MANUFACTURING & RE-MANUFACTURING INVENTORY MODELS WITH IMPRECISE AMOUNT OF DEFECTIVE ITEMS

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ETHICAL DECLARATION

I hereby declare that I am the sole author of this thesis and that I have conducted my work in accordance with academic rules and ethical behaviour at every stage from the planning of the thesis to its defence. I confirm that I have cited all ideas, information and findings that are not specific to my study, as required by the code of ethical behaviour, and that all statements not cited are my own.

ABSTRACT

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Ph.D. Program in Applied Mathematics and Statistics

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In this thesis, we developed an integrated inventory model in supply chain environment for a single product with complete backordering, partial backordering or complete lost sales. The production process is not totally reliable. Therefore, the system generates an imprecise number of defective items within the production cycle. Moreover, only a random proportion of these are reworkable. To see their effect in the model, four case combinations are studied, where both defective rate and rework rate are deterministic and stochastic. As one of the main extensions, two investment functions are considered to improve the supplier's production process quality and reworking power. Additionally, customer time sensitivity term is investigated for partial backordering. The goal was to minimise the total combined annual costs of

the integrated system. The model defines the optimal reorder point and order quantity based on the expected total annual cost, and a solution algorithm was presented for solving the model. With a numerical example, it was shown that the proposed integrated model provides reduced cost in comparison to a model that considers only buyer's decision under stochastic demand with partial backordering. That is, cooperation between buyer and supplier is beneficial when there is stochastic defective and rework rate, time-sensitive customer behaviour, and investment in production and rework. In addition, with further sensitivity analysis one can see that supplier investment decreases the production rate of defective items and increases rework power. Therefore, smaller lot sizes are produced.

Keywords: inventory optimisation, supply chain optimisation, reliability, optimisation of inventory.

ÖZET

KUSURLU ÜRÜN MİKTARININ BELİRSİZ OLDUĞU DURUMLAR ICIN
İMALAT VE YENİDEN İMALAT ENVANTER MODELLERİ

Yüce, Gizem

Uygulamalı Matematik ve İstatistik Doktora Programı

Tez Danışmanı: Prof. Dr. Gözde Yazgı TÜTÜNCÜ

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Bu tezde, yok satma, gecikmiş üretim ve kısmi gecikmiş üretim ile tek bir ürün için tedarik zinciri ortamında entegre envanter modeli geliştirilmistir. Üretim süreci tamamen güvenilir olmadığından, döngü sırasında belirsiz sayıda hatalı ürün üretilmektedir. Ayrıca, bunların sadece rastgele bir kısmı yeniden işlenebilir. Modelin hassasiyetini görmek için, hem hatalı oranı hem de yeniden işleme oranı belirli ve rastgele olan dört durum kombinasyonu incelenmiştir. Ana katkılardan biri olarak, tedarikçinin üretim süreci kalitesini ve yeniden işleme gücünü iyileştirmek için iki yatırım fonksiyonu dikkate alınmıştır. Ayrıca, kısmi gecikmiş üretim için müşteri zaman hassasiyeti terimi incelenmiştir. Amaç, entegre sistem için toplam yıllık maliyetlerini en aza indirmektir. Optimal sipariş miktarı ve yeniden sipariş noktası beklenen toplam yıllık maliyetten belirlenmiş ve model çözümü için bir çözüm algoritması önerilmiştir. Bir sayısal örnek ile, önerilen yeni entegre modelin, stokastik hatalı ve yeniden işleme oranı, zaman hassasiyetli müşteri davranışı ve üretim ve yeniden işleme yatırımı durumlarında alıcı ve tedarikçi arasındaki işbirliğine fayda sağladığı gösterilmiştir. Ek olarak, daha fazla hassasiyet analizi ile tedarikçinin yatırım yapması, hatalı öğelerin üretim verimlilik oranını azalttıgı ve yeniden işleme gücünü arttırdıgı gözlemlenmistir. Bu nedenle, daha küçük lotlar üretilmesi önerilir.

Anahtar Kelimeler: envanter optimizasyonu, tedarik zinciri optimizasyonu, güvenilir-lik.

This thesis work is dedicated to my family...

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CHAPTER 1: INTRODUCTION

Inventory is one of the key factors that must be overseen in the supply chain. It incorporates an immense range of materials that is being produced, sold, stored during business. In general, planning, storing, and moving of inventory have its own financial value. One of the main aims for each firm is to keep the level of inventory in the supply chain at certain level such that it lowers significant costs yet protects a company (customer) from stock-out. For the case of managing inventories, one needs to decide how much it should be ordered for replenishment and when the inventory should be replenished. The main goal is to acquire the lowest possible total cost.

Mathematically, inventory model is a tool for determining the optimum level of inventories that business should provide in a production process, keep up with frequency of ordering, make a decision on quantity of goods to be stored, tracking flow of supply for continuous service to customers without any delay in delivery. There are certain types of costs for inventory such as purchase cost, backorder cost, holding cost and ordering cost. Ordering cost is total expenses of processing an order, no matter how much the order quantity is. Holding cost defines the sum of the costs from insurance, security, taxes, warehousing and other related expenses. Backorder cost emerges when there is a stock-out case under the demand of an item and purchase cost is the actual price for the items.

The traditional inventory models, including *the economic order quantity (EOQ), the economic production lot size,* and *EOQ with planned shortages*, operate under the assumption that demand and other variables are constant and predictable. In other words, these models assume that the demand for a product is known with certainty and does not vary over time, and other parameters such as setup costs, holding costs, and production rates remain unchanged. The deterministic inventory model known as the economic order quantity (EOQ) is considered one of the most fundamental models. In this case, it is assumed that there is one product and the demand is known&constant during the year. In general, there are two costs; the cost of ordering and receiving the product, and holding cost as having the product in inventory for a year. Here, unit cost does not play a role, so price is same for any quantity. Also, shortages are not allowed. The objective is minimising the total costs of inventory that is summation of annual holding cost and annual ordering cost.

To create more accurate and practical inventory models, we can consider various factors such as demand, lead time, products, capacity, and service level. For example, we can model demand as constant and predictable, or as a random variable with a probability distribution that reflects the uncertainty of future demand. We can also consider lead time as either zero, fixed, or stochastic to account for the variability in the time it takes to receive goods after placing an order. Moreover, we can model inventory systems for single or multiple products, taking into account different lead times, demand rates, and costs. Additionally, we can consider capacity constraints, such as order or inventory limits, or assume no capacity constraints in the model. Lastly, we can incorporate service levels into the inventory model, such as meeting all demand (no shortages) or allowing for shortages to occur. These different factors can create various inventory models that provide more accurate and practical solutions to inventory management. Additionally, in real life, production process might have imperfect quality of items, therefore it is unrealistic to assume that every item is produced with good quality. In contrast to classical economic order/production quantity (EOQ/EPQ) models, real-world situations may involve defective items due to imperfections in the production process, spoilage during transit, or other factors. It is essential to consider the impact of defective items on customer satisfaction levels as their impact cannot be ignored. In addition, the presence of a random number of defective items can reduce the original order size of the buyer (customer), leading to the possibility of a stock-out situation. There are three possible responses for stock-out case; (partial or complete) backordering, substitution, and (partial or complete) lost sales. Backordering is a separate order which is requested by customer and is prepared as soon as the product is available by supplier. Substitution happens when there is another product that is acceptable instead of the one is not available and lost sales occur when customer invalidates the order.

Defective items can be collected and remanufactured, then considered as perfect items, or disposed as scrap, or priced and sold as lower quality products. It is possible to combine those steps or examine them individually with corresponding system costs. Also, it is highly possible to repair only proportional amount of defective items, so in general, the idea of working on defective items requires separate and attentive consideration. It is important to decide when to switch the system from production to remanufacturing since managing both at the same time may not be possible.

In production, it is inevitable to have defective items therefore process quality becomes more of an issue. It can help to supplier for producing smaller batches with perfect quality items. Investment in process quality can have significant effect on reduction the number of defective items produced and its corresponding costs. Some examples of investments could be purchasing new equipment, enhancing machine maintenance, increasing repair frequency, and providing training to employees. Additionally when there is reworking process, we can include investing on machinery & workers to have higher efficiency. Since production and reworking are two separate operations, handling them can require different qualities.

When shortage happens for various reasons, first problem is customer behaviour. That is, if they are willing to wait before receiving their items so shortage will be backordered or not. One of the main questions is about the length of waiting period, because some customers may agree to wait for short period of time. Hence, backlogged demand is strongly dependent to customer time-sensitivity.

In this work, we aim to decide which inventory model would be the best under stochastic demand and imperfect production when shortage is allowed. It is essential to find a possible expression for case such as lost sales, backorder or mixture of them in the mathematical model with reasonable assumptions. How we can manage the production phase when there is randomness in defective items and when remanufacturing should be started are another step of this study. We will propose the most favourable strategies under the fact that we can repair only proportional amount of defective items and two different form of investments. Considering various possible scenarios such as demand variability, defective production, reworking and time-sensitive customers, optimal lot size and reorder point with total expected cost will be examined in detail.

The rest of the thesis is organised as follows. The next section is devoted to the literature review. Independent model papers for EOQ and EPQ with integrated model papers are discussed. In Chapter 2, general models and basic definitions are introduced. Related to the study, continuous review models are discussed with different conditions. The first contribution of this work is proposed in Chapter 3. Independent models for buyer and supplier are built. Starting to build buyer's models for deterministic and stochastic demand, we also analyse lost-sales, partial backlogging and complete backlogging. For supplier, two parameters are studied in the model. Their stochasticity and its effect on the cost function is proposed. Second section is devoted to building the integrated model and different distribution cases for two parameters. In Chapter 4, a example is presented with sensitivity analysis for buyer's model and case figures for supplier's model. Then real life scenarios is shown. Then overall analysis is done to compare the optimal values.

1.1. Motivation and Objective

Modelling of an inventory system is one of the most important tasks in supplier-buyer chain and more realistic models enable us to interpret future of the production systems. While there are common models for certain scenarios, considering additional assumptions can make them more realistic. Of course, for the beginning oversimplified assumptions such as all products have perfect quality or no shortage make the model easier to deal with mathematically but may not match up with the real life problems. That is why modification of existing models -with random defective proportionality, remanufacturing, shortage, or stochastic demand consideration- is fundamental. With different review process, adding those creates more reasonable models. Here, the essential point is to achieve sufficient tractability and acceptable realism in the formulated models.

The purpose of this thesis is building an integrated inventory model for single buyer single supplier chain with stochastic demand, defective items and reworking when shortage at the buyer is completely lost/partially backordered/completely backordered. Although supplier reworks the defective items, buyer may still receive imperfect items (not necessarily defective) and sends those to the outlet shops. While the parameter values and initial conditions determine the result for deterministic models, with stochasticity there is inherent randomness which have same parameter values and initial conditions with different results. Therefore, assuming stochastic demand on buyer's side, can help modelling the supply-chain flow more pragmatically. Moreover, due to the nature of stochastic demand, imperfect products or simply not handling inventory effectively, shortage may occur. When it happens, sales can be completely or partially backordered or lost. Among those cases, the most realistic one is considered as partial backordering since some of the customers are time-sensitive. That is, part of those are willing to wait during shortage while others are not. To model that behaviour, it is necessary to include a parameter for length of shortage, lead-time demand, reorder point and duration for customer waiting period. Linear function for the decline in backordering may be good start however considering exponential structure makes it more realistic. Also, having backorder parameter depends on reorder level gives a robust insight for analysis of their relationship.

At the end, three real life scenarios will be studied for continuous review inventory models with defective items. First scenario includes proportional defective items as random variable with full reworking rate. After remanufacturing all imperfect items are considered as brand new. Shortage is partially backordered. The second one is about random defective items and not every defective item can be reworkable. This time reworking process on them is also not perfect, that is, after remanufacturing there will be brand new and lower quality items. Demand is random variable and shortage is partially backordered. In the third scenario, defective rate is deterministic with random reworking rate and after remanufacturing, items will be all in lower quality. With all those scenarios, our goal is to minimise the total expected cost of the system, therefore we will compare integrated model with vendor's independent cost and buyer's independent cost.

In the existing literature, the metrics we just mentioned such as stochastic demand, rework on defective items, investment and customer behaviour parameters are not considered in one study simultaneously. Despite being neglected too often, customer behaviour is specifically substantial since it is one of the core pieces for supply-chain. Moreover, considerable efficiency on production and reworking requires rational investment to the process. Besides its financial value, less scrap at the end of production/rework also means being more environmentally-kind. Therefore, in this study we aimed to fill the gap in the existing literature and to extent it by examining an integrated, single-supplier, single-buyer inventory model with stochastic demand, investment for production and rework quality, and time-sensitive customer behaviour. Overall, the goal is bridging all the existing models with addition of these components and finding the most updated inventory model. Table 1 summarises a comparison between the proposed model and several other relevant studies conducted in the past ten years. It is clear that realistic case combinations of key parameters has not been proposed.

Table 1. A comparison of the present model with few related works in the literature

Paper	Model	Demand	Production	Investment	Defective rate	Rework	Time-sensitive customer
Taleizadeh et al. (2015)	EPQ	Linear	Imperfect	No	Stochastic	Yes	No
Hsu and Hsu (2016)	EPQ	Constant	Imperfect	No	Both	No	No
Al-Salamah (2019)	EPQ	Constant	Imperfect	No	Deterministic	Yes	No
Hsu and Hsu (2013a)	EOQ	Constant	Imperfect	No	Stochastic	No	No
Sicilia et al. (2012)	EOQ	Constant	Perfect	No	No	No	Yes
Hsu and Hsu (2013b)	Integrated	Constant	Imperfect	No	Stochastic	No	No
Sarkar et al. (2017)	Integrated	Constant	Imperfect	No	Deterministic	Yes	No
Gutgutia and Jha (2018)	Integrated	Unknown	Imperfect	No	Stochastic	No	No
Dey (2019)	Integrated	Stochastic	Imperfect	Yes	Deterministic	No	No
Present model	Integrated	Stochastic	Imperfect	Yes	Stochastic	Yes	Yes

1.2. Literature Review

To understand the flow in supply chain management, there have been various models proposed in the literature with different perspective and requirements. Economic Order Quantity (EOQ) has been one substantial tool so the buyer knows when to order to keep the inventory at certain level for reduced overall cost. On the other hand, in production, Economic Production Quantity (EPQ) models are used to regulate production so supplier can meet the continuous demand. EOQ and EPQ models share several assumptions, such as the nature of the demand, item quality and inspection. When supplier and buyer are linked, joint optimisation of the production and inventory can be more efficient in many ways.

1.2.1. Papers for EOQ models

EOQ models have been studied with numerous assumptions in the literature for the last decade (Pentico and Drake (2011)). As a start point, one of the assumptions that basic EOQ model has is receiving perfect quality items. However, production is not always perfect, that is, reliability of the production process and quality of received items are connected. As a result, it is possible that the manufacturing process will degrade and produce defective or low-quality products. Porteus (1986) introduced a simple EOQ model with defective items. He showed that there is a strong relationship between quality and lot size, therefore investment in process was studied. With the quality investment, it is possible to reduce the out-of-control probability and setup cost. Another point in his paper is explicit optimal solutions could be derived thanks to logarithmic form of investment cost function. Salameh and Jaber (2000) studied a model with random proportion of imperfect quality items -not necessarily defective- when EOQ/EPQ formulae is used. They considered the lower quality items to be sold at the end of screening process as single batch and there is error-free complete screening process. Their study showed that with increased amount of imperfect items, lot size also increased. Among the works that modified or extended the paper of Salameh and Jaber (2000), Rezaei (2005) considered shortage problem with complete back-ordering due to defective items in the classical EOQ/EPQ model. Yu et al.

(2005) extended their model with deterioration and partial back-ordering. Since not all customers are willing to wait during shortage, they define lost sales with impatient customers. Additionally, a lower bound on the back-ordered ratio is obtained for concave profit function. Wee et al. (2007) examined the model in Salameh and Jaber (2000) for the case of shortage with complete back-ordering in each cycle. Eroglu and Ozdemir (2007) proposed an inventory model that considers a random defective rate and allows for shortages, which are completely back-ordered.. They also assumed a screening process to separate good and defective items. Cheikhrouhou et al. (2018) presented an inventory model with sample inspection that detects defective lots. The goal is to minimise the system costs with optimal sample size and optimal order size. The defective items are assumed as random variable which may follow standard uniform distribution. After receiving shipment, sampling process starts. Then a quality inspection is applied to n items in each lot. The inspection is also imperfect so there are Type-I and Type-II errors. The demand rate is assumed constant and uniform, and shortage is not allowed. There are two cases discussed; any defective item is promptly returned to the supplier and they are kept till next shipment. Finally, they showed that first case was more profitable. Sharifi et al. (2015) examined a model that expands upon previous literature on the economic order quantity (EOQ) model with imperfect items and partial backordering, incorporating screening errors. The aim of this model is to optimise profits by determining the optimal order size and the maximum number of backordered units. The few assumptions are; there is instantaneous replenishment, fixed proportion of defective items, and items of inferior quality are offered at a discounted price and there are Type-I and Type-II errors according to misclassification. The proposed model is solved analytically and they showed that it is concave. Therefore there are unique values of optimal order size and the maximum number of backorder units which maximise the expected profit. Annadurai and Uthayakumar (2010) presented a continuous review inventory model with defective items and partial backorders. The authors initially made the assumption that lead time demand follows a normal distribution. However, they later relaxed this assumption by utilising a minmax distribution-free approach. The decision variables are order quantity, setup cost, reorder point and length of lead time for an inventory model. The main few assumptions are; the number of defective items follows a binomial distribution, any defective items are identified and returned to the supplier during the next shipment after complete and error-free inspection. The objective is to analyse the effect of defective items with a mixture of backorders by reducing the setup cost to minimise the order quantity, reorder point, and lead time. Skouri et al. (2014) studied an economic order quantity model with backorders when a fraction of all supply is imperfect. They considered "all or none" inspection policy, so if the batch is below quality standards it is assumed as defective then rejected. Two-dimensional constrained optimisation problem was presented and solved. Corresponding optimal cost, optimal planned order quantity and backorder values are obtained in closed form. Öztürk et al. (2015) investigated the EOQ model for defective items and rework option. Here, demand, rework rate and inspection rate is constant and known with both the rework rate and inspection rate are higher than the demand. Full inspection is processed and defective items include scrap, imperfect quality and reworkable items of proportions. Shortages are allowed and backordered. Reworking starts right after the inspection process and results in scrap and good items. The model is solved analytically and the optimum order quantity and the optimum backorder quantity. Rezaei (2016) defined an inspection plans for imperfect items by using the economic order quantity model with three different possible scenarios; full inspection, rejection, and no inspection. These plans are determined by the outcome of sampling inspection plans that is the imperfect rate may be either below the minimum limit, or between upper and lower limit, or above a maximum limit. The goal is to formulate the total revenue and EOQ model of those three cases. In this study, Hsu and Hsu (2013a) proposed a model when there is inspection errors, sales returns, imperfect quality, and shortage backordering. The aim is to maximise the total profit per cycle. The main assumptions are; constant annual demand rate, imperfect production process, imperfect screening process at buyer's side, shortage. They studied on two models as the one with if shortages are allowed and the one with no shortages. The closed form solution is obtained for the optimal order size, the optimal order point and the maximum backorder units.

