ on the interval $[-1,1]$. Using the inequality

$$\left| \int_{-1}^1 R_N(t) dt \right| \leq \int_{-1}^1 |R_N(t)| dt \quad (18)$$

and the mean value theorem, the upper bound of the mean error \bar{R}_N is obtained as

$$R_N(c) \leq \frac{1}{2} \int_{-1}^1 |R_N(t)| dt = \bar{R}_N \quad (19)$$

4. Applications

Example 4.1. Consider the nonlinear differential equation

$$y'' - 2y' + y + y^2 - y''y' = 2 + 4e^t; \quad y(0) = 3, \quad y'(0) = 2$$

We assume that the problem has a Legendre polynomial solution in the form for $N = 2$

$$y_2(t) = \sum_{n=0}^2 a_n P_n(t); \quad -1 \leq t \leq 1$$

Performing the necessary procedures and considering the collocation points $P_0(t)=1, P_1(t)=t, P_2(t)=\frac{3}{2}t^2 - \frac{1}{2}$, then, we find the fundamental matrix equation in the following:

$$(F_0 P \Pi^0 + F_1 P \Pi + F_2 P \Pi^2) A + (Q_{00} P^*_{0,0} + Q_{21} P^*_{2,1}) \bar{A} = G$$

where

$$F_0 P \Pi^0 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}, F_1 P \Pi = \begin{bmatrix} 0 & -2 & 6 \\ 0 & -2 & 0 \\ 0 & -2 & -6 \end{bmatrix}, F_2 P \Pi^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}, G = \begin{bmatrix} g(-1) \\ g(0) \\ g(1) \end{bmatrix} = \begin{bmatrix} 2+4e^{-1} \\ 6 \\ 2+4e \end{bmatrix}$$

$$Q_{00} P^*_{0,0} + Q_{21} P^*_{2,1} = \begin{bmatrix} 1 & -1 & 1 & 1 & -1 & 1 & -2 & 2 & -2 \\ 1 & 0 & -1/2 & 0 & 0 & 0 & 1/2 & -3 & 1/4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -8 \end{bmatrix}$$

$$[W; \tilde{V} : \tilde{G}] = \begin{bmatrix} 1 & -3 & 10 & : & 1 & -1 & 1 & 1 & -1 & 1 & -2 & 2 & -2 & : & 2+4e^{-1} \\ 1 & 0 & -1/2 & : & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & 2 \end{bmatrix}$$

Then, the solution of the problem is obtained as the first three term in Maclaurin expansion of the exact solution $y = 1 + 2e^t$.

Table 1. Numerical results for Example 4.1.

t_i	Exact Solution	$y(t)$ for $N = 2$	Absolute Errors
0.0	3.0	3.0	0
0.2	3.4428	3.44	0.0028
0.4	3.9836	3.96	0.2364
0.6	4.6442	4.56	0.8423
0.8	5.4510	5.24	0.2110
1.0	6.4365	6.0	0.4365

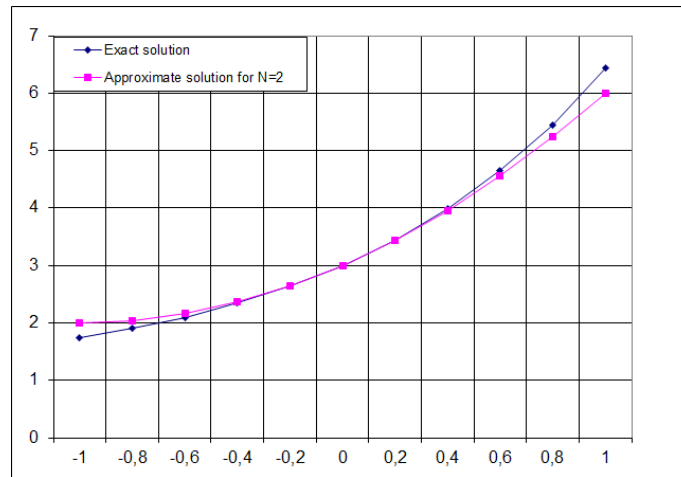


Figure 1. The comparison of numerical and exact solutions of Example 4.1 for $N = 2$.

Example 4.2. [9,13] Consider Bratu-type problem

$$y''(x) - 2e^{y(x)} = 0; 0 < x < 1$$

with $y(0) = 0$ and $y'(0) = 0$. The exact solution of this problem is $y(x) = -2\ln(\cos x)$. Expanding $e^{y(x)}$, we can write this equation [9] as

$$y''(x) - 2y(x) - y^2(x) - 2 = 0.$$

Using Legendre matrix-collocation method with truncation limit $N = 4$ and 5 , we solve this problem. The following solutions are obtained as

$$y_4(x) = -1.04 \times 10^{-17} + 0.965317x^2 - 0.0160093x^3 + 0.196559x^4$$

$$y_5(x) = 2.08 \times 10^{-17} + 0.971444x^2 + 0.00924656x^3 + 0.146425x^4 + 0.0314368x^5$$

Table 2. Comparison of the absolute errors for different N of Example 4.2.

x_i	N=4	N=5	x_i	N=4	N=5
0.1	3.60e-04	2.78e-04	0.6	1.44e-02	1.08e-02
0.2	1.47e-03	1.09e-03	0.7	2.15e-02	1.66e-02
0.3	3.34e-03	2.44e-03	0.8	3.27e-02	2.60e-02
0.4	6.00e-03	4.36e-03	0.9	5.17e-02	4.26e-02
0.5	9.56e-03	7.02e-03	-	-	-

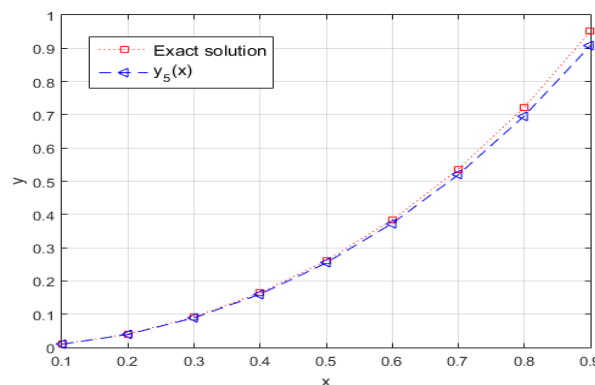


Figure 2. Comparison of the Legendre polynomial solution $y_5(x)$ and exact solution for Example 4.2.

As seen from Fig. 2 and Table 2, the Legendre polynomial solutions $y_4(x)$ and $y_5(x)$ coincide with the exact solution. Also, the absolute error decreases as N is increased. The upper bounds of the mean errors \bar{R}_4 and \bar{R}_5 are obtained as $2.07e-01$ and $1.82e-01$, respectively. It is clearly seen that the mean errors are consistent with results in tables and figures.

5. Conclusions and Recommendations

In this study, we propose a matrix-collocation method based on Legendre polynomials to obtain the approximate solutions of nonlinear ordinary differential equations with quadratic terms. Besides, the error analysis is introduced to indicate the accuracy of the method. The present method and its error analysis are applied on some examples. Comparison of the obtained results with exact solutions displays that the present method is impressive and suitable. Also, the method can be extended on different type of mathematical models together with some modifications.

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Authors' contributions: All authors read and approved the final manuscript.

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