1.2.2. Papers for EPQ models

In production scheduling, EPQ model with defective items have been studied with several different assumptions for the last decade Pentico and Drake (2011). In a very early study, Rosenblatt and Lee (1986) examined an economic production quantity model with imperfect production where defective rate is defined as linear, exponential, and multi-state as a function of setup cost. When reworking is possible, utilisation of production process can be more effective. Hayek and Salameh (2001) studied an EPQ model with the effect of constant rework rate on random defective proportion where shortage is allowed. Reworking process is assumed as perfect and defective rate equals to rework rate. They obtained the optimal production quantity and maximum backorder level allowed in a production cycle that minimise the total cost. Chiu (2003)'s research highlighted the influence of reworking defective items on the EPQ model with backordering. In this study, not all defective items are restorable, therefore scrap items are considered with its cost. The renewal reward theorem is utilised to examine cycle length when it is variable. Optimal lot size and maximal backorder level are obtained to minimise the total cost under allowed backordering. Tsai (2009) presented an economic production quantity model for imperfect production process with addition of learning effect to determine the optimal production quantity. This effect helps to produce a single item in n batches at an increasing rate. A random variable is used to represent the percentage of defective items and the optimal lot size is derived from solution procedure. Revisiting Chiu's paper, Taleizadeh et al. (2015) considered an EPQ inventory model with rework process through multiple shipments policy with addition of pricing. The aim is to determine the selling price, lot size, and number of shipments that will yield maximum profit. The demand is assumed price-sensitive, and the production is imperfect so there are defective items with certain ratio. The reworking process is also assumed as imperfect that means there is scrapping rate. By showing that the average benefit function is concave, they proved the existence of an replenishment lot size, an optimal price, and number of shipments. Hsu and Hsu (2016) developed a model for optimal production lot size and backorder quantity with defective items. Several scenarios such as randomness on defective rate and drawn time of defective items from inventory are discussed. Al-Salamah (2019) examined economic production quantity models with imperfect production process and flexible rework rate and two types of rework process as asynchronous and synchronous. The synchronous rework provides immediate reworking on defective items while asynchronous keeps them until the completion of manufacturing of lot. With the flexible rework rate, there are two possible cases; either the rework rate is higher than the demand rate, or the demand rate is higher than the rework rate. The goal is to minimise the associated cost for each model and obtain the optimal lot sizes and backorders for different assumptions for rework rate and rework process. It is assumed that the demand rate and production rate are constant and known, and the proportion of defective items is also known. There is screening process which classifies items as either defective or nondefective, the rework process is perfect, and backorders are allowed. Example analysis shows that lot size and backorder are sensitive to different assumptions for rework rate and rework process. Chiu et al. (2011), combined reworking process and multiple shipments for an imperfect economic manufacturing quantity (EMQ) model. The classical EMQ model assumes that all items produced are of perfect quality and there is continuous issuing policy. On the other hand, in real vendor-supplier environment, production of random defective items is inevitable. The integrated EMQ model here, includes a random defective rate during production process, production setup cost, reworking with scrap rate, fixed and variable transportation costs, and inventory holding cost for manufacturer and customer. In addition, the reworking of defective items takes place after the regular production process in each cycle. Once the quality of the entire lot has been verified at the end of the rework process, the items can be shipped to customers. With mathematical modelling of problem, they formulated cost functions and optimal replenishment lot size. Moreover, due to success of repairing process, specific cases are shown for cost functions. Chiu et al. (2008) studied on an expediting decisive rule about either rework the repairable defective items or not in economic production quantity model with proportional repairing success and no backlogging as assumptions. EPQ model is used to conclude the optimal production size when when the company produce the items internally instead of obtaining from a supplier.

According to the decision, reworking the defective items starts right after regular process ends. It is possible that the rework process is imperfect, which can result in some defective items failing to be reworked and becoming scrap items. The proposed mathematical system is used for the exact critical point of repair cost and assistance for determining whether it is beneficial to rework the imperfect items. Krishnamoorthi and Panayappan (2012) presented an imperfect quality inventory model and defect sales returns which determines an optimal production lot size by using the economic production quantity model. The minimisation of the total cost derived with optimal production lot size for a single type of product. The assumption is that any defective items produced can be reworked, and the outcome of the reworking process is either a good item or a scrap item. Shortages are backordered and met by the next possible replenishment. In this study, inspection cost is ignored and sensitivity analysis is observed for various system performance measures. As a different perspective, Ritha and Priya (2016) examined the costs of transporting materials, the energy used in the production process, and the cost of waste generated by defective items during the rework process. They used the extended form of the EPQ model with defective items that are reworkable. The model is created with assumptions such as no shortages are allowed, only non-defective items are used to meet the demand during production, and reworking of defective items occurs at a fixed rate. As a side note, screening occurs after the production period has ended and after identifying the defective items are reworked before they are returned to the inventory. Mathematical formulas for calculating the ideal order quantities and overall profit per unit time are presented Khanna et al. (2017) studied a finite production model that observes the imperfect environment including the concept of inspection errors and imperfect rework process. The definition of problem is finding the optimum production quantity by maximising the difference between total revenues and costs per unit time. The cost components are production cost, inspection cost, Type-I&II error costs, inventory holding cost, rework cost, and disposal cost. Demand rate is assumed as constant, uniform and deterministic and reworking starts after the end of production process. Additionally, a portion of the defective items is sent for rework, while the remaining defective items are disposed of at a lower cost. Unlike their most recent paper Khanna et al. (2017), AIP (2016) studied on the problem about finding the optimal production and backorder quantity. In addition to the assumptions made in their previous study, this study allows for shortages, which are entirely backlogged. Moreover, they considered a shortage cost in their model this time.

Sarkar et al. (2014) developed inventory models based on three different distribution density functions: beta, triangular, and uniform, all of which incorporate a variable defective production rate. Demand and production rate are assumed constant, no shortage is allowed, there is full screening process with negligible cost, the proportion of defective products is random variable and it follows three distribution density functions. After rework process, all items are assumed to have perfect quality, as a backorder cost linear and fixed backorder cost is considered. They derived closed-form of solutions of the models. As a goal, it was shown that minimum cost was obtained from triangular distribution. Mukhopadhyay and Goswami (2014) developed an imperfect EPQ inventory model with rework and learning process. The goal is to reduce the total production inventory cost. A constant demand rate is assumed, and the production process is considered to be imperfect. With screening process there are perfect, imperfect, and defective are obtained, and then defective items are reworked, defective items are sold at a discounted price, while perfect items are sold at full price. The fraction of non-reworkable imperfect items are uniformly distributed random variables. There is learning process from experience that concludes less setup time and cost, and no shortages are allowed. They specified the setup cost as a function of production run length for the case of learning. Total cost is shown as convex function so there is optimal value of production lot size.

1.2.3. Papers for integrated models

The supply-chain coordination for inventory management is one of the many tasks in competitive markets. As one of the early studies, Goyal (1977) studied the integrated optimisation problem for single buyer and single vendor where vendor's production rate is infinite. Later, Goyal and Nebebe (2000) examined the economic production model and shipment policy for supplier-buyer chain to obtain minimum total joint cost. Wu and Ouyang (2003) derived an algebraic approach to single vendor single buyer inventory system with shortage instead of using differential calculus. they showed that optimal integrated cost is lower when there is shortage. Hsu and Hsu (2013b) conducted research on a model that integrates production inventory with imperfect quality items and planned backorders, involving a single vendor and a single buyer. The aim was to minimise the total joint annual cost. The main assumptions are; constant and known demand rate, percentage of defective items has a probability density function, error-free screening process at buyer's side, and complete backordering. The integrated model for expected annual joint cost is derived and the optimal solution is provided. Since there are independent models given, example showed joint model has reduced cost compared to individual models. In this paper, Sarkar et al. (2017) developed an integrated inventory model with defective items and two-stage inspection. First-stage inspection is for detection of defective items and second-stage is about ensuring reworked products have perfect quality. The goal is to reduce the total system cost. There is single vendor and single buyer for single item with production of defective items. The model follows make-to-order policy, fixed setup cost is assumed in the model, and no shortage is allowed. The variable transportation cost is used in a form of power function in the model and it is solved analytically. Gutgutia and Jha (2018) studied an integrated inventory model with single vendor single buyer supply chain using service level constraint (SLC) approach that aims to model the stock-out case in an inventory system. Also, lead time reduction and random defective items are also considered for minimising the total expected cost of the system. They allowed partial backordering and lost sales for stock-out situation. They derived closed form expressions for the optimal order quantity, safety factor, and shipment frequency. Hsien-Jen (2013) studied an integrated single supplier-buyer inventory system with stochastic defective items under continuous reviewing. The study considers the lead time demand to be known only for the first two moments and unknown distribution afterwards. Therefore, a minmax distribution-free approach is employed to determine the optimal order quantity, reorder point, lead time, and number of lots delivered. The primary objective of this approach is to minimise the expected total system cost. They considered the possibility of crashing the components of lead time one at a time at a certain cost. They also assumed that defective items are random variables that follow a binomial distribution, and that these items are returned to the vendor upon delivery of the next lot. Furthermore, they allowed for shortages and partial backorders in their model. Here, both vendor and buyer's expected average total cost per unit time are calculated individually then jointly. The parameters that have effects on decision making process are also studied and as a conclusion, integrated model was decided more beneficial for both sides. Kang et al. (2018) studied an inventory model by including safety stock with imperfect production. The mathematical model is optimised for lot size, planned backorder quantity, and a safety stock for minimising the average cost of an imperfect production setup. The few main assumptions are; it is a single stage manufacturing setup for a single item, production is not perfect so there are imperfect items which are reworked, there are additional units produced as safety stock with their associated cost, shortages are allowed and backordered, demand and production rates are known and there is inspection process but not for reworked products as they are considered good without inspection process. In this study, Lopes (2018) constructed an integrated model for production system with imperfect inspection that is fractional of items. Additionally, there are defective items from inspection process which are reworked with a fixed cost. The goal is to minimise the total expected cost per item. As one of decision variables, there is buffer stock for a demand when preventive maintenance is completed. The few main assumptions are; defective items are detected by inspection process, since inspection is imperfect there are Type-I and Type-II errors. The model assumes that the probability of defective items is lower when the system is in-control than when it is out-of-control. Additionally, the producer offers a free minimal repair warranty. The holistic approach to a joint optimisation model is given to see the relationship between different elements and system productivity and decrease in cost. Moshrefi and Jokar (2012) developed an integrated inventory model that has stock-dependent demand, shortages, and a function for customer impatience about backorder. The objective is to minimise the overall cost by finding the optimal inventory cycles that balance inventory, ordering, and shortage costs for both the supplier and the customer in the supply chain. The main assumptions in this paper are the function for the shortage period that shows fewer customers are willing to wait until replenishment, and another function for none-shortage periods that shows the demand rate is dependent on inventory volume at the retailer's shelf area. It was shown that there is unique local minimum for the integrated model for a given number of shipments. Yu and Hsu (2017) investigated an integrated model for single-supplier and single-buyer with immediate return of imperfect items under the unequal sized shipments. Here, the aim is to maximise the annual integrated profit by optimal number of shipments and optimal sizes of the shipments in a cycle. The first shipment is small size and the rest of them are equal sized. The demand rate is assumed as constant and uniform, and for reduced holding cost the production quantities during the time intervals between successive shipments is taken greater than the size of each shipment. Lots have a proportional defective units and their percentage is uniformly distributed. There is an inspection process with a fixed rate at buyer's side and defective items are sent to vendor immediately, and shortages are not allowed. They showed the benefits of this model compared to the integrated model under equal sized policy. Dey (2019) studied an integrated single vendor-buyer production inventory model with fuzziness and randomness. Also, production process quality control is defined and included in the model. The goal is to find the minimum of total cost of integrated system by obtaining optimal values of the safety stock, number of deliveries, the order quantity and the probability of 'out-of-control' of the production system.The study assumes that the annual demand is a discrete fuzzy random variable and that the buyer follows a continuous review inventory policy. The lead time demand is also considered a fuzzy random variable, and shortages are allowed and fully backlogged. Furthermore, imperfect items have a warranty cost associated with them. The vendor has made an investment in the production process quality, which is described as a logarithmic function. The objective of this study is to determine the optimal inventory policy that minimises costs while considering these various factors. The model has both fuzziness and stochastic uncertainty, yet it demonstrates the same trends as in deterministic and stochastic models that have similar assumptions. In his later study, Taleizadeh (2018) conducted research on an economic production quantity model that involves a single machine and multiple products, and incorporates a rework process and preventive maintenance. The objective was to minimise the total cost of the production system by determining the best time for preventive maintenance, optimal production and backordered quantities for each product, and cycle length. The model is also including partial backordering and service level constraint.The developed model considers preventive maintenance to occur when the inventory level is positive or negative. To account for this, two separate models are formulated and solved. Then with those results, the new model is solved by classical optimisation method.

CHAPTER 2: PRELIMINARIES AND GENERAL MODELS

2.1. Elements of Inventory Models

To determine profitability, there are several factors that need to be considered, including the cost of ordering or producing a product, the cost of holding inventory such as storage space, insurance, protection, taxes, etc., and the cost of shortages which includes delayed revenue and storage space. Additionally, revenue, discount rates, salvage costs for selling a product at a lower price, and lead time, which is the amount of time between placing an order and receiving it into inventory, are all important factors.

There are two types of inventory models; deterministic and stochastic due to randomness of demand. Deterministic models assume that demand is constant and known over a specific time period, while stochastic models consider demand as a random variable with a known probability distribution. Moreover, there is another type of classification related with the inventory review; continuous and periodic. When the stock level drops below the certain reorder point, continuous model requires to place an order. On the other hand, in periodic review, discrete intervals are more important to decide an order placement rather than reorder point.

2.2. Notation list for mathematical models in the literature

The parameters mentioned here are only for continuous review models in the literature. For our study, new notation list will be given.

x = demand per unit time,

D = expected total demand per unit time,

P = production in units per unit time,

Q = order quantity per cycle,

T = order cycle,

r = reorder point,

K = order setup cost,

c = unit ordering cost,

h_B = unit holding cost per unit time,

S = level of inventory when Q units is added under planned shortage,

μ_x = mean demand per unit time,

L = lead time,

w = additional cost related with storage space,

P' = finite replenishment rate per unit time,

M = maximum stock level,

k = total production time per cycle,

m = unit selling price of good items,

Z = proportion for satisfied demand,

N = net revenue,

g = scrap value of an unsold unit,

c_d = unit direct cost,

c_{sh} = cost for shortage per unit short per unit of time short,

c_{ro} = reorder cost,

c_{var} = variable cost,

c_S = unit disposal cost for scrap items,

c_r = unit reworking cost,

c_e = delivery cost per shipment,

q = proportion of defective items in a lot,

θ_3 = proportion of reworkable items in defective items,

2.3. Deterministic Continuous Review Inventory Models

Usually, inventory levels decrease as products are sold or consumed, and then they are restocked or replenished by purchasing new batches of products. The basic model showing this is economic order quantity (EOQ) model. It assumes known and constant demand and lead time, instantaneous receipt of product without any quantity discounts, order cost and holding cost only, and no stock-out. The goal is to decide when and how much to order so the total of those costs is minimised. Continuous reviewing is assumed, so when the inventory drops low enough then it can be replenished. Figure 1 shows the pattern for inventory levels for demand rate D, and order quantity Q to replenish inventory, specifically for this model the inventory level falls to 0.

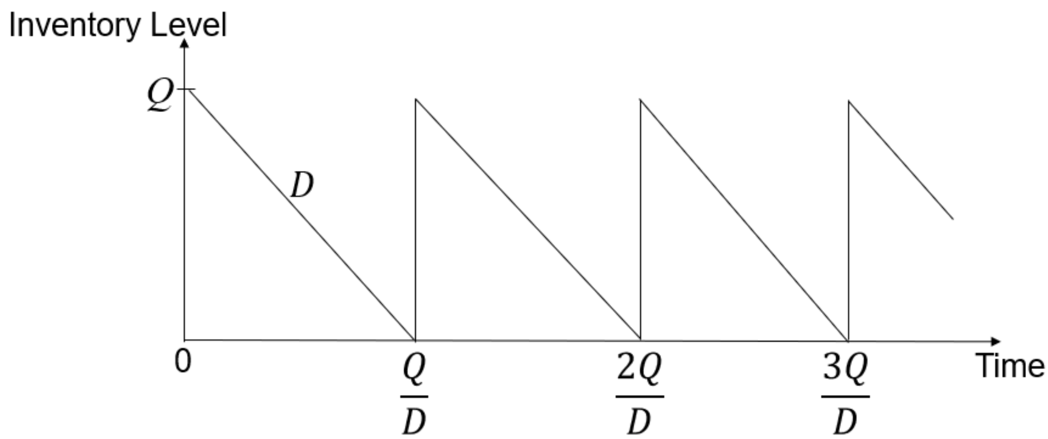

Figure 1. Inventory level in terms of time for EOQ model.

Another important term here is *reorder point* which shows the next order placement is required. The time for consecutive replenishment of inventory is called *cycle*. Now, the total cost per unit time TC can be formulated with follows:

The cost of ordering or producing per cycle $= K + cQ$.

The average inventory level during a cycle is $\frac{(Q+0)}{2} = \frac{Q}{2}$ units with corresponding holding cost $\frac{h_B Q}{2}$ per unit time.

The cycle length is $\frac{Q}{D}$, therefore holding cost per cycle is $\frac{h_B Q^2}{2D}$.

The total cost per cycle $= K + cQ + \frac{h_B Q^2}{2D}$ and the total cost per unit time is

$$TC = \frac{K + cQ + \frac{h_B Q^2}{2D}}{\frac{Q}{D}} = \frac{KD}{Q} + Dc + \frac{h_B Q}{2} \tag{1}$$

Once we take the first derivative of TC with respect to Q to find Q^* that minimises TC;

$$\frac{dTC}{dQ} = -\frac{KD}{Q^2} + \frac{h_B}{2} = 0 \tag{2}$$

and

$$Q^* = \sqrt{\frac{2KD}{h_B}} \tag{3}$$

that is the EOQ formula. The cycle time, let t^* is

$$T^* = \frac{Q^*}{D} = \sqrt{\frac{2K}{Dh_B}}. \tag{4}$$

From above, we can see how Q^* and T^* change according to K, h_B, and D.

2.3.1. The EOQ Model with Planned Shortages

This model deviates from the basic EOQ model by considering planned shortages as an allowed occurrence. Customers are aware of the product unavailability and are willing to wait for the product to become available again. Consequently, their backorders are fulfilled as soon as the new order arrives in the inventory. In this case, the pattern is shown in Figure 2.

As a difference from the previous figure, here we have negative values for the number of units of the backordered product where S shows inventory level after batch of Q units is added and $(Q - S)$ is shortage in inventory before a batch of Q units is added.

This time, the total cost per unit time is given as follows:

The cost of ordering or producing per cycle $= K + cQ$.

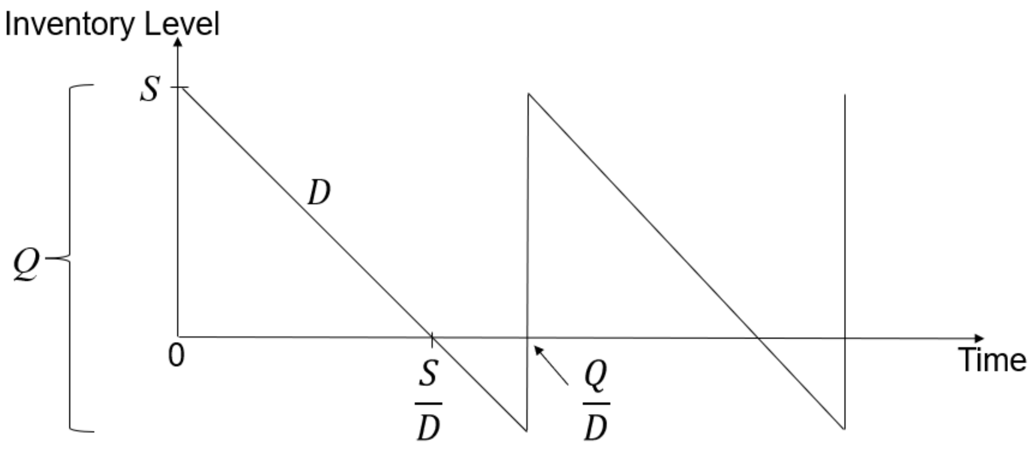

Figure 2. Inventory level in terms of time for EOQ model with planned shortages allowed.

For each cycle, the inventory level is positive only for $\frac{S}{D}$ time. The average inventory level during a cycle is $\frac{(S+0)}{2} = \frac{S}{2}$ units with corresponding cost $\frac{h_B S}{2}$ per unit time where h_B is holding cost.

The cycle length is $\frac{S}{D}$, therefore holding cost per cycle is $\frac{h_B S^2}{2D}$.

For shortage time, we have $\frac{(Q-S)}{D}$. The average amount of shortages during this time is $\frac{(0+Q-S)}{2} = \frac{(Q-S)}{2}$ units, and the cost is $\frac{c_{sh}(Q-S)}{2}$ per unit time. So, shortage cost per cycle is

$$\frac{c_{sh}(Q-S)}{2}\frac{Q-S}{D} = \frac{c_{sh}(Q-S)^2}{2D} \tag{5}$$

The total cost per cycle $= K + cQ + \frac{h_B S^2}{2D} + \frac{c_{sh}(Q-S)^2}{2D}$ and the total cost per unit time is

$$\frac{K + cQ + \frac{h_B S^2}{2D} + \frac{c_{sh}(Q-S)^2}{2D}}{\frac{Q}{D}} = \frac{KD}{Q} + Dc + \frac{h_B S^2}{2Q} + \frac{c_{sh}(Q-S)^2}{2Q} \tag{6}$$

Once more, we take the partial derivative of TC with respect to Q and S to find Q^* and S^* then set them equal to zero for minimising TC;

$$\frac{\partial TC}{\partial S} = \frac{h_B S}{Q} - \frac{c_{sh}(Q-S)}{Q} = 0. \tag{7}$$

and

$$\frac{\partial TC}{\partial Q} = -\frac{KD}{Q^2} - \frac{h_B S^2}{2Q^2} + \frac{c_{sh}(Q-S)}{Q} - \frac{c_{sh}(Q-S)^2}{2Q^2} = 0. \tag{8}$$

Solutions to these equations give

$$S^* = \sqrt{\frac{2KD}{h_B}}\sqrt{\frac{c_{sh}}{c_{sh}+h_B}}, \qquad Q^* = \sqrt{\frac{2KD}{h_B}\frac{c_{sh}+h_B}{c_{sh}}}. \tag{9}$$

The optimal cycle time, T^* is

$$T^* = \frac{Q^*}{D} = \sqrt{\frac{2K(c_{sh}+h_B)}{Dh_B c_{sh}}}. \tag{10}$$

2.4. Stochastic Continuous Review Inventory Model

For the uncertainty about demand rate, stochastic inventory models are more meaningful. With stochastic demand case shown in Figure 3, the inventory level is controlled continuously so new order is placed immediately when inventory level falls below the reorder point. This system is based on two fundamental components; *reorder point* and *order quantity*. For this model, a single product is considered, inventory level is always known because of the nature of continuous review, and choosing reorder point and order quantity are the only goals. Another important assumption is, if there is stock-out case, the demand is backlogged. Also for that case, there is certain shortage cost (c_{sh}) for each unit backordered per unit time until the backorder is filled.

This model is pretty similar to the EOQ model with planned shortages, with only one different assumption; instead of having unknown demand, that model assumes known demand rate. Next thing to consider is deciding the order quantity (Q) and the reorder point (r). Because of the close relationship with the EOQ model with planed shortages, choosing Q is straightforward as follows;

$$Q^* = \sqrt{\frac{2K\mu_L(c_{sh}+h_B)}{c_{sh}h_B}}, \tag{11}$$

where μ_L denotes the average demand per unit time. This formula is an approximation for the optimal order quantity since there is no formula for the exact value of it. To

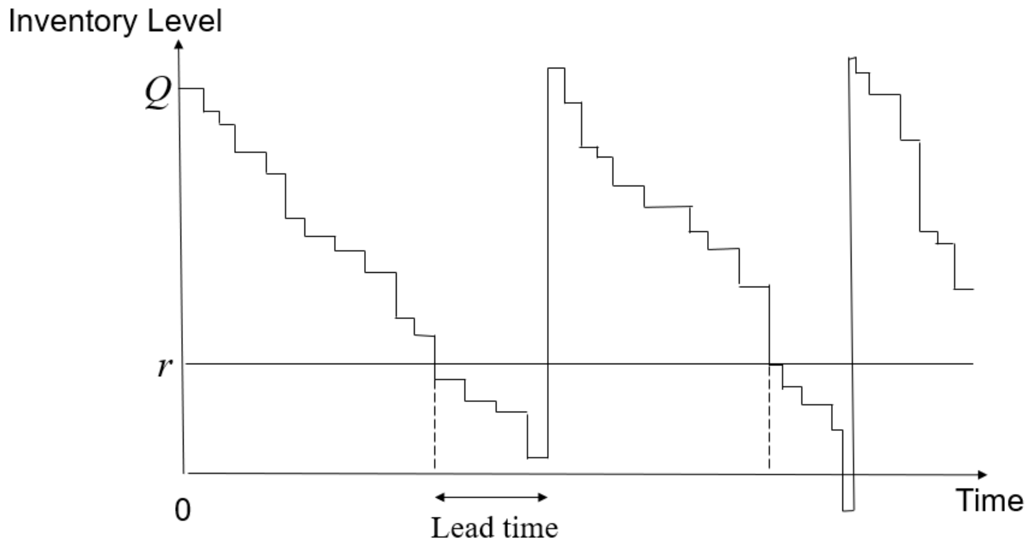

Figure 3. Stochastic demand with reorder point.

choose the reorder point (r), we need to know desired service level which can be defined in several ways such as stock-out probability, number of stock-out, average delay etc. Once the probability distribution is known, it is possible to find safety stock that is the expected inventory level just before the order quantity is received.

2.5. Models for continuous review method

Various quantitative models have been developed for inventory control with the goal of determining an order size that minimizes costs. In this section, we discuss several models for continuous review method from Waters (2008) as shown with Table 2. We start with the basic model, Economic Order Quantity (EOQ) and then remove its assumptions to develop new models.

To analyse all models, we can start with EOQ model. Its assumptions and cost components are given here again just to have consistency in parameters notation set from Waters (2008). The main assumptions for EOQ model is;

- known, continuous and constant demand,
- fixed and known costs,
- no shortages,

Table 2. Models with continuous review system for known and uncertain demand from Waters (2008).

Continuous Review Method
DEMAND

Known	**Uncertain**
Economic Order Quantity (EOQ)	Models for marginal analysis with discrete demand
Models with finite lead time	Models with discrete demand and shortages
Models with variable cost	Models with uncertain demand and constant lead time
Models with finite replenishment rate	Models with constant demand and uncertain lead time
Models with backorders	Models with uncertain demand and lead time
Models with lost sales	
Models with constraints on storage space	
Models with constraints on average investment in stocks	
Models with discrete, variable demand	

- zero lead time,
- single item is considered,
- each order has single delivery,
- instantaneous replenishment.

Addition to these assumptions, there are four costs variables in the analysis; unit cost (c), reorder cost (c_{ro}), holding cost (h_B), and shortage cost (c_{sh}) with three other variables; order quantity (Q), cycle time (T), and demand (x).

To find total cost per unit time, we add these components and substitute $Q = xT$, since the amount of entering stock in cycle should be equal to the amount of leaving stock in cycle. Therefore, we have the total cost as

$$TC = xc + \frac{xc_{ro}}{Q} + \frac{h_B Q}{2} \tag{12}$$

To obtain the minimum cost for inventory control, the equation above is differentiated with respect to Q as follows:

$$\frac{d(TC)}{dQ} = -\frac{c_{ro}x}{Q^2} + \frac{h_B}{2} = 0 \tag{13}$$

The optimal order size (economic order quantity), Q^*:

$$Q^* = \sqrt{\frac{2c_{ro}x}{h_B}} \tag{14}$$

with optimal cycle length, T^*:

$$T^* = Q^*/x = \sqrt{\frac{2c_{ro}}{xh_B}} \tag{15}$$

Another important term is optimal cost per unit time, TC^* for Q^* which can be obtained from last two terms of TC as variable cost (c_{var}):

$$c_{var} = \frac{c_{ro}x}{Q} + \frac{h_B Q}{2} \tag{16}$$

Here, if we substitute Q^* into the equation for optimal value, c^*_{var}

$$c^*_{var} = \sqrt{2c_{ro}h_B x} \tag{17}$$

The optimal total cost per unit time in inventory control consists of both variable and fixed costs, that is

$$TC^* = cx + c^*_{var} \tag{18}$$

2.5.1. Models with finite lead time

With EOQ model, we made the assumption that there is no lead time involved. This means that as soon as an order is placed, the items are immediately available for use and do not need to wait for any delivery or processing time. In order to make the inventory models more realistic, a non-zero lead time can be considered, which indicates that there is a finite amount of wait time for materials to become available for use. To plan stock successfully, we need to place an order so that existing stock is at certain level and we need delivery. At this point, it is beneficial to define a *reorder level* to show that it is necessary to place an order for inventory. Here, what we have already as stock needs to be sufficient till next order arrives. In a situation where the demand is known as well as the lead time, the required amount of inventory that is necessary for the lead time is a known value obtained by multiplying the constant demand rate with the constant lead time. Therefore, the reorder level is

$$\text{reorder level} = \text{lead time} \times \text{demand per unit time}$$
$$r = LD \tag{19}$$

This shows the level for stock to order a batch of size Q^*.

2.5.2. Models with variable costs

In previous two models, we assumed that costs are constant and known, but actually cost may vary according to quantity ordered, such as lower prices for larger orders. In general, there is more than one discounted price with larger orders so the bigger order means the less cost for buyer. Our objective is to find the order quantity that minimises the total cost per unit time, which will be the optimal value of Q. From Figure 4, it can be seen that continuous line shows the valid total cost for each order quantity, that is, if we place an order between 0 and Q_1 the unit cost will be c_1 and something between Q_1 an Q_2 gives lower cost as c_2 and so on. The broken line shows invalid cost for given order quantity.

In general, we have

$$Q^* = \sqrt{\frac{2c_{ro}x}{h_B}} \tag{20}$$

The holding cost can be defined by proportion of the unit cost as I, and there is a minimum point of the cost curve Q_i^* for each unit cost c_i. That is, Q_1^* shows the lowest point on the total cost curve for c_1, and so on. Hence, we can show Q_i^* as follows:

$$Q_i^* = \sqrt{\frac{2c_{ro}x}{Ic_i}} \tag{21}$$

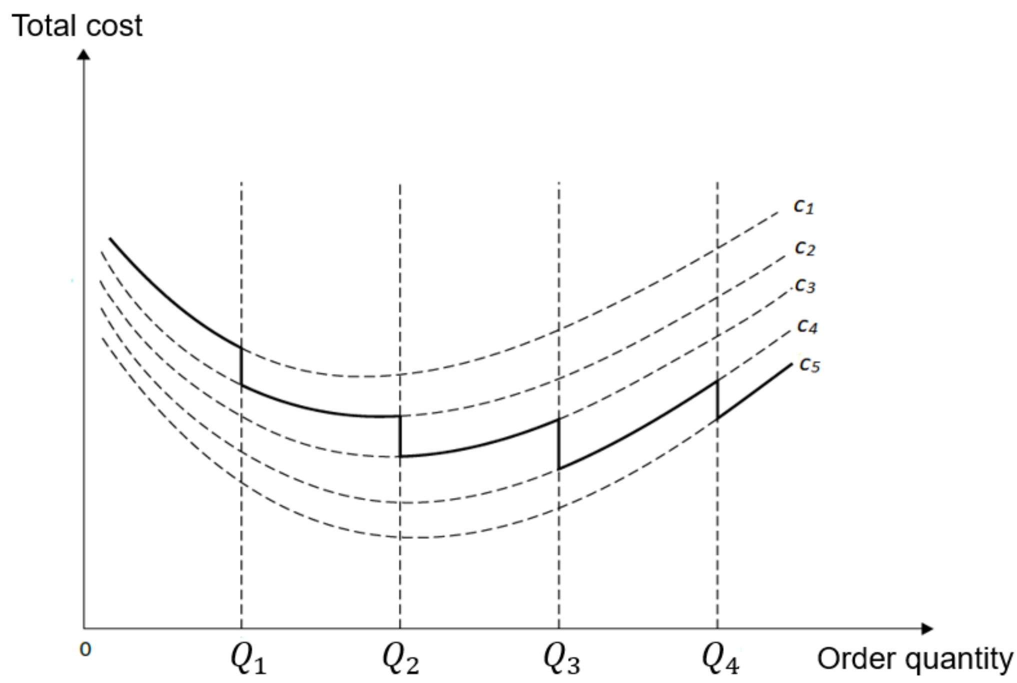

Figure 4. The valid cost curve for five unit costs.

The objective is to find the optimal order quantity that minimises the total cost per unit time for each unit cost c_i. There are two types of minimum values: valid and invalid. A valid minimum is within the range of valid order quantities for a particular unit cost, whereas an invalid minimum is not.

2.5.3. Models with finite replenishment rate

When the production rate for goods exceeds the demand rate, the inventory level increases at a certain rate, and the goods begin to accumulate. At some point, there should be decision made to stop production and switch facilities to making other items. This model is concerned with finding the optimal batch size which determines the best time for transferring goods between two different locations. The assumptions for EOQ model are still valid for this model except instantaneous replenishment rate, this time replenishment rate is P and demand rate is D, with inventory increasing as $P-D$. The concept of finding the optimal batch size is similar to the EOQ model, but in this case, the reorder cost may be associated with the cost of setting up production. Therefore the total cost becomes

$$TC = cD + \frac{c_{ro}D}{Q} + \frac{h_B Q}{2}\frac{(P-D)}{P} \tag{22}$$

From here, we differentiate TC with respect to Q and set it equal to zero to get an optimal order size;

$$Q^* = \sqrt{\frac{2c_{ro}D}{h_B}}\sqrt{\frac{P}{P-D}} \tag{23}$$

Also, we have cycle length as

$$T^* = \sqrt{\frac{2c_{ro}}{h_B D}}\sqrt{\frac{P}{P-D}} \tag{24}$$

and variable cost

$$c^*_{var} = \sqrt{2c_{ro}h_B D}\sqrt{\frac{P-D}{P}} \tag{25}$$

so that the total cost is

$$TC^* = cD + c^*_{var} \tag{26}$$

and finally the production time

$$k^* = Q^*/P \tag{27}$$

2.5.4. Models for planned shortages with backorders

The previous models are based on no shortages and all demand is met. This is beneficial when shortages are expensive but under some conditions, planned shortages are also reasonable. In general, if the cost of holding inventory (i.e. storing, maintaining, and financing it) exceeds the profit, then a planned shortage may be a better option. Additionally, when unit cost is high or holding the stocks is too expensive or lead time from suppliers are reasonably short and customers are accepting to wait, then back-ordering is another option.

Defining the shortage cost, denoted as c_{sh}, is a crucial first step. The shortage cost is a time-dependent cost that represents the cost per unit time of not meeting demand. Again, to find the optimal order size, we will follow the standard approach.

In the initial phase of the cycle, the entire demand is fulfilled using the available inventory, which implies that the quantity received by the customer is $Q-S$. This is equivalent to xt_1. During the second phase of the cycle, all demand goes unfulfilled, resulting in backorders. Hence, the shortage, denoted as S, is equal to the unmet demand of xt_2. Therefore, if we add these and substitute t_1 and t_2 and divide it by t, we will have the total cost per unit time as

$$TC = cx + \frac{c_{ro}x}{Q} + \frac{h_B(Q-S)}{2Q} + \frac{c_{sh}S^2}{2Q} \tag{28}$$

In this equation, we have two variables, Q and S, so we differentiate with respect to both and set the results to zero to get optimal order size and optimal amount to be back-ordered;

$$Q^* = \sqrt{\frac{2c_{ro}x(h_B + c_{sh})}{h_B c_{sh}}} \tag{29}$$

$$T^* = \sqrt{\frac{2c_{ro}h_B x}{c_{sh}(h_B + c_{sh})}} \tag{30}$$

Also, we know

$$t_1 = \frac{(Q^* - S^*)}{x} = \text{time for fulfilled demand with available inventory} \tag{31}$$

$$t_2 = \frac{S^*}{x} = \text{time for backordered demand} \tag{32}$$

$$T = t_1 + t_2 = \text{cycle time} \tag{33}$$

2.5.5. Models with lost sales

Lost sales occur when a customer refuses to wait for back-ordered items to become available and its analysis for minimising the cost is not equal to maximising revenue anymore. The goal of this model is to maximise the profit. There is new parameter for this as selling price per unit, m. In this model, it is necessary to consider the cost of lost sales as two parts: the loss of profit, which can be represented as a notional cost of $(m-c)$ per unit of sales lost, and the direct cost, c_d.

The profit N is

$$\begin{aligned} c_{total} &= \text{cost for lost sales per unit (with loss of profits)} \\ &= c_d + m - c \end{aligned} \tag{34}$$

and

$$\begin{aligned} Z &= \text{fulfilled demand proportion} \\ &= \frac{Q}{(xT)} \end{aligned} \tag{35}$$

Therefore, we have

$$N = Z\left[xc_{total} - \frac{c_{ro}x}{Q} - \frac{h_B Q}{2} - dx\right] \tag{36}$$

To get maximum profit, we differentiate the above equation:

$$Q^* = \sqrt{\frac{2c_{ro}x}{h_B}} \tag{37}$$

The optimal value for N is

$$N^* = Z\left[xc_{total} - \sqrt{2c_{ro}h_B x}\right] \tag{38}$$

Since the goal is to maximise the revenue, N_O, we use the following argument:

- If $xc_{total} > \sqrt{2xc_{ro}h_B}$, then Z should be 1. It means there are no shortages.
- If $xc_{total} < \sqrt{2xc_{ro}h_B}$, then Z should be the smallest possible, so $Z = 0$. It means no inventory item.
- If $xc_{total} = \sqrt{2xc_{ro}h_B}$, then the revenue is zero regardless of Z.

2.5.6. Models with constraints on storage space

When applying the EOQ model to all items in an inventory, it is possible for the total stock to exceed the available capacity. Therefore, it is necessary to find a way to decrease the inventory for allowable range. To reduce the inventory, we can introduce a new cost term for used space. The original holding cost, denoted as h_B, and an new cost for storage area, denoted as w. The total holding cost for each unit becomes:

$$\text{Total holding cost} = h_B + ws_i \tag{39}$$

where s_i represents the space required by a single item. Now we have

$$Q_i = \sqrt{\frac{2c_{ro_i}x_i}{h_{B_i} + ws_i}} \tag{40}$$

Since it is possible to have different value for each item, there is subscripts for all variables.

2.5.7. *Models with constraints on average investment in stock*

Consider an organisation that keeps n items. It also has total average investment upper limit u'. The objective is to minimise the total variable cost subject to the constraint that the average investment does not exceed the upper limit. That is:

$$\begin{aligned} \text{Minimise} : c_{var} &= \sum_{i=1}^{n} \frac{c_{ro_i}x_i}{Q_i} + \frac{h_{B_i}Q_i}{2} \\ \text{Subject to} : \sum_{i=1}^{n} \frac{c_iQ_i}{2} &\geq u' \end{aligned} \tag{41}$$

To solve this problem, we can include Lagrange multiplier, then differentiate the objective. The optimal order size will be

$$Q_i = Q_i^* \frac{2u'h_B}{c\sum_{i=1}^{n} V_i^*} \tag{42}$$

2.5.8. *Models with discrete and variable demand*

For the case of known and small size demands, it is possible to use deterministic model to find an optimal ordering policy. If the order quantity is smaller than the ideal number, it would result in frequent orders and a high reorder cost. Conversely, if the order quantity is greater than the ideal number, it would result in high stock levels and a high holding cost. The aim is to determine the optimal period number for single ordered demand. We set a single order that would suffice for the next N periods. For discrete demand in period i is represented with x_i. We have:

$$M = \sum_{i=1}^{N} x_i \tag{43}$$

where M is highest actual stock level. The variable cost c_{var_N} for inventory for N periods is the total of reorder and holding cost is

$$c_{var_N} = \frac{c_{ro}}{N} + \frac{h_B \sum_{i=1}^{N} x_i}{2} \tag{44}$$

To find the optimal value and minimal cost, we replace $N+1$ for N:

$$c_{var_{N+1}} = \frac{c_{ro}}{N+1} + \frac{h_B \sum_{i=1}^{N+1} x_i}{2} \tag{45}$$

And the idea is to get the point where V_{N+1} is larger than c_{var_N} :

$$\frac{c_{ro}}{N+1} + \frac{h_B \sum_{i=1}^{N+1} x_i}{2} > \frac{c_{ro}}{N} + \frac{h_B \sum_{i=1}^{N} x_i}{2} \tag{46}$$

or

$$N(N+1)x_{N+1} > \frac{2c_{ro}}{h_B} \tag{47}$$

An effective application of this model requires foreknowledge of the demand pattern, enabling the development of ordering policy that can be utilised in each subsequent inventory cycle.

2.5.9. Models with uncertain demand

In this particular model, we are looking at an inventory item that follows a Normal distribution, with an average demand of μ_x per unit of time and a standard deviation of σ_x. Additionally, the item has a fixed lead time of L. The mean lead time demand is calculated by multiplying the average demand by the lead time, resulting in $L\mu_x$. The variance of the lead time demand is found by multiplying the variance of the demand distribution by the lead time squared, giving $\sigma_x^2 L$. Finally, the standard deviation of the lead time demand can be obtained by multiplying the standard deviation of the demand distribution by the lead time, resulting in $\sigma_x L$. The service level shows the possibility for the reorder level is above the lead time demand, therefore the Normal distribution can be used to have

$$
\begin{aligned}
\text{safety stock} &= Z \times \text{standard deviation of lead time} \\
&= Z\sigma_x\sqrt{L}
\end{aligned}
$$

Z defines standard deviation counts for specified service level. Because of safety stock, reorder level is higher as:

$$
\begin{aligned}
\text{reorder level} &= \text{lead time demand} + \text{safety stock} \\
&= L\mu_x + Z\sigma_x\sqrt{L}
\end{aligned}
$$

2.5.10. Models with uncertain lead time

This model describes the model with constant demand and uncertain lead time with normal distribution. Safety stock is added to the reorder level and service level is defined in terms of the probability that lead time demand is greater than the reorder level:

$$
\begin{aligned}
\text{Service level} &= Pr(L\mu_x < r) \\
&= Pr(L < r/\mu_x)
\end{aligned}
$$

2.5.11. Models with uncertain demand and uncertain lead time

Assuming that both the demand and lead time for an item are Normally distributed, we can use standard calculations. The average demand for the item is denoted by μ_x, while its standard deviation is denoted by σ_x. On the other hand, the lead time has an average denoted by μ_{LT} and a standard deviation of σ_{LT}. The average lead time demand is the product of the average lead time and the average demand, i.e., $\mu_{LT}\mu_x$. The standard deviation of lead time demand can be calculated using the formula

$$\sqrt{\mu_{LT}\sigma_x^2 + \mu_x^2\sigma_{LT}^2}.$$

CHAPTER 3: METHODOLOGY

3.1. The Inventory Models

In this section, we propose two new models namely, continuous review inventory model for buyer and continuous review inventory model for supplier.

3.1.1. Notations and assumptions

The following notations and assumptions are used for these models. Some of the notations are from the list previously defined for continuous models given in the literature, and the rest is regulated accordingly.

Decision variables

Q_B = the order quantity for the buyer,

Q_V = the production lot size for the supplier,

r = reorder point for the buyer

Parameters

Stochastic parameters;

x = the demand during lead time at the buyer's side,

q = the indicator of production reliability where $0 < q < 1$ that shows the proportion of defective items in produced order lot,

θ_3 = the proportion of reworkable items in defective items,

Deterministic parameters;

D = the annual expected demand at the buyer's side,

P = the production rate of the supplier,

p = the proportion of imperfect items in an order lot received by the buyer where $0 < p < 1$,

θ_1 = the proportion of scrap items in defective items,

θ_2 = the proportion of imperfect items in defective items,

θ_l = the proportion of lower quality items after reworking,

K_B = the constant ordering cost per order for the buyer,

K_V = the setup cost of production system for the supplier,

F = the transportation cost per delivery that includes the delivery from the supplier to buyer,

h_B = the holding cost per unit per year for the buyer,

h_V = the holding cost per unit per year for the supplier,

h_{V_l} = the holding cost per lower quality unit per year for the supplier,

d = the fixed backordering cost per unit at the buyer,

c_b = the backordering cost per unit per unit of time at the buyer,

c_l = the lost sales cost per unit per unit of time at the buyer,

c = the unit variable cost,

c_P = the cost of production and inspection per unit for the supplier,

c_r = the rework cost of a defective item,

η = the fractional opportunity cost of capital per cycle,

α = backlogging intensity that denotes the maximum proportion of backlogged demand, $(0 \leq \alpha \leq 1)$,

b = backlogging resistance for the shape of time-sensitive customer function, $(b > 0)$,

L = the length of lead time for the buyer,

τ = the expected waiting time for customer during shortage,

T = time interval between successive shipments of Q units,

$E(\cdot)$ = mathematical expectation,

$*$ = the superscript representing optimal value.

Assumptions:

- There is single supplier and single buyer for one product.
- The buyer follows continuous review inventory policy and places an order when on-hand inventory reaches the reorder point r.
- The production rate is known, constant and continuous.
- Each lot Q_V, contains proportion of defective units q and Q_B contains proportion of imperfect units p.
- For defective items, there is rework process. After reworking, items have perfect quality. For the study, it is assumed that rework process is perfect. Moreover, for the three cases, reworking results lower quality items.
- There is 100% and error-free inspection process at the supplier's side with its cost. After production, reworking starts for the defective items those can be reworkable. At the end of the process, all items are considered as perfect items.
- At the buyer's side, there is inspection process during packing in cycle. Then at the end of cycle, the imperfect items (due to transportation, mishandling etc.) are sent to outlet shops.
- For the shortage at buyer's side, there is function defined to show that less customers are waiting for next replenishment as time passes, Sicilia et al. (2012);

$$B_p(r) = \alpha e^{-b\left(\frac{\tau}{L-\frac{rL}{\mu_L}}\right)} \tag{1}$$

where $0 \leq \alpha \leq 1$ and $b > 0$ and τ shows the expected waiting time when shortage occurs, so that if it is longer it will decrease the proportion of backlogged demand.

- There are two types of investment in the production process quality such as new equipment purchase, advanced maintenance etc to decrease defective items and increase rework power. For these factors, the logarithmic investment functions $I(q)$ and $I(\theta_3)$ from Dey (2019) is considered as follows:

$$I(q) = \frac{1}{\delta_1} ln\left(\frac{q_0}{q}\right) \tag{2}$$

$$I(\theta_3) = \frac{1}{\delta_2} ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \tag{3}$$

  where δ_1 is the percentage decrease in q per unit amount increase in investment and q_0 is the original percentage for defective state before any investment. Additionally, δ_2 is the percentage increase in θ_3 per unit amount increase in investment and θ_{3_0} is the original reworking proportion before the investment.

- The lead time is constant and known.

In the next two sections, we will see the independent costs for the buyer and supplier. Buyer's model has two cases; deterministic and stochastic demand. According to those, two separate functions for expected total costs are defined. For the supplier, two parameters, q and θ_3, are the main figures for model and according to their stochasticity, four models are characterised for the expected total cost. Since the proofs are similar for each case, only one version of each total cost's convexity has shown in the Appendix.

3.1.2. Continuous Inventory Review Model for Buyer

When the buyer does not cooperate with the supplier for maximisation of their mutual benefits, it means that the buyer decides independently and the behaviour of inventory level over time changes accordingly. As the nature of continuous review, the shipment will be processed when the inventory level drops to the reorder point r. Figure 5 shows the behaviour of the perfect item inventory level and reorder point for the buyer. Here, p is the proportion of imperfect items in the lot (because of mishandling, transportation etc.) which are sent to outlet shops later. The cost incurred by the buyer for inventory in a single cycle includes various components such as order placement cost, transportation cost, inspection cost, and holding cost(since lead time demand is known in the deterministic demand case there is no shortage cost).

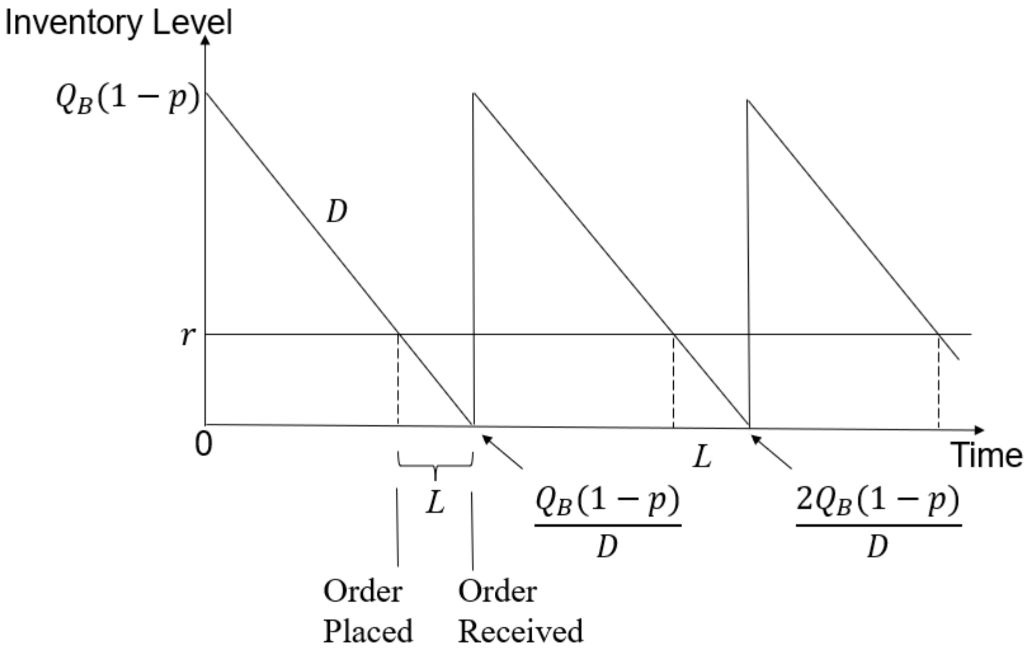

Figure 5. Perfect item inventory level and reorder point for the buyer.

3.1.2.1 The demand is deterministic

The buyer places an order when the inventory level is r and the order arrives after L time periods. Here, Q_B is the only decision variable since r can be calculated easily because of deterministic demand rate D. Note that for the beginning of the each cycle, inventory level has the order quantity Q_B, then as perfect items we have $Q_B(1-p)$. The maximum inventory level is $Q_B(1-p)$, and as imperfect items we have Q_Bp. Moreover, $T = \frac{Q_B(1-p)}{D}$. Since we have deterministic demand rate and lead time for this model, reorder point becomes $r = LD$. Hence, the buyer's average inventory during cycle becomes

$$\begin{aligned} I_{average} &= \left(\frac{1}{2}Q_B(1-p)T + Q_BpT\right) \\ &= \left(\frac{1}{2}\frac{(Q_B(1-p))^2}{D} + \frac{Q_B^2p(1-p)}{D}\right) \end{aligned} \quad (4)$$

and the total cost for the buyer per cycle is

$$TC_B(Q_B) = K_B + F + cQ_B + h_B\left(\frac{1}{2}\frac{(Q_B(1-p))^2}{D} + \frac{Q_B^2 p(1-p)}{D}\right). \quad (5)$$

Since the replenishment cycle length is $T = \frac{Q_B(1-p)}{D}$, we have average annual cost

$$TC_B(Q_B) = \left(K_B + F + cQ_B + h_B\left(\frac{1}{2}\frac{(Q_B(1-p))^2}{D} + \frac{Q_B^2 p(1-p)}{D}\right)\right)\frac{D}{Q_B(1-p)} \quad (6)$$

Taking the first derivative of $TC_B(Q_B)$ with respect to Q_B, we have

$$\frac{d(TC_B)}{dQ_B} = \frac{2D(F+K_B) + h_B(p^2-1)Q_B^2}{2(p-1)Q_B^2} = 0 \quad (7)$$

Taking the second derivative, we have

$$\frac{d^2(TC_B)}{dQ_B^2} = \frac{2D(F+K_B)}{(1-p)Q_B^3}. \quad (8)$$

Since all parameters in the above derivative are positive, $\frac{d^2(TC_B)}{dQ_B^2} > 0$, which implies that the total annual cost is a convex function and there exists a unique value of Q_B, that is given as

$$Q_B^* = \sqrt{\frac{2D(F+K_B)}{h_B(1-p^2)}}. \quad (9)$$

3.1.2.2 The demand is stochastic with lost-sales

With the previous model, we defined the cost function for the buyer under known demand case. When the demand is stochastic, that is, we do not know if it is more or less than reorder point during the lead time, shortage may occur. Lead time demand is non-negative continuous random variable x with pdf as $f(x)$ and cdf as $F(x)$, and mean= μ_L and standard deviation= σ_L. Moreover,

$$\text{if} \quad x > r \implies \text{shortage} = x - r$$

$$\text{if} \quad x < r \implies \text{shortage} = 0$$

At the end of each cycle, the expected shortage will be

$$
\begin{aligned}
\bar{s}(r) &= \int_0^r 0 f(x)dx + \int_r^\infty (x-r)f(x)dx \\
&= \int_r^\infty (x-r)f(x)dx
\end{aligned} \tag{10}
$$

At this point, for holding cost we need to calculate the expected inventory level before an order arrives differently:

$$
\begin{aligned}
\bar{n}(r) &= \int_0^r (r-x)f(x)dx \\
&= \int_0^\infty (r-x)f(x)dx + \int_r^\infty (x-r)f(x)dx \\
&= \bar{s}(r) - \mu_L + r
\end{aligned} \tag{11}
$$

where μ_L is the expected demand during lead time. The average inventory level during cycle becomes

$$
\begin{aligned}
I_{average} &= \left(\frac{1}{2}Q_B(1-p)T + Q_B pT + \bar{n}(r)T\right) \\
&= \left(\frac{1}{2}\frac{(Q_B(1-p))^2}{D} + \frac{Q_B^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q_B(1-p)}{D}\right)
\end{aligned} \tag{12}
$$

The total average cost for the buyer per cycle is

$$
\begin{aligned}
TC_B(Q_B, r) = K_B + h_B &\left(\frac{1}{2}\frac{(Q_B(1-p))^2}{D} + \frac{Q_B^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q_B(1-p)}{D}\right) \\
&+ F + cQ_B + c_l\bar{s}(r)
\end{aligned} \tag{13}
$$

The expected total annual cost will be

$$
\begin{aligned}
ETC_B(Q_B, r) &= \frac{K_B D}{Q_B(1-p)} + \frac{FD}{Q_B(1-p)} + \frac{cD}{(1-p)} + \frac{c_l\bar{s}(r)D}{Q_B(1-p)} \\
&+ h_B\left(\frac{1}{2}(Q_B(1-p)) + Q_B p + (r-\mu_L+\bar{s}(r))\right)
\end{aligned} \tag{14}
$$

To minimise $ETC_B(Q_B, r)$ we first take its partial derivatives with respect to Q_B and r as follows:

$$\frac{\partial(ETC_B)}{\partial Q_B} = \frac{2D(F + K_B + c_l\bar{s}(r)) + h_B(p^2 - 1)Q_B^2}{2(p-1)Q_B^2} = 0 \tag{15}$$

$$\frac{\partial(ETC_B)}{\partial r} = \frac{h_B(1-p)Q_B(\bar{s}'(r) + 1) + c_l D\bar{s}'(r)}{(1-p)Q_B} = 0 \tag{16}$$

For a given r, we can obtain the optimal value of Q_B from Eq.15 and the complimentary cumulative distribution of x at optimal $r : F(r)$ from Eq.16 and the derivative of $\bar{s}(r)$. That is

$$Q_B^* = \sqrt{\frac{2D(K_B + F + c_l\bar{s}(r))}{h_B(1-p^2)}} \tag{17}$$

$$1 - F(r^*) = \frac{h_B(1-p)Q_B}{Dc_l + (1-p)Q_B h_B} \tag{18}$$

The optimal value of r is

$$r^* = F^{-1}\left(1 - \frac{h_B Q_B(1-p)}{Dc_l + (1-p)Q_B h_B}\right) \tag{19}$$

For the solution process, we will use iteration to pull Q_B and r from these equations. Starting with Q_{B_0} value and iterative until the Q_B values converge (Figure 6).

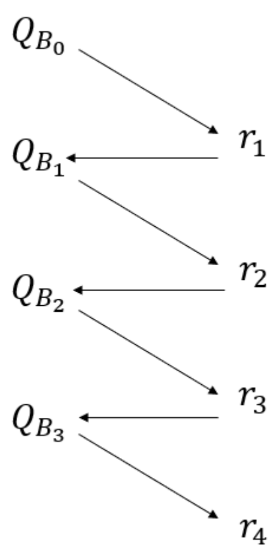

Figure 6. Iteration for solution process

3.1.2.3 The demand is stochastic with partial backlogging

When shortage happens, it may be an option to wait for customers. That is, as soon as inventory arrives, their demand will be filled. However, not every customer is willing to wait due to their urgency, timing, or simply they do not want to. Therefore in this section, we analysed partially backordering case with time-sensitive customers. We propose a parameter that represents the proportion $B_p(r)$ of backlogged demand defined as a negative exponential function of reorder point r. Here, τ shows the expected waiting time when shortage occurs, so that if it is long it will decrease. The proportion of backlogged demand:

$$B_p(r) = \alpha e^{-b\left(\frac{\tau}{L-\frac{rL}{\mu_L}}\right)} \tag{20}$$

where $0 \leq \alpha \leq 1$ and $b > 0$. With new parameter, the expected value of the amount of backlogged demand will be

$$\bar{B}(r) = \int_r^\infty B_p(r)(x-r)f(x)dx \tag{21}$$

Moreover, the expected lost-sales from shortage becomes

$$\bar{s}(r) = \int_r^\infty (1-B_p(r))(x-r)f(x)dx \tag{22}$$

At the end of each cycle, on-hand inventory, $\bar{n}(r)$, is

$$\begin{aligned}
\bar{n}(r) &= \int_0^r (r-x)f(x)dx \\
&= \int_0^\infty (r-x)f(x)dx + \int_r^\infty (x-r)f(x)dx \\
&= \int_0^\infty (r-x)f(x)dx + \int_r^\infty (1-B_p(r)+B_p(r))(x-r)f(x)dx \\
&= \int_0^\infty (r-x)f(x)dx + \int_r^\infty (1-B_p(r))(x-r)f(x)dx + \int_r^\infty B_p(r)(x-r)f(x)dx \\
&= \bar{B}(r) + \bar{s}(r) - \mu_L + r
\end{aligned} \tag{23}$$

The total cycle cost for buyer becomes

$$TC_B(Q_B,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q_B(1-p))^2}{D} + \frac{Q_B^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q_B(1-p)}{D}\right) \\ + F + cQ_B + d + c_b\bar{B}(r) + c_l\bar{s}(r) \tag{24}$$

The expected total annual cost will be

$$ETC_B(Q_B,r) = \frac{K_B D}{Q_B(1-p)} + h_B\left(\frac{1}{2}(Q_B(1-p)) + Q_B p + (r-\mu_L+\bar{s}(r))\right) \\ + \frac{FD}{Q_B(1-p)} + \frac{cD}{(1-p)} + \frac{dD}{Q_B(1-p)} + \frac{c_b\bar{B}(r)D}{Q_B(1-p)} + \frac{c_l\bar{s}(r)D}{Q_B(1-p)} \tag{25}$$

To minimise $ETC_B(Q_B,r)$ we first take its partial derivatives with respect to Q_B and r as follows:

$$\frac{\partial(ETC_B)}{\partial Q_B} = \frac{2D(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B) + h_B\left(p^2-1\right)Q_B^2}{2(p-1)Q_B^2} = 0 \tag{26}$$

$$\frac{\partial(ETC_B)}{\partial r} = \frac{D(c_b\bar{B}'(r) + c_l\bar{s}'(r)) + h_B(1-p)Q_B(1+\bar{s}'(r))}{(1-p)Q_B} = 0 \tag{27}$$

Proposition 1. *$ETC_B(Q_B,r)$ expected annual total cost is convex in (Q_B,r) when the following condition is satisfied:*

$$2(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B)\left(\bar{s}''(r)(h_B(1-p)Q + c_l D) + c_b D\bar{B}''(r)\right) \\ \geq D\left(c_b\bar{B}'(r) + c_l\bar{s}'(r)\right)^2 \tag{28}$$

The proof and condition is presented in Appendix.

For a given r, we can obtain the optimal value of Q_B from Eq. 26.

$$Q_B^* = \sqrt{\frac{2D(c_b\bar{B}(r) + c_l\bar{s}(r) + d + F + K_B)}{h_B\left(1-p^2\right)}} \tag{29}$$

For the optimal value of r from Eq. 27:

$$\bar{B}'(r)\frac{Dc_b}{(1-p)Q_B} + \bar{s}'(r)\left(\frac{Dc_l}{(1-p)Q_B} + h_B\right) + h_B = 0 \tag{30}$$

where

$$\bar{B}'(r) = \frac{\partial}{\partial r}\int_r^\infty B_p(r)(x-r)f(x)dx$$
$$= B'_p(r)\int_r^\infty xf(x)dx - \Big(B'_p(r)r + B_p(r)\Big)(1-F(r)) \tag{31}$$

Here for the integral part, expected value of truncated random variable is used. That is, the expected value of a random variable with probability density function $f(x)$ and cumulative distribution function $F(x)$ given that the random variable is greater than some known value. Thus

$$E(x|x>r) = \frac{\int_r^\infty xg(x)dx}{1-F(r)}$$
$$\int_r^\infty xf(x)dx = (1-F(r))E(x|x>r) \tag{32}$$

where $g(x) = f(x)$ for all $x > r$ and $g(x) = 0$ otherwise.

As we mentioned above, the term $E(x|x>r)$ is the expected value of x that is greater than r, so it is shortage amount which in this case will be partially backordered. Therefore it is equal to $(\bar{B}(r)+\bar{s}(r))$. Therefore

$$\bar{B}'(r) = \frac{\partial}{\partial r}\int_r^\infty B_p(r)(x-r)f(x)dx$$
$$= B'_p(r)(1-F(r))(\bar{B}(r)+\bar{s}(r)) - (B'_p(r)r + B_p(r))(1-F(r))$$
$$= (1-F(r))\Big(B'_p(r)(\bar{B}(r)+\bar{s}(r)-r) - B_p(r)\Big) \tag{33}$$

and

$$\bar{s}'(r) = \frac{\partial}{\partial r}\int_r^\infty (1-B_p(r))(x-r)f(x)dx$$
$$= \frac{\partial}{\partial r}\Big(\int_r^\infty (x-r)f(x)dx - \int_r^\infty B_p(r)(x-r)f(x)dx\Big)$$
$$= (1-F(r))\Big(B_p(r) - B'_p(r)(\bar{B}(r)+\bar{s}(r)-r) - 1\Big) \tag{34}$$

Hence, the optimal value of r is

$$F(r)=\left(1-\frac{h_B(1-p)Q_B}{\left(B'_p(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ_B(1-p)\right)+\left(c_lD+h_BQ_B(1-p)\right)}\right) \tag{35}$$

$$r^*=F^{-1}\left(1-\frac{h_B(1-p)Q_B}{\left(B'_p(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ_B(1-p)\right)+\left(c_lD+h_BQ_B(1-p)\right)}\right) \tag{36}$$

3.1.2.4 The demand is stochastic with backlogging

For this case, at the end of each cycle, all shortage will be backlogged. The expected backorder amount becomes

$$\bar{B}(r)=\int_r^{\infty}(x-r)f(x)dx \tag{37}$$

The expected on-hand inventory level will be

$$\begin{aligned}\bar{n}(r)&=\int_0^{r}(r-x)f(x)dx\\&=\int_0^{\infty}(r-x)f(x)dx+\int_r^{\infty}(x-r)f(x)dx\\&=\bar{B}(r)-\mu_L+r\end{aligned} \tag{38}$$

Total average cost for buyer per cycle is

$$\begin{aligned}TC_B(Q_B,r)=K_B+h_B&\left(\frac{1}{2}\frac{(Q_B(1-p))^2}{D}+\frac{Q_B^2p(1-p)}{D}+(r-\mu_L)\frac{Q_B(1-p)}{D}\right)\\&+F+cQ_B+d+c_b\bar{B}(r)\end{aligned} \tag{39}$$

and the expected total annual cost is defined as follows:

$$\begin{aligned}ETC_B(Q_B,r)=\frac{K_BD}{Q_B(1-p)}+h_B&\left(\frac{1}{2}(Q_B(1-p))+Q_Bp+(r-\mu_L)\right)\\&+\frac{FD}{Q_B(1-p)}+\frac{cD}{(1-p)}+\frac{dD}{Q_B(1-p)}+\frac{c_b\bar{B}(r)D}{Q_B(1-p)}\end{aligned} \tag{40}$$

To minimise $ETC_B(Q_B, r)$ we first take its partial derivatives with respect to Q_B and r as follows:

$$\frac{\partial(ETC_B)}{\partial Q_B} = \frac{2Dc_b\bar{B}(r) + 2D(d+F+K_B) + h_B\left(p^2-1\right)Q_B^2}{2(p-1)Q_B^2} = 0 \tag{41}$$

$$\frac{\partial(ETC_B)}{\partial r} = \frac{Dc_b\bar{B}'(r) + h_B(1-p)Q_B}{(1-p)Q_B} = 0 \tag{42}$$

For a given r, we can obtain the optimal value of Q_B from Eq. 41.

$$Q_B^* = \sqrt{\frac{2D(c_b\bar{B}(r) + d + F + K_B)}{h_B(1-p^2)}} \tag{43}$$

For the optimal value of r from Eq. 42:

$$\bar{B}'(r)\frac{Dc_b}{(1-p)Q_B} + h_B = 0 \tag{44}$$

where

$$\begin{aligned} \bar{B}'(r) &= \frac{\partial}{\partial r}\int_r^{\infty}(x-r)f(x)dx \\ &= -(1-F(r)) \end{aligned} \tag{45}$$

Hence, the optimal value of r is

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q_B}{Dc_b}\right) \tag{46}$$

3.1.3. Continuous Review Inventory Model for Supplier

The main difference of supplier model from Buyer's model is that there is a grouping process of items according to their conditions. At the end of this process, these items can be grouped as nondefective, defective or reworkable items. The parameter for the proportion of defective items is given as q and reworkable items is given as θ_3. At the supplier side, to meet the demand, rate of production of non-defective items is greater than or equal to demand, $P(1-q) \geq D$. Therefore, for some time after the start of a new production run, the inventory level starts to increase with a rate $(P-D)$ $((1-q)P-D$ for nondefective items). The length of each production run is showed by t_1 which is the time required to produce order quantity, Q_V. So,

$$t_1 = \frac{Q_V}{P} \tag{47}$$

After production, there is an inspection and reworking process on those defective items that can be reworkable. For the reworking, we have qQ_V defective units, however only certain amount of them (θ_3) is reworkable. Therefore the time required to rework on those items is

$$t_2 = \frac{q\theta_3 Q_V}{P} \tag{48}$$

And t_3 which is the time to build up the inventory, will be defined during non-defective inventory calculations in the next section.

The supplier's inventory cost per cycle has production setup, holding, reworking, and cost of quality improvement (investment). Figure 7 and Figure 8 show the behaviour of the inventory level of nondefective items and reworkable defective items at the supplier respectively.

The reworking cost includes all defective items that can be reworkable, therefore it is calculated as $c_r Q_V q \theta_3$.

In general, the supplier's goal is to reduce the defective items produced and/or rework on those as effective as possible. By investment in process quality control, it is possible to have more non-defective items, smaller lot size, and less set-up cost etc. There is close relationship between optimal policy and process quality, so we included two terms for this case. Here, q_0 is the probability of production process for original defective case and investment is all about to lower that probability. Additionally, q is given as defective probability so we have $0 \leq q \leq q_0$. With same logic, θ_3 is the probability of reworking on defective items and θ_{3_0} is the original reworking probability before investment with $0 \leq \theta_{3_0} \leq \theta_3$. For supplier's independent cost, we will examine four cases of q and θ_3.

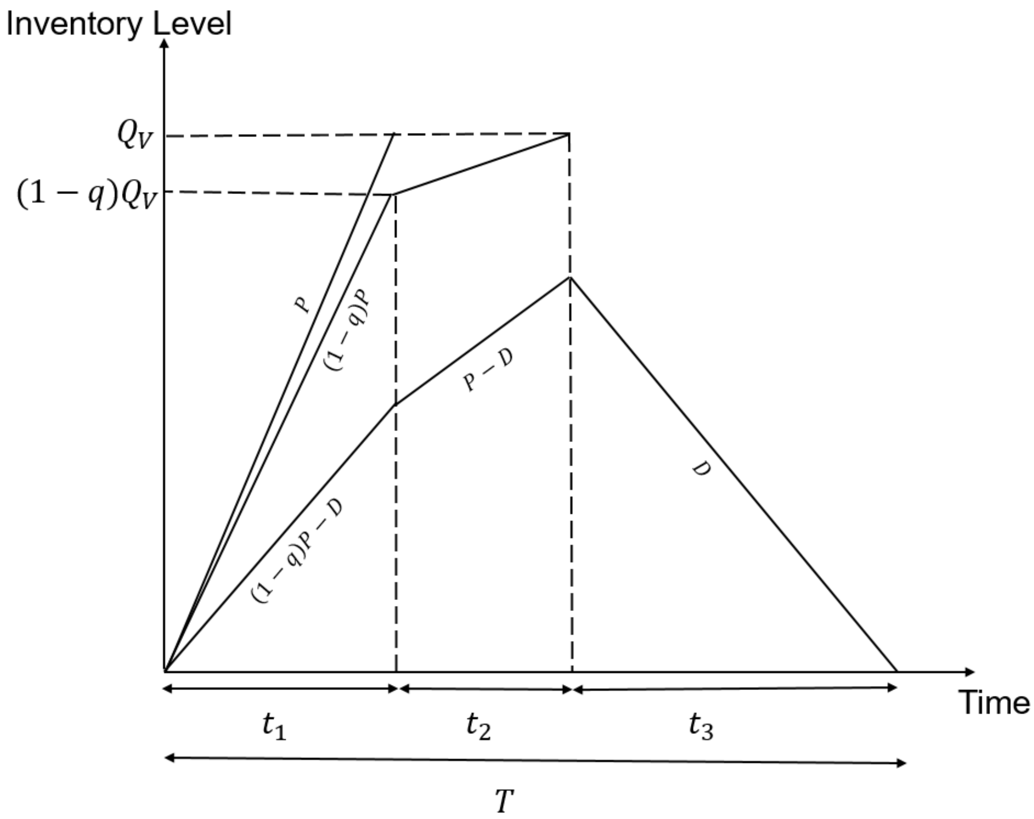

Figure 7. Inventory behaviour of the nondefective items

3.1.3.1 Case 1: q and θ_3 are both deterministic

The inventory function for nondefective items during t_1 can be defined as

$$I_1(t) = ((1-q)P - D)t \tag{49}$$

therefore the total inventory will be

$$\begin{aligned}\Delta_1 &= \int_0^{t_1} I_1(t)dt = \int_0^{t_1} ((1-q)P-D)tdt \\ &= \frac{((1-q)P-D)t_1^2}{2} \\ &= \frac{((1-q)P-D)Q_V^2}{2P^2}\end{aligned} \tag{50}$$

Here,

$$I_2(0) = I_1(t_1) = ((1-q)P - D)t_1 \tag{51}$$

and

$$I_2(t) = (P-D)t + ((1-q)P - D)t_1 \tag{52}$$

Total inventory during rework process,

$$\begin{aligned}\Delta_2 &= \int_0^{t_2} I_2(t)dt = \frac{1}{2}(P-D)t_2^2 + ((1-q)P-D)t_2t_1 \\ &= \frac{qQ_V^2\theta_3(P(q(\theta_3-2)+2) - D(q\theta_3+2))}{2P^2}\end{aligned} \tag{53}$$

The inventory curve during t_3 can be shown by

$$I_3(t) = Dt \tag{54}$$

with the terminal value

$$I_3(t_3) = I_2(t_2) = (P-D)t_2 + ((1-q)P-D)t_1 \tag{55}$$

It can be shown as:

$$t_3 = \frac{(P-D)t_2 + ((1-q)P-D)t_1}{D} \tag{56}$$

Total inventory during t_3 will be:

$$\begin{aligned}\Delta_3 &= \int_0^{t_3} I_3(t)dt = \frac{1}{2}Dt_3^2 \\ &= \frac{Q_V^2(P(-q\theta_3+q-1)+q\theta_3D+D)^2}{2P^2D}.\end{aligned} \tag{57}$$

The inventory curve for defective items during the period t_1 can be shown as (Fig. 8)

$$J_1(t) = q\theta_3Pt \tag{58}$$

where $J_1(t_1) = Q_V q$, and the total inventory during t_1 can be found as

$$\Gamma_1 = \int_0^{t_1 = \frac{Q_V}{P}} J_1(t)dt$$
$$= \frac{1}{2} q\theta_3 P \left(\frac{Q_V}{P}\right)^2 = \frac{q\theta_3 Q_V^2}{2P} \tag{59}$$

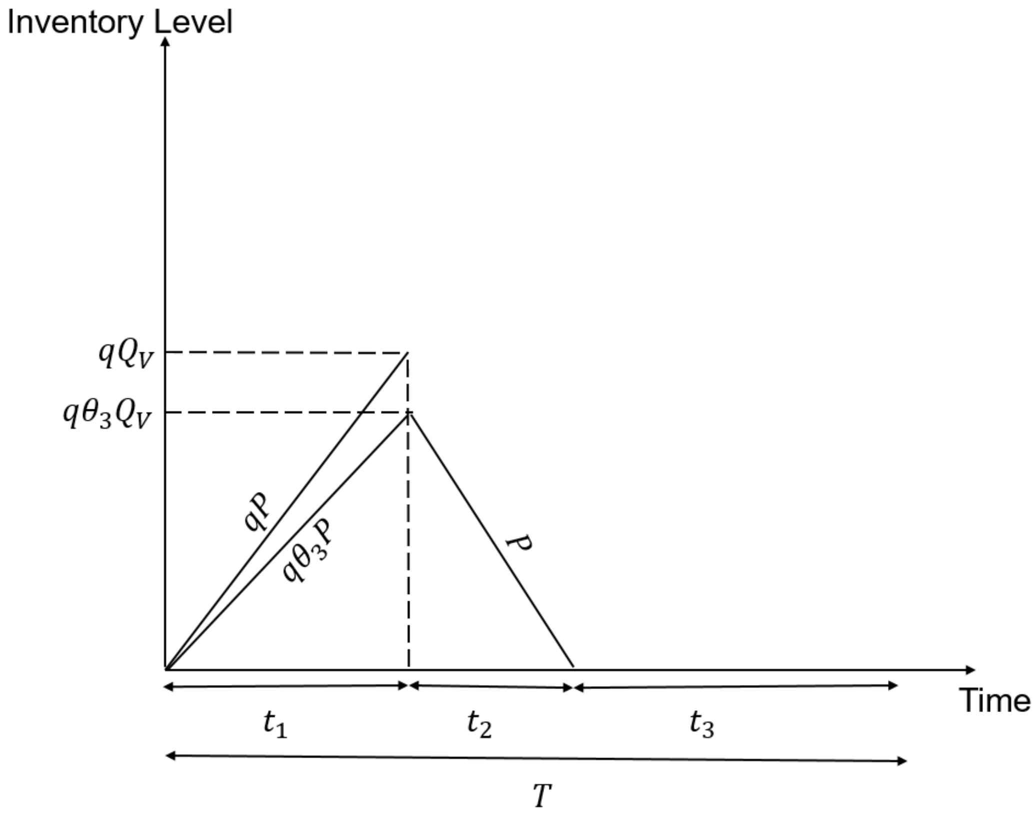

Figure 8. Inventory behaviour of the reworkable defective items

The inventory curve of the defective items during t_2 is $J_2(t) = Pt$ with total inventory

$$\Gamma_2 = \int_0^{t_2} J_2(t)dt$$
$$= \frac{1}{2} P \left(\frac{qQ_V\theta_3}{P}\right)^2 = \frac{q^2 Q_V^2 \theta_3^2}{2P} \tag{60}$$

The supplier's total cost per cycle becomes

$$TC_V(Q_V) = K_V + h_V(\Delta_1 + \Delta_2 + \Delta_3 + \Gamma_1 + \Gamma_2) + c_P Q_V + c_r Q_V q \theta_3 + \frac{Q_V \eta}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{Q_V \eta}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \tag{61}$$

Then its average annual cost is

$$TC_V(Q_V) = \left(K_V + h_V\left(\frac{Q_V^2(q(\theta_3 - 1) + 1)(P(q(\theta_3 - 1) + 1) - D)}{2PD}\right) + \frac{Q_V \eta}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{Q_V \eta}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_P Q_V + c_r q Q_V \theta_3\right) \frac{D}{Q(1-p)} \tag{62}$$

Proposition 2. *$TC_V(Q_V)$ annual total cost is strictly convex in Q_V.*

The proof is presented in Appendix.

To minimise $TC_V(Q_V)$, we take first derivative with respect to Q_V

$$\frac{d(TC_V)}{dQ_V} = \frac{2PDK_V - h_V Q_V^2(q(\theta_3 - 1) + 1)(P(q(\theta_3 - 1) + 1) - D)}{2(p-1)PQ_V^2} \tag{63}$$

Taking the second derivative, we have

$$\frac{d^2(TC_V)}{dQ_V^2} = \frac{2K_V D}{(1-p)Q_V^3} \tag{64}$$

All parameters are positive, and $\frac{d^2(TC_V)}{dQ_V^2} > 0$. Therefore, there exists unique value for Q_V^* given as

$$Q_V^* = \sqrt{\frac{2PDK_V}{h_V(q(\theta_3 - 1) + 1)(P(q(\theta_3 - 1) + 1) - D)}} \tag{65}$$

with condition $0.5 \leq \theta_3 \leq 1$.

3.1.3.2 Case 2: q is deterministic and θ_3 is stochastic with standard uniform distribution

With θ_3 given as random variable, it has an upper (U_2) and lower (L_2) bounds with the probability distribution function of g_2. The expected value of the total inventory of nondefective items during t_1 will be

$$
\begin{aligned}
E(\Delta_1) &= \int_{L_2}^{U_2} \frac{((1-q)P-D)Q_V^2}{2P^2} g_2 d\theta_3 \\
&= \frac{((1-q)P-D)Q_V^2}{2P^2} \int_{L_2}^{U_2} \left(\frac{1}{U_2-L_2}\right) d\theta_3 \qquad \left(g_2 = \frac{1}{U_2-L_2}\right) \\
&= \frac{((1-q)P-D)Q_V^2}{2P^2} \qquad (66)
\end{aligned}
$$

And the inventory during t_2 is

$$
\begin{aligned}
E(\Delta_2) &= \int_{L_2}^{U_2} \left(\frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2}\right) g_2 d\theta_3 \\
&= \int_{L_2}^{U_2} \frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} \left(\frac{1}{U_2-L_2}\right) d\theta_3 \\
&+ \int_{L_2}^{U_2} \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2} \left(\frac{1}{U_2-L_2}\right) d\theta_3 \\
&= \frac{qQ_V^2 \left(L_2^2 q(P-D) + L_2(P(q(U_2-3)+3) - D(qU_2+3))\right)}{6P^2} \\
&+ \frac{qQ_V^2 U_2(P(q(U_2-3)+3) - D(qU_2+3))}{6P^2} \qquad (67)
\end{aligned}
$$

During t_3, the inventory becomes

$$
\begin{aligned}
E(\Delta_3) &= \int_{L_2}^{U_2} \frac{1}{2} D \left(\frac{(P-D)q\theta_3 Q_V + ((1-q)P-D)Q_V}{PD}\right)^2 g_2 d\theta_3 \\
&= \frac{Q_V^2 \left((P(-qU_2+q-1) + qU_2 D + D)^3 - (P(-L_2 q + q - 1) + L_2 qD + D)^3\right)}{6qP^2 D(U_2-L_2)(D-P)} \\
&\qquad (68)
\end{aligned}
$$

The expected value of the total inventory of defective items during t_1 will be

$$
\begin{aligned}
E(\Gamma_1) &= \int_{L_2}^{U_2} \frac{q\theta_3 Q_V^2}{2P} g_2 d\theta_3 \\
&= \frac{qQ_V^2(L_2+U_2)}{4P} \qquad (69)
\end{aligned}
$$

and during t_2

$$E(\Gamma_2) = \int_{L_2}^{U_2} \frac{q^2 Q_V^2 \theta_3^2}{2P} g_2 d\theta_3$$
$$= \frac{q^2 Q_V^2}{2P} \int_{L_2}^{U_2} \theta_3^2 \left(\frac{1}{U_2 - L_2} \right) d\theta_3$$
$$= \frac{q^2 Q_V^2}{2P} \frac{(U_2^2 + U_2 L_2 + L_2^2)}{3} \tag{70}$$

The expected annual cost is

$$ETC_V(Q_V) = \frac{K_V D}{Q_V(1-p)} + \frac{h_V D}{Q_V(1-p)} (E(\Delta_1) + E(\Delta_2) + E(\Delta_3) + E(\Gamma_1) + E(\Gamma_2))$$
$$+ \frac{\eta D \ln\left(\frac{q_0}{q}\right)}{\delta_1(1-p)} + \frac{\eta D E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right]}{\delta_2(1-p)} + \frac{c_P D}{(1-p)} + \frac{c_r q \mu_{\theta_3} D}{(1-p)} \tag{71}$$

where $\mu_{\theta_3} = \frac{U_2+L_2}{2}$ and $E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right] \left(= \mu_{\ln(\theta_3/\theta_{3_0})}\right)$ is derived as follows:

$$E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right] = E\left[\ln(\theta_3) - \ln(\theta_{3_0})\right]$$
$$= E[\ln(\theta_3)] - E[\ln(\theta_{3_0})] \tag{72}$$

Here,

$$E[\ln(\theta_{3_0})] = \ln(\theta_{3_0}) \tag{73}$$

since θ_{3_0} is constant. However,

$$E[\ln(\theta_3)] \neq \ln[E(\theta_3)] \tag{74}$$

Therefore, we need to find $E(\ln(\theta_3))$ which is a function of θ_3.

$$
\begin{aligned}
E(\ln\theta_3) &= \int_{-\infty}^{\infty} \ln\theta_3 \cdot g_2 d\theta_3 \\
&= \int_{L_2}^{U_2} \ln\theta_3 \left(\frac{1}{U_2 - L_2}\right) d\theta_3 \\
&= \left(\frac{1}{U_2 - L_2}\right) \int_{L_2}^{U_2} \ln\theta_3 d\theta_3
\end{aligned} \tag{75}
$$

With integration by part, the expected value becomes

$$
E(\ln\theta_3) = \left(\frac{1}{U_2 - L_2}\right)\Big[\ln(U_2)U_2 - \ln(L_2)L_2 - U_2 + L_2\Big] \tag{76}
$$

Finally,

$$
E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right] = \left(\frac{1}{U_2 - L_2}\right)\Big[\ln(U_2)U_2 - \ln(L_2)L_2 - U_2 + L_2\Big] - \ln(\theta_{3_0}) \tag{77}
$$

To minimise $ETC_V(Q_V)$, we take first derivative with respect to Q_V

$$
\begin{aligned}
\frac{d(ETC_V)}{dQ_V} &= \frac{h_V\left(-2P\left(q^2\left(L_2^2 + L_2(U_2 - 3) + (U_2 - 3)U_2 + 3\right) + 3q(L_2 + U_2 - 2) + 3\right)\right)}{12(p-1)P} \\
&\quad + \frac{h_V Q_V^2\left(D(q(L_2 + U_2 - 2) + 2)\right) + 4K_V PD}{4(p-1)PQ_V^2}
\end{aligned} \tag{78}
$$

Taking the second derivative, we have

$$
\frac{d^2(ETC_V)}{dQ_V^2} = \frac{2K_V D}{(1-p)Q_V^3} \tag{79}
$$

All parameters are positive, and $\frac{d^2(ETC_V)}{dQ_V^2} > 0$. Therefore, there exists unique value for Q_V^* given as

$$
Q_V^* = \sqrt{\frac{12K_V PD}{2h_V P\left(q^2\left(L_2^2 + (L_2 + U_2)(U_2 - 3) + 3\right) + 3qA + 3\right) - 3h_V D(qA + 2)}} \tag{80}
$$

where $A = (L_2 + U_2 - 2)$.

3.1.3.3 Case 3: q is stochastic with standard uniform distribution and θ_3 is deterministic

The expected value of the total inventory of nondefective items during t_1 will be

$$\begin{aligned} E(\Delta_1) &= \int_{L_1}^{U_1} \frac{((1-q)P-D)Q_V^2}{2P^2} g_1 dq \qquad \left(g_1 = \frac{1}{U_1 - L_1}\right) \\ &= -\frac{Q_V^2 (P(L_1+U_1-2)+2D)}{4P^2} \end{aligned} \tag{81}$$

And during t_2

$$\begin{aligned} E(\Delta_2) &= \int_{L_1}^{U_1} \left(\frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2} \right) g_1 dq \\ &= \frac{Q_V^2 \theta_3 \left(P\left(L_1^2(\theta_3-2)+(L_1+U_1)((\theta_3-2)U_1+3)\right)\right)}{6P^2} \\ &- \frac{Q_V^2 \theta_3 \left(D\left(L_1^2\theta_3+(L_1+U_1)(\theta_3 U_1+3)\right)\right)}{6P^2} \end{aligned} \tag{82}$$

And during t_3

$$\begin{aligned} E(\Delta_3) &= \int_{L_1}^{U_1} \frac{1}{2} D \left(\frac{(P-D)q\theta_3 Q_V + ((1-q)P-D)Q_V}{PD} \right)^2 g_1 dq \\ &= \frac{Q_V^2 \left((P(-\theta_3 U_1+U_1-1)+\theta_3 U_1 D+D)^3\right)}{6P^2 D(U_1-L_1)(-P\theta_3+P+\theta_3 D)} \\ &- \frac{Q_V^2 \left(-(P(-L_1\theta_3+L_1-1)+L_1\theta_3 D+D)^3\right)}{6P^2 D(U_1-L_1)(-P\theta_3+P+\theta_3 D)} \end{aligned} \tag{83}$$

The inventory curve for defective items during the period t_1

$$\begin{aligned} E(\Gamma_1) &= \int_{L_1}^{U_1} \frac{q\theta_3 Q_V^2}{2P} g_1 dq \\ &= \frac{Q_V^2 \theta_3}{2P} \frac{1}{U_1-L_1} \int_{L_1}^{U_1} q dq \\ &= \frac{Q_V^2 \theta_3}{2P} \frac{U_1+L_1}{2} \end{aligned} \tag{84}$$

and during t_2, it will be

$$
\begin{aligned}
E(\Gamma_2) &= \int_{L_1}^{U_1} \frac{q^2 Q_V^2 \theta_3^2}{2P} g_1 dq \\
&= \frac{Q_V^2 \theta_3^2}{2P} \frac{1}{U_1 - L_1} \int_{L_1}^{U_1} q^2 dq \\
&= \frac{Q_V^2 \theta_3^3}{2P} \frac{(U_1^2 + U_1 L_1 + L_1^2)}{3} \qquad (85)
\end{aligned}
$$

The supplier's expected annual cost becomes

$$
\begin{aligned}
ETC_V(Q_V) = {} & \frac{K_V D}{Q_V(1-p)} + \frac{h_V D}{Q_V(1-p)} (E(\Delta_1) + E(\Delta_2) + E(\Delta_3) + E(\Gamma_1) + E(\Gamma_2)) \\
& + \frac{\eta D E\left[\ln\left(\frac{q_0}{q}\right)\right]}{\delta_1(1-p)} + \frac{\eta D \ln\left(\frac{\theta_3}{\theta_{3_0}}\right)}{\delta_2(1-p)} + \frac{c_P D}{(1-p)} + \frac{c_r \mu_q \theta_3 D}{(1-p)} \qquad (86)
\end{aligned}
$$

where $\mu_q = \frac{U_1 + L_1}{2}$ and $E\left[\ln\left(\frac{q_0}{q}\right)\right] \left(= \mu_{\ln(q_0/q)}\right)$ is derived as follows:

$$
\begin{aligned}
E\left[\ln\left(\frac{q_0}{q}\right)\right] &= E\left[\ln(q_0) - \ln(q)\right] \\
&= E[\ln q_0] - E[\ln q] \qquad (87)
\end{aligned}
$$

Here,

$$E[\ln q_0] = \ln(q_0) \qquad (88)$$

since q_0 is constant. However,

$$E[\ln q] \neq \ln[E(q)] \qquad (89)$$

Therefore, we need to find $E(\ln q)$ which is a function of q:

$$\begin{aligned}
E(\ln q) &= \int_{-\infty}^{\infty} \ln q \cdot g_1 dq \\
&= \int_{L_1}^{U_1} \ln q \left(\frac{1}{U_1 - L_1}\right) dq \\
&= \left(\frac{1}{U_1 - L_1}\right) \int_{L_1}^{U_1} \ln q dq
\end{aligned} \tag{90}$$

With integration by part, the expected value becomes

$$E(\ln q) = \left(\frac{1}{U_1 - L_1}\right) \Big[\ln(U_1)U_1 - \ln(L_1)L_1 - U_1 + L_1\Big] \tag{91}$$

Finally,

$$E\left[\ln\left(\frac{q_0}{q}\right)\right] = ln(q_0) - \left(\frac{1}{U_1 - L_1}\right)\Big[\ln(U_1)U_1 - \ln(L_1)L_1 - U_1 + L_1\Big] \tag{92}$$

To minimise $ETC_V(Q_V)$, we take first derivative with respect to Q_V

$$\begin{aligned}
\frac{d(ETC_V)}{dQ_V} &= \frac{6K_V D - h_V Q_V^2 \left((L_1^2(\theta_3 - 1)^2 + (L_1 + U_1)(\theta_3 - 1)((\theta_3 - 1)U_1 + 3) + 3)\right)}{6(p-1)Q_V^2} \\
&\quad + \frac{h_V D((\theta_3 - 1)(L_1 + U_1) + 2)}{4(p-1)P}
\end{aligned} \tag{93}$$

Taking the second derivative, we have

$$\frac{d^2(ETC_V)}{dQ_V^2} = \frac{2K_V D}{(1-p)Q_V^3} \tag{94}$$

All parameters are positive, and $\frac{d^2(ETC_V)}{dQ_V^2} > 0$. Therefore, there exists unique value for Q_V^* given as

$$Q_V^* = \sqrt{\frac{12PDK_V}{2h_V P\left(L_1^2 B^2 + B(BU_1 + 3)(L_1 + U_1) + 3\right) - 3h_V D(B(L_1 + U_1) + 2)}} \tag{95}$$

where $B = (\theta_3 - 1)$.

3.1.3.4 Case 4: qnd $\theta_3$re both stochastic with standard uniform distribution

The expected value of the total inventory of nondefective items during t_1 will be

$$\begin{aligned} E(\Delta_1) &= \int_{L_2}^{U_2}\int_{L_q}^{U_q} \frac{((1-q)P-D)Q_V^2}{2P^2} g_2 g_1\, dq\, d\theta_3 \\ &= -\frac{Q_V^2(P(L_q+U_q-2)+2D)}{4P^2} \end{aligned} \tag{96}$$

Then, the expected value of total inventory during rework process t_2 is

$$\begin{aligned} E(\Delta_2) &= \int_{L_2}^{U_2}\int_{L_q}^{U_q} \left(\frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2}\right) g_2 g_1\, dq\, d\theta_3 \\ &= -\frac{Q_V^2 U_q^3\left(P\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2\right)-D\left(L_2^2+L_2U_2+U_2^2\right)\right)}{18P^2(L_q-U_q)} \\ &- \frac{Q_V^2 U_q^2(L_2+U_2)(P-D)}{4P^2(L_q-U_q)} + \frac{Q_V^2 L_q^2(L_2+U_2)(P-D)}{4P^2(L_q-U_q)} \\ &+ \frac{Q_V^2 L_q^3\left(P\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2\right)-D\left(L_2^2+L_2U_2+U_2^2\right)\right)}{18P^2(L_q-U_q)} \end{aligned} \tag{97}$$

During t_3, the total inventory becomes

$$\begin{aligned} E(\Delta_3) &= \int_{L_2}^{U_2}\int_{L_q}^{U_q} \frac{1}{2}D\left(\frac{(P-D)q\theta_3 Q_V + ((1-q)P-D)Q_V}{PD}\right)^2 g_2 g_1\, dq\, d\theta_3 \\ &= \frac{-Q_V^2 U_q^3\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D(L_q-U_q)} \\ &- \frac{Q_V^2 U_q^3\left(P\left(-2L_2^2+L_2(3-2U_2)+U_2(3-2U_2)\right)+D\left(L_2^2+L_2U_2+U_2^2\right)\right)}{18P^2(L_q-U_q)} \\ &- \frac{Q_V^2 U_q\left(U_q(P-D)(P(L_2+U_2-2)-D(L_2+U_2))+2(P-D)^2\right)}{4P^2D(L_q-U_q)} \\ &+ \frac{Q_V^2 L_q^3\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D(L_q-U_q)} \\ &+ \frac{Q_V^2 L_q^3\left(P\left(-2L_2^2+L_2(3-2U_2)+U_2(3-2U_2)\right)+D\left(L_2^2+L_2U_2+U_2^2\right)\right)}{18P^2(L_q-U_q)} \\ &+ \frac{Q_V^2 L_q\left(L_q(P-D)(P(L_2+U_2-2)-D(L_2+U_2))+2(P-D)^2\right)}{4P^2D(L_q-U_q)} \end{aligned} \tag{98}$$

The expected value of total inventory during t_1 will be

$$
\begin{aligned}
E(\Gamma_1) &= \int_{L_2}^{U_2}\int_{L_q}^{U_q} \frac{q\theta_3 Q_V^2}{2P} g_1 g_2 dq d\theta_3 \\
&= \frac{Q_V^2}{2P}\left(\frac{U_q+L_q}{2}\right)\left(\frac{U_2+L_2}{2}\right)
\end{aligned} \tag{99}
$$

and during t_2

$$
\begin{aligned}
E(\Gamma_2) &= \int_{L_2}^{U_2}\int_{L_q}^{U_q} \frac{q^3 Q_V^2 \theta_3^3}{2P} g_1 g_2 dq d\theta_3 \\
&= \frac{Q_V^2}{2P}\left(\frac{U_q^2+U_qL_q+L_q^2}{3}\right)\left(\frac{U_2^2+U_2L_2+L_2^2}{3}\right)
\end{aligned} \tag{100}
$$

The expected annual cost becomes

$$
\begin{aligned}
ETC_V(Q_V) &= \frac{K_V D}{Q_V(1-p)} + \frac{h_V D}{Q_V(1-p)}(E(\Delta_1)+E(\Delta_2)+E(\Delta_3)+E(\Gamma_1)+E(\Gamma_2)) \\
&+ \frac{\eta D \mu_{\ln(q_0/q)}}{\delta_1(1-p)} + \frac{\eta D \mu_{\ln(\theta_3/\theta_{3_0})}}{\delta_2(1-p)} + \frac{c_P D}{(1-p)} + \frac{c_r \mu_q \mu_{\theta_3} D}{(1-p)}
\end{aligned} \tag{101}
$$

To minimise $ETC_V(Q_V)$, we take first derivative with respect to Q_V

$$
\begin{aligned}
\frac{d(ETC_V)}{dQ_V} &= \frac{18K_V D - h_V Q_V^2\left(L_q^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)\right)}{18(p-1)Q_V^2} \\
&- \frac{h_V\left(L_q\left(2U_q(L_2^2+(U_2-3)U_2+3)+L_2(2U_q(U_2-3)+9)+9(U_2-2)\right)\right)}{36(p-1)} \\
&- \frac{h_V\left(2U_q^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_q(L_2+U_2-2)+18\right)}{36(p-1)} \\
&+ \frac{h_V\left(D(L_2+U_2-2)(L_q+U_q)+4D\right)}{8(p-1)P}
\end{aligned} \tag{102}
$$

Taking the second derivative, we have

$$
\frac{d^2(ETC_V)}{dQ_V^2} = \frac{2K_V D}{(1-p)Q_V^3} \tag{103}
$$

All parameters are positive, and $\frac{d^2(ETC_V)}{dQ_V^2} > 0$. Therefore, there exists unique value for Q_V^* given as

$$Q_V^* = \sqrt{\frac{72K_V PD}{h_V\left(2P\left(2L_q^2C + L_q\left(2U_qC + 9A\right) + 2U_q^2C + 9U_qA + 18\right) - 9DA(L_q + U_q) - 36D\right)}} \tag{104}$$

where $C = (L_2^2 + L_2(U_2 - 3) + (U_2 - 3)U_2 + 3)$ and $A = (L_2 + U_2 - 2)$.

3.2. Integrated Continuous Review Inventory Models

3.2.1. Integrated Models with lost-sales case

For the integrated case, we will analyse the buyer's cost under deterministic demand case with four cases of the supplier and then buyer's cost with stochastic demand with those four cases. That is, we will have eight integrated models in total.

The supplier's total cost in a cycle when q and θ_3 are deterministic is

$$\begin{aligned} TC_V(Q) = K_V + h_V\left(\frac{Q^2(q(\theta_3 - 1) + 1)(P(q(\theta_3 - 1) + 1) - D)}{2PD}\right) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) \\ + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_P Q + c_r qQ\theta_3 \end{aligned} \tag{105}$$

and the buyer's total cost with deterministic demand in a supplier's cycle is

$$TC_B(Q) = K_B + F + cQ + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D}\right) \tag{106}$$

The total cost for integrated system is,

$$\begin{aligned} TC(Q) &= TC_V(Q) + TC_B(Q) \\ TC(Q) &= K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3 - 1) + 1)(P(q(\theta_3 - 1) + 1) - D)}{2PD}\right) + c_P Q \\ &\quad + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D}\right) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \\ &\quad + c_r Qq\theta_3 + cQ \end{aligned} \tag{107}$$

We have cycle time $T = \frac{Q(1-p)}{D}$, so the average total annual cost would be

$$\begin{aligned} TC(Q) = \Bigg(& K_V + K_B + F + h_V \left(\frac{Q^2(q(\theta_3 - 1) + 1)(P(q(\theta_3 - 1) + 1) - D)}{2PD} \right) \\ & + h_B \left(\frac{1}{2} \frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} \right) + \frac{Q\eta}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \\ & + c_P Q + c_r Q q \theta_3 + cQ \Bigg) \frac{D}{Q(1-p)} \end{aligned} \tag{108}$$

To minimise $TC(Q)$, we take first derivative with respect to Q

$$\begin{aligned} \frac{d(TC)}{dQ} = & \frac{2D(F + K_B + K_V) + h_B\left(p^2 - 1\right) Q^2}{2(p-1)Q^2} \\ & - \frac{h_V(q(\theta_3 - 1) + 1)(Pq(\theta_3 - 1) + P - D)}{2(p-1)P} \end{aligned} \tag{109}$$

Taking the second derivative, we have

$$\frac{d^2(TC)}{dQ^2} = \frac{2D(F + K_B + K_V)}{(1-p)Q^3} \tag{110}$$

All parameters are positive, and $\frac{d^2(TC)}{dQ^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-2PD(F + K_B + K_V)}{h_B\left(p^2 - 1\right)P + h_V(q(\theta_3 - 1) + 1)(P(q(1 - \theta_3) - 1) + D)}} \tag{111}$$

As the second model, the supplier's expected total cost in a cycle when q is deterministic and θ_3 is stochastic with standard uniform distribution defined as

$$\begin{aligned} ETC_V(Q) = & K_V + h_V \left(\frac{Q^2\left(q^2\left(L_2^2 + (L_2 + U_2)(U_2 - 3) + 3\right) + 3q(L_2 + U_2 - 2) + 3\right)}{6D} \right) \\ & - h_V \left(\frac{Q^2(q(L_2 + U_2 - 2) + 2)}{4P} \right) + \frac{\eta Q}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2} \mu_{\ln(\theta_3/\theta_{3_0})} \\ & + c_P Q + c_r q Q \mu_{\theta_3} \end{aligned} \tag{112}$$

and the buyer's expected total cost with deterministic demand in a supplier's cycle is

$$ETC_B(Q) = K_B + F + cQ + h_B \left(\frac{1}{2} \frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} \right) \tag{113}$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q) &= ETC_V(Q) + ETC_B(Q) \\
ETC(Q) &= K_V + K_B + F + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(q^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right) + 3q(L_2+U_2-2) + 3\right)}{6D}\right) \\
&\quad - h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&\quad + c_PQ + cQ + c_rqQ\mu_{\theta_3} \qquad (114)
\end{aligned}
$$

The expected total annual cost becomes

$$
\begin{aligned}
ETC(Q) &= \Bigg(K_V + K_B + F + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(q^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right) + 3q(L_2+U_2-2) + 3\right)}{6D}\right) \\
&\quad - h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&\quad + c_PQ + cQ + c_rqQ\mu_{\theta_3}\Bigg)\frac{D}{Q(1-p)} \qquad (115)
\end{aligned}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{d(ETC)}{dQ} &= \frac{D\left(4P(F+K_B+K_V) + h_VQ^2(q(L_{\theta_3}+U_{\theta_3}-2)+2)\right)}{4(p-1)PQ^2} + \frac{h_B(p+1)}{2} \\
&\quad - \frac{h_V\left(q^2\left(L_2^2 + (L_2+U_2)(U_2-3) + 3\right) + 3q(L_2+U_2-2) + 3\right)}{6(p-1)} \qquad (116)
\end{aligned}
$$

Taking the second derivative, we have

$$
\frac{d^2(TC)}{dQ^2} = \frac{2D(F+K_B+K_V)}{(1-p)Q^3} \qquad (117)
$$

All parameters are positive, and $\frac{d^2(TC)}{dQ^2} > 0$. Therefore, there exists unique value for Q^* given as

$$
Q^* = \sqrt{\frac{-12PD(F+K_B+K_V)}{2P(3h_B(p^2-1) - h_V(q^2C + 3qA + 3)) + 3h_VD(qA+2)}} \qquad (118)
$$

where $C = (L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3)$ and $A = (L_2 + U_2 - 2)$.

For the model where q is stochastic with standard uniform distribution and θ_3 is deterministic, the supplier's cost function per cycle will be

$$
\begin{aligned}
ETC_V(Q) = K_V + h_V &\left(\frac{Q^2 \left(L_1^2(\theta_3-1)^2 + (L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3 \right)}{6D} \right) \\
&+ h_V \left(\frac{Q^2\left(-3(\theta_3-1)D(L_1+U_1)-6D\right)}{12PD} \right) + c_P Q + c_r Q \theta_3 \mu_q \\
&+ \frac{\eta Q}{\delta_1} \mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \qquad (119)
\end{aligned}
$$

and the buyer's expected total cost with deterministic demand in a supplier's cycle is

$$
ETC_B(Q) = K_B + F + cQ + h_B \left(\frac{(Q(1-p))^2}{2D} + \frac{Q^2 p(1-p)}{D} \right) \qquad (120)
$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q) &= ETC_V(Q) + ETC_B(Q) \\
ETC(Q) &= K_V + K_B + F + cQ + h_B \left(\frac{(Q(1-p))^2}{2D} + \frac{Q^2 p(1-p)}{D} \right) \\
&+ h_V \left(\frac{Q^2 \left((\theta_3-1)(L_1^2(\theta_3-1) + (L_1+U_1)((\theta_3-1)U_1+3)) + 3 \right)}{6D} \right) \\
&+ h_V \left(\frac{Q^2\left((1-\theta_3)(L_1+U_1)-2\right)}{4P} \right) + c_P Q + c_r Q \theta_3 \mu_q \\
&+ \frac{\eta Q}{\delta_1} \mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \qquad (121)
\end{aligned}
$$

With the buyer's cycle length, the expected total annual cost is

$$
\begin{aligned}
ETC(Q) = \Bigg(K_V + K_B + F + cQ + h_B &\left(\frac{(Q(1-p))^2}{2D} + \frac{Q^2 p(1-p)}{D} \right) \\
&+ h_V \left(\frac{Q^2 \left((\theta_3-1)(L_1^2(\theta_3-1) + (L_1+U_1)((\theta_3-1)U_1+3)) + 3 \right)}{6D} \right) \\
&+ h_V \left(\frac{Q^2\left((1-\theta_3)(L_1+U_1)-2\right)}{4P} \right) + c_P Q + c_r Q \theta_3 \mu_q \\
&+ \frac{\eta Q}{\delta_1} \mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \Bigg) \frac{D}{Q(1-p)} \qquad (122)
\end{aligned}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$\frac{d(ETC)}{dQ}=\frac{h_V\left(2P\left(L_1^2(\theta_3-1)^2+(L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)\right)}{12(1-p)P}\\-\frac{h_V\left(D((\theta_3-1)(L_1+U_1)+2)\right)}{4(1-p)P}+\frac{2D(F+K_B+K_V)+h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2} \tag{123}$$

Taking the second derivative, we have

$$\frac{d^2(ETC)}{dQ^2}=\frac{2D(F+K_B+K_V)}{(1-p)Q^3} \tag{124}$$

All parameters are positive, and $\frac{d^2(ETC)}{dQ^2}>0$. Therefore, there exists unique value for Q^* given as

$$Q^*=\sqrt{\frac{-12PD(F+K_B+K_V)}{6h_B\left(p^2-1\right)P-h_V\left(2P\left(L_1^2B^2+(L_1+U_1)B(BU_1+3)+3\right)-3BD(L_1+U_1)-6D\right)}} \tag{125}$$

where $B=(\theta_3-1)$.

When q and θ_3 are both stochastic with standard uniform distribution, the supplier's cost function per cycle becomes

$$\begin{aligned}ETC_V(Q)=&K_V+c_PQ+c_rQ\mu_q\mu_{\theta_3}+\frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)}+\frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})}\\&+h_V\left(\frac{Q^2L_q^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D}\right)\\&+h_V\left(\frac{Q^2L_1\left(2L_2^2U_q+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36D}\right)\\&+h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36D}\right)\\&-h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right)\end{aligned} \tag{126}$$

and the buyer's expected total cost with deterministic demand in a supplier's cycle is

$$ETC_B(Q)=K_B+F+cQ+h_B\left(\frac{(Q(1-p))^2}{2D}+\frac{Q^2p(1-p)}{D}\right) \tag{127}$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q) &= ETC_V(Q) + ETC_B(Q) \\
ETC(Q) &= K_B + K_V + F + h_B\left(\frac{(Q(1-p))^2}{2D} + \frac{Q^2 p(1-p)}{D}\right) \\
&\quad + h_V\left(\frac{Q^2 L_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right)}{18D}\right) + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&\quad + h_V\left(\frac{Q^2 L_1\left(2L_2^2 U_1 + L_2(2U_1(U_2-3)+9) + 2U_1((U_2-3)U_2+3) + 9(U_2-2)\right)}{36D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right) + 9U_1(L_2+U_2-2) + 18\right)}{36D}\right) \\
&\quad - h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right) + cQ + c_P Q + c_r Q\mu_q\mu_{\theta_3}
\end{aligned}
\tag{128}
$$

The expected total annual cost is

$$
\begin{aligned}
ETC(Q) = \Bigg(& K_B + K_V + F + h_B\left(\frac{(Q(1-p))^2}{2D} + \frac{Q^2 p(1-p)}{D}\right) \\
&+ h_V\left(\frac{Q^2 L_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right)}{18D}\right) \\
&+ h_V\left(\frac{Q^2 L_1\left(2L_2^2 U_1 + L_2(2U_1(U_2-3)+9) + 2U_1((U_2-3)U_2+3) + 9(U_2-2)\right)}{36D}\right) \\
&+ h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right) + 9U_1(L_2+U_2-2) + 18\right)}{36D}\right) \\
&- h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right) + cQ + c_P Q + c_r Q\mu_q\mu_{\theta_3} \\
&+ \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})}\Bigg)\frac{D}{Q(1-p)}
\end{aligned}
\tag{129}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{d(ETC)}{dQ} &= \frac{-h_V L_q^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right)}{18(p-1)} \\
&\quad - \frac{h_V L_1\left(2L_2^2 U_1 + L_2(2U_1(U_2-3)+9) + 2U_1((U_2-3)U_2+3) + 9(U_2-2)\right)}{36(p-1)} \\
&\quad - \frac{h_V\left(2U_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right) + 9U_1(L_2+U_2-2) + 18\right)}{36(p-1)} \\
&\quad + \frac{h_V Q^2\left(D(L_2+U_2-2)(L_1+U_1) + 4D\right) + 8PD(F+K_B+K_V) + 4h_B\left(p^2-1\right)PQ^2}{8(p-1)PQ^2}
\end{aligned}
\tag{130}
$$

Taking the second derivative, we have

$$\frac{d^2(ETC)}{dQ^2} = \frac{2D(F+K_B+K_V)}{(1-p)Q^3} \tag{131}$$

All parameters are positive, and $\frac{d^2(ETC)}{dQ^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-72PD(F+K_B+K_V)}{36h_B(p^2-1)P - h_V\left(2P\left(2C(L_1^2+U_1^2)+L_1(2U_1C+9A)+9U_1A+18\right)-9D(AB-4)\right)}} \tag{132}$$

where $C = (L_2^2 + L_2(U_2-3)+(U_2-3)U_2+3)$, $B = L_1+U_1$ and $A = (L_2+U_2-2)$.

After the models for deterministic demand at buyer's, it is time to analyse for stochastic demand with four cases for supplier.

The supplier's total cost per cycle for deterministic q and θ_3 is

$$\begin{aligned} TC_V(Q) = K_V + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) \\ + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_PQ + c_rqQ\theta_3 \end{aligned} \tag{133}$$

and the buyer's total average cost with stochastic demand in a supplier's cycle is

$$\begin{aligned} TC_B(Q,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\ + F + cQ + c_l\bar{s}(r) \end{aligned} \tag{134}$$

The total cost for integrated system is,

$$\begin{aligned} TC(Q,r) &= TC_V(Q) + TC_B(Q,r) \\ TC(Q,r) &= K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + c_PQ \\ &+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) + c_l\bar{s}(r) \\ &+ \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_rQq\theta_3 + cQ \end{aligned} \tag{135}$$

We have cycle time $T = \frac{Q(1-p)}{D}$, so the expected total annual cost would be

$$ETC(Q,r) = \left(K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right)\right.$$
$$+cQ + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)$$
$$\left.+\frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_PQ + c_rQq\theta_3 + c_l\bar{s}(r)\right)\frac{D}{Q(1-p)} \tag{136}$$

To minimise $ETC(Q,r)$, we take first derivative with respect to Q and r,

$$\frac{\partial(ETC)}{\partial Q} = \frac{2D(F+K_B+K_V)+2c_lD\bar{s}(r)+h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2} + \frac{h_V(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2(1-p)P} \tag{137}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D\left(F+K_B+K_V+c_l\bar{s}(r)\right)}{(1-p)Q^3} \tag{138}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-2PD\left(F+c_l\bar{s}(r)+K_B+K_V\right)}{\left(h_B\left(p^2-1\right)P+h_V(q(\theta_3-1)+1)(P(q(1-\theta_3)-1)+D)\right)}} \tag{139}$$

The derivative of $ETC(Q,r)$ with respect to r is

$$\frac{\partial(ETC)}{\partial r} = \frac{\bar{s}'(r)(h_B(p-1)Q-c_lD)+h_B(p-1)Q}{(p-1)Q} \tag{140}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial r^2} = \frac{\bar{s}''(r)(h_B(p-1)Q-c_lD)}{(p-1)Q} \tag{141}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial r^2} > 0$. Therefore, there exists unique value for

r^* given as

$$
\begin{aligned}
1-F(r^*) &= \frac{h_B Q(1-p)}{c_l D + h_B Q(1-p)} \\
r^* &= F^{-1}\left(1 - \frac{h_B Q(1-p)}{c_l D + h_B Q(1-p)}\right) \qquad (142)
\end{aligned}
$$

For the rest of the models, r^* equation will be the same since the integrated cost function is changing only by constant.

In the model where q is deterministic and θ_3 is stochastic with standard uniform distribution, the supplier's cost function per cycle will be

$$
\begin{aligned}
ETC_V(Q) = K_V &- h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&+ h_V\left(\frac{Q^2\left(q^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+3q(L_2+U_2-2)+3\right)}{6D}\right) \\
&+ c_P Q + c_r q Q \mu_{\theta_3} \qquad (143)
\end{aligned}
$$

and the buyer's total average cost with stochastic demand in a supplier's cycle is

$$
\begin{aligned}
ETC_B(Q,r) = K_B + h_B &\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
&+ F + cQ + c_l\bar{s}(r) \qquad (144)
\end{aligned}
$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_B + K_V + F + h_B\left(\frac{(1-p)Q(-2\mu_L+pQ+Q+2\bar{s}(r)+2r)}{2D}\right) \\
&+ h_V\left(\frac{Q^2\left(q^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+3q(L_2+U_2-2)+3\right)}{6D}\right) \\
&- h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + c_l\bar{s}(r) + cQ + c_P Q + c_r q Q \mu_{\theta_3} \\
&+ \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \qquad (145)
\end{aligned}
$$

The expected total annual cost is

$$ETC(Q,r) = \Bigg(K_B + K_V + F + h_B\left(\frac{(1-p)Q(-2\mu_L + pQ + Q + 2\bar{s}(r) + 2r)}{2D}\right)$$
$$+ h_V\left(\frac{Q^2\left(q^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right) + 3q(L_2+U_2-2)+3\right)}{6D}\right)$$
$$- h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + c_l\bar{s}(r) + cQ + c_PQ + c_rqQ\mu_{\theta_3}$$
$$+ \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})}\Bigg)\frac{D}{Q(1-p)} \tag{146}$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$\frac{\partial(ETC)}{\partial Q} = \frac{D\left(4P(F+K_B+K_V) + h_VQ^2(q(L_2+U_2-2)+2)\right) + 4c_lPD\bar{s}(r)}{4(p-1)PQ^2}$$
$$- \frac{h_V\left(q^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right) + 3q(L_2+U_2-2)+3\right)}{6(p-1)}$$
$$+ \frac{h_B(p+1)}{2} \tag{147}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(F + K_B + K_V + c_l\bar{s}(r))}{(1-p)Q^3} \tag{148}$$

All parameters are positive, and $\frac{d^2(ETC)}{dQ^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-12PD(F + c_l\bar{s}(r) + K_B + K_V)}{6h_B(p^2-1)P - 2h_VP\left(q^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right) + 3qA + 3\right) + 3h_VD(2+qA)}} \tag{149}$$

where $C = (L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3)$, $A = (U_2 + L_2 - 2)$ and

$$r^* = F^{-1}\left(1 - \frac{h_BQ(1-p)}{c_lD + h_BQ(1-p)}\right) \tag{150}$$

For the next model, q is stochastic with standard uniform distribution and θ_3 is deterministic. Therefore, supplier's cost function per cycle is

$$
\begin{aligned}
ETC_V(Q) = K_V + h_V \left(\frac{Q^2 \left(L_1^2(\theta_3-1)^2 + (L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3 \right)}{6D} \right) \\
+ h_V \left(\frac{Q^2 \left(-3(\theta_3-1)D(L_1+U_1)-6D\right)}{12PD} \right) + c_P Q + c_r Q \theta_3 \mu_q \\
+ \frac{\eta Q}{\delta_1} \mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right)
\end{aligned}
\tag{151}
$$

and the buyer's expected total cost with stochastic demand in the supplier's cycle is

$$
\begin{aligned}
ETC_B(Q,r) = K_B + h_B \left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D} \right) \\
+ F + cQ + c_l \bar{s}(r)
\end{aligned}
\tag{152}
$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_V + K_B + F + cQ + c_P Q + c_r Q \theta_3 \mu_q + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \\
&+ h_V \left(\frac{Q^2 \left(L_1^2(\theta_3-1)^2 + (L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3 \right)}{6D} \right) \\
&+ h_V \left(\frac{Q^2 \left(-3(\theta_3-1)D(L_1+U_1)-6D\right)}{12PD} \right) + c_l \bar{s}(r) \\
&+ h_B \left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D} \right)
\end{aligned}
\tag{153}
$$

The expected total annual cost is

$$
\begin{aligned}
ETC(Q,r) = \Bigg(& K_V + K_B + F + cQ + c_P Q + c_r Q \theta_3 \mu_q + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \\
&+ h_V \left(\frac{Q^2 \left(L_1^2(\theta_3-1)^2 + (L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3 \right)}{6D} \right) \\
&+ h_V \left(\frac{Q^2 \left(-3(\theta_3-1)D(L_1+U_1)-6D\right)}{12PD} \right) + c_l \bar{s}(r) \\
&+ h_B \left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D} \right) \Bigg) \\
&\frac{D}{Q(1-p)}
\end{aligned}
\tag{154}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$\frac{\partial(ETC)}{\partial Q} = \frac{2D(F+K_B+K_V)+h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2} - \frac{h_V\left(L_1^2(\theta_3-1)^2+(L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6(p-1)} + \frac{h_V Q^2\left((\theta_3-1)D(L_1+U_1)+2D\right)+4c_l PD\bar{s}(r)}{4(p-1)PQ^2} \tag{155}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D\left(F+K_B+K_V+c_l\bar{s}(r)\right)}{(1-p)Q^3} \tag{156}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-12PD\left(F+c_l\bar{s}(r)+K_B+K_V\right)}{6h_B(p^2-1)P-h_V\left(2P\left(L_1^2B^2+(L_1+U_1)B(BU_1+3)+3\right)-3BD(L_1+U_1)-6D\right)}} \tag{157}$$

where $B=(\theta_3-1)$ and

$$r^* = F^{-1}\left(1-\frac{h_B Q(1-p)}{c_l D+h_B Q(1-p)}\right) \tag{158}$$

In the final model, where q and θ_3 are both stochastic with standard uniform distribution, supplier's cost function per cycle is

$$\begin{aligned}
ETC_V(Q) = {} & K_V + c_P Q + c_r Q\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
& + h_V\left(\frac{Q^2L_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D}\right) \\
& + h_V\left(\frac{Q^2L_1\left(2L_2^2U_1+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36D}\right) \\
& + h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36D}\right) \\
& - h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right)
\end{aligned} \tag{159}$$

and the buyer's expected total cost with deterministic demand in a supplier's cycle is

$$\begin{aligned}ETC_B(Q,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)\\ + F + cQ + c_l\bar{s}(r) \qquad (160)\end{aligned}$$

The expected total cost for integrated system is,

$$\begin{aligned}&ETC(Q,r) = ETC_V(Q) + ETC_B(Q,r)\\ &ETC(Q,r) = K_B + K_V + F + cQ + c_PQ + c_rQ\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})}\\ &\quad - h_B\left(\frac{(p-1)Q(-2M_L+pQ+Q+2\bar{s}(r)+2r)}{2D}\right) + c_l\bar{s}(r)\\ &\quad + h_V\left(\frac{Q^2L_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D}\right)\\ &\quad + h_V\left(\frac{Q^2L_1\left(2L_2^2U_1+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36D}\right)\\ &\quad + h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36D}\right)\\ &\quad - h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right) \qquad (161)\end{aligned}$$

The expected total annual cost becomes

$$\begin{aligned}ETC(Q) = \Bigg(&K_B + K_V + F + cQ + c_PQ + c_rQ\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})}\\ &- h_B\left(\frac{(p-1)Q(-2M_L+pQ+Q+2\bar{s}(r)+2r)}{2D}\right) + c_l\bar{s}(r)\\ &+ h_V\left(\frac{Q^2L_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D}\right)\\ &+ h_V\left(\frac{Q^2L_1\left(2L_2^2U_1+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36D}\right)\\ &+ h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36D}\right)\\ &- h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right)\frac{D}{Q(1-p)} \qquad (162)\end{aligned}$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$\begin{aligned}\frac{\partial(ETC)}{\partial Q} &= \frac{-h_V L_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18(p-1)} + \frac{c_l D\bar{s}(r)}{(p-1)Q^2} \\ &- \frac{h_V L_1\left(2L_2^2U_1+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36(p-1)} \\ &- \frac{h_V\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36(p-1)} \\ &+ \frac{h_V Q^2\left(D(L_2+U_2-2)(L_1+U_1)+4D\right)+8PD(F+K_B+K_V)+4h_B\left(p^2-1\right)PQ^2}{8(p-1)PQ^2}\end{aligned} \tag{163}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D\left(F+K_B+K_V+c_l\bar{s}(r)\right)}{(1-p)Q^3} \tag{164}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-72PD\left(F+c_l\bar{s}(r)+K_B+K_V\right)}{36h_B(p^2-1)P-h_V\left(2P\left(2C(L_1^2+U_1^2)+L_1\left(2U_1C+9A\right)+9U_1A+18\right)-9D(AB-4)\right)}} \tag{165}$$

where $C=(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3)$, $B=(L_1+U_1)$ and $A=(U_2+L_2-2)$, and

$$r^* = F^{-1}\left(1-\frac{h_B Q(1-p)}{c_l D+h_B Q(1-p)}\right) \tag{166}$$

3.2.2. Integrated Models with partial backordering

When there is deterministic demand, there is no shortage occurring. Therefore in this section we only examine the stochastic demand. This time, there is partial backordering at buyer's side. As the first model, supplier's total cost per cycle for deterministic q and θ_3 is

$$\begin{aligned}TC_V(Q) &= K_V + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) \\ &+ \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_P Q + c_r Q q\theta_3\end{aligned} \tag{167}$$

and the buyer's total average cost with stochastic demand and partial backordering is

$$TC_B(Q,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\ +F+cQ+d+c_b\bar{B}(r)+c_l\bar{s}(r) \tag{168}$$

The total cost for integrated system is,

$$\begin{aligned}
TC(Q,r) &= TC_V(Q) + TC_B(Q,r) \\
TC(Q,r) &= K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + c_PQ \\
&\quad + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) + c_rQq\theta_3 \\
&\quad + cQ + d + c_b\bar{B}(r) + c_l\bar{s}(r) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)
\end{aligned} \tag{169}$$

We have cycle time $T = \frac{Q(1-p)}{D}$, so the expected total annual cost would be

$$\begin{aligned}
ETC(Q,r) = \Bigg(& K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + c_PQ \\
&+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) + c_rQq\theta_3 \\
&+ cQ + d + c_b\bar{B}(r) + c_l\bar{s}(r) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\Bigg)\frac{D}{Q(1-p)}
\end{aligned} \tag{170}$$

Proposition 3. *$ETC(Q,r)$ annual total cost is strictly convex in (Q,r).*

The proof is presented in Appendix A.

To minimise $ETC(Q,r)$, we take first derivative with respect to Q and r,

$$\begin{aligned}
\frac{\partial(ETC)}{\partial Q} = &\frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)+h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2} \\
&- \frac{h_V(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2(p-1)P}
\end{aligned} \tag{171}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+K_B+K_V+d)}{(1-p)Q^3} \tag{172}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-2PD(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{h_B\,(p^2-1)\,P+h_V(q(\theta_3-1)+1)(P(q(-\theta_3)+q-1)+D)}} \tag{173}$$

The derivative of $ETC(Q,r)$ with respect to r is

$$\frac{\partial(ETC)}{\partial r} = \frac{Dc_b\bar{B}'(r)+Dc_l\bar{s}'(r)+h_B(1-p)Q(1+\bar{s}'(r))}{(1-p)Q} \tag{174}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial r^2} = \frac{Dc_b\bar{B}''(r)+\bar{s}''(r)(h_B(1-p)Q+c_lD)}{(1-p)Q} \tag{175}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial r^2} > 0$. Therefore, there exists unique value for r^* given as

$$\begin{aligned} F(r) &= \left(1-\frac{h_B(1-p)Q}{\Big(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\Big)\Big(D(c_l-c_b)+h_BQ(1-p)\Big)+\Big(c_lD+h_BQ(1-p)\Big)}\right)\\ r^* &= F^{-1}\left(1-\frac{h_B(1-p)Q}{\Big(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\Big)\Big(D(c_l-c_b)+h_BQ(1-p)\Big)+\Big(c_lD+h_BQ(1-p)\Big)}\right) \end{aligned} \tag{176}$$

Similar to the previous case, the rest of the models, r^* equation will be the same since the integrated cost function is changing only by constant. In the model where q is deterministic and θ_3 is stochastic with standard uniform distribution, the supplier's cost function per cycle will be

$$
\begin{aligned}
ETC_V(Q) = K_V - h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
+ h_V\left(\frac{Q^2\left(q^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+3q(L_2+U_2-2)+3\right)}{6D}\right) \\
+ c_PQ + c_rqQ\mu_{\theta_3}
\end{aligned} \tag{177}
$$

and the buyer's total average cost with stochastic demand in a supplier's cycle is

$$
\begin{aligned}
ETC_B(Q,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
+ F + cQ + d + c_b\bar{B}(r) + c_l\bar{s}(r)
\end{aligned} \tag{178}
$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
&+ h_V\left(\frac{Q^2(q^2(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3)+3q(L_2+U_2-2)+3)}{6D}\right) \\
&- h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + K_V + F + cQ + c_PQ + c_rqQ\mu_{\theta_3} \\
&+ \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} + d + c_b\bar{B}(r) + c_l\bar{s}(r)
\end{aligned} \tag{179}
$$

The expected total annual cost is

$$
\begin{aligned}
ETC(Q,r) = \Bigg(K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
+ h_V\left(\frac{Q^2(q^2(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3)+3q(L_2+U_2-2)+3)}{6D}\right) \\
- h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + K_V + F + cQ + c_PQ + c_rqQ\mu_{\theta_3} \\
+ \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} + d + c_b\bar{B}(r) + c_l\bar{s}(r)\Bigg)\frac{D}{Q(1-p)}
\end{aligned} \tag{180}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$\frac{\partial(ETC)}{\partial Q} = \frac{D\left(4P(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)+h_VQ^2(q(L_2+U_2-2)+2)\right)}{4(p-1)PQ^2}$$
$$-\frac{h_V\left(q^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+3q(L_2+U_2-2)+3\right)}{6(p-1)}$$
$$+\frac{h_B(p+1)}{2} \tag{181}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{(1-p)Q^3} \tag{182}$$

All parameters are positive, and $\frac{d^2(ETC)}{dQ^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-12PD\,(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{6h_B(p^2-1)P-2h_VP\,(q^2C+3qA+3)+3h_VD(qA+2)}} \tag{183}$$

where $C = (L_2^2+L_2(U_2-3)+(U_2-3)U_2+3)$ and $A = (U_2+L_2-2)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1-\frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right) \tag{184}$$

For the next model, q is stochastic with standard uniform distribution and θ_3 is deterministic. Therefore, supplier's cost function per cycle is

$$ETC_V(Q) = K_V + h_V\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2+(L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6D}\right)$$
$$+h_V\left(\frac{Q^2\left((1-\theta_3)(L_1+U_1)-2\right)}{4P}\right)+c_PQ+c_rQ\theta_3\mu_q$$
$$+\frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)}+\frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \tag{185}$$

and the buyer's expected total cost with stochastic demand in the supplier's cycle is

$$
\begin{aligned}
ETC_B(Q,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
+F+cQ+d+c_b\bar{B}(r)+c_l\bar{s}(r) \qquad (186)
\end{aligned}
$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q)+ETC_B(Q,r)\\
ETC(Q,r) &= K_V+K_B+F+cQ+c_PQ+c_rQ\theta_3\mu_q+\frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)}+\frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\\
&+h_V\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2+(L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6D}\right)\\
&+h_V\left(\frac{Q^2\left(-3(\theta_3-1)D(L_1+U_1)-6D\right)}{12PD}\right)+d+c_b\bar{B}(r)+c_l\bar{s}(r)\\
&+h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \qquad (187)
\end{aligned}
$$

The expected total annual cost is

$$
\begin{aligned}
ETC(Q,r) = &\left(K_V+K_B+F+cQ+c_PQ+c_rQ\theta_3\mu_q+\frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)}+\frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right.\\
&+h_V\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2+(L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6D}\right)\\
&+h_V\left(\frac{Q^2\left(-3(\theta_3-1)D(L_1+U_1)-6D\right)}{12PD}\right)\\
&+h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)\\
&\left.+d+c_b\bar{B}(r)+c_l\bar{s}(r)\right)\frac{D}{Q(1-p)} \qquad (188)
\end{aligned}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} = &\frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)+h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2}\\
&-\frac{h_V\left(L_1^2(\theta_3-1)^2+(L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6(p-1)}\\
&+\frac{h_VD((\theta_3-1)(L_1+U_1)+2)}{4(p-1)P} \qquad (189)
\end{aligned}
$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V)}{(1-p)Q^3} \tag{190}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-12PD(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V)}{6h_B(p^2-1)P - h_V\left(2P\left(L_1^2B^2 + (L_1+U_1)B(BU_1+3)+3\right) - 3BD(L_1+U_1) - 6D\right)}} \tag{191}$$

where $B = (\theta_3 - 1)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B'_p(r)(\bar{B}(r)+\bar{s}(r)-r) - B_p(r)\right)\left(D(c_l-c_b) + h_BQ(1-p)\right) + \left(c_lD + h_BQ(1-p)\right)}\right) \tag{192}$$

As our final model, where q and θ_3 are both stochastic with standard uniform distribution, supplier's cost function per cycle is

$$\begin{aligned} ETC_V(Q) = {} & K_V + c_PQ + c_rQ\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\ & + h_V\left(\frac{Q^2L_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3\right)}{18D}\right) \\ & + h_V\left(\frac{Q^2L_1\left(2L_2^2U_1 + L_2(2U_1(U_2-3)+9) + 2U_1((U_2-3)U_2+3) + 9(U_2-2)\right)}{36D}\right) \\ & + h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2+3\right) + 9U_1(L_2+U_2-2) + 18\right)}{36D}\right) \\ & - h_V\left(\frac{Q^2((L_2+U_2-2)(L_1+U_1)+4)}{8P}\right) \end{aligned} \tag{193}$$

and the buyer's expected total cost with deterministic demand in a supplier's cycle is

$$\begin{aligned} ETC_B(Q,r) = {} & K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r - \mu_L + \bar{s}(r))\frac{Q(1-p)}{D}\right) \\ & + F + cQ + d + c_b\bar{B}(r) + c_l\bar{s}(r) \end{aligned} \tag{194}$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_B + K_V + F + cQ + c_P Q + c_r Q \mu_q \mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&\quad + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
&\quad + h_V\left(\frac{Q^2 L_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2+3\right)}{18D}\right) + d + c_b\bar{B}(r) + c_l\bar{s}(r) \\
&\quad + h_V\left(\frac{Q^2 L_1\left(2L_2^2U_1 + L_2(2U_1(U_2-3)+9) + 2U_1((U_2-3)U_2+3) + 9(U_2-2)\right)}{36D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2+3\right) + 9U_1(L_2+U_2-2) + 18\right)}{36D}\right) \\
&\quad - h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right)
\end{aligned}
\tag{195}
$$

The expected total annual cost becomes

$$
\begin{aligned}
ETC(Q) = \Bigg(& K_B + K_V + F + cQ + c_P Q + c_r Q \mu_q \mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
& + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
& + h_V\left(\frac{Q^2 L_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2+3\right)}{18D}\right) + d + c_b\bar{B}(r) + c_l\bar{s}(r) \\
& + h_V\left(\frac{Q^2 L_1\left(2L_2^2U_1 + L_2(2U_1(U_2-3)+9) + 2U_1((U_2-3)U_2+3) + 9(U_2-2)\right)}{36D}\right) \\
& + h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2+3\right) + 9U_1(L_2+U_2-2) + 18\right)}{36D}\right) \\
& - h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right)\Bigg)\frac{D}{Q(1-p)}
\end{aligned}
\tag{196}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} &= \frac{D(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V)}{(p-1)Q^2} \\
&\quad - \frac{h_V L_1\left(2L_2^2U_1 + L_2(2U_1(U_2-3)+9) + 2U_1((U_2-3)U_2+3) + 9(U_2-2)\right)}{36(p-1)} \\
&\quad - \frac{h_V\left(2U_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2+3\right) - 9U_1(L_2+U_2-2) - 18\right)}{36(p-1)} \\
&\quad + \frac{h_V\left(D(L_2+U_2-2)(L_1+U_1) + 4D\right) + 4h_B\left(p^2-1\right)P}{8(p-1)P} \\
&\quad - \frac{h_V L_1^2\left(L_2^2 + L_2(U_2-3) + (U_2-3)U_2+3\right)}{18(p-1)}
\end{aligned}
\tag{197}
$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{(1-p)Q^3} \tag{198}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-72PD\left(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V\right)}{36h_B\left(p^2-1\right)P - h_V\left(2P\left(2L_1^2C+\left(L_1+U_1\right)\left(2U_1C+9A\right)+18\right)-9DAB-36D\right)}} \tag{199}$$

where $C = (L_2^2 + L_2(U_2-3) + (U_2-3)U_2+3)$, $B = (L_1+U_1)$ and $A = (U_2+L_2-2)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right) \tag{200}$$

3.2.3. *Integrated Models with partial backordering when q and θ_3 follow beta distribution*

Different than standard uniform distribution, in this section we will examine the four models when q and θ_3 follows beta distribution. As the first model, supplier's total cost per cycle for deterministic q and θ_3 is

$$\begin{aligned} TC_V(Q) = K_V + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) \\ + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_PQ + c_rQq\theta_3 \end{aligned} \tag{201}$$

and the buyer's total average cost with stochastic demand and partial backordering is

$$\begin{aligned} TC_B(Q,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\ + F + cQ + d + c_b\bar{B}(r) + c_l\bar{s}(r) \end{aligned} \tag{202}$$

The total cost for integrated system is,

$$
\begin{aligned}
TC(Q,r) &= TC_V(Q) + TC_B(Q,r) \\
TC(Q,r) &= K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + c_P Q \\
&\quad + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) + c_r Q q \theta_3 \\
&\quad + d + c_b\bar{B}(r) + c_l\bar{s}(r) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + cQ \qquad (203)
\end{aligned}
$$

We have cycle time $T = \frac{Q(1-p)}{D}$, so the expected total annual cost would be

$$
\begin{aligned}
ETC(Q,r) = \Bigg(&K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + c_P Q \\
&+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) + c_r Q q \theta_3 \\
&+ d + c_b\bar{B}(r) + c_l\bar{s}(r) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + cQ\Bigg)\frac{D}{Q(1-p)}
\end{aligned}
\qquad (204)
$$

To minimise $ETC(Q,r)$, we take first derivative with respect to Q and r,

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} &= \frac{2D(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V) + h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2} \\
&\quad - \frac{h_V(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2(p-1)P}
\end{aligned}
\qquad (205)
$$

Taking the second derivative, we have

$$
\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r) + K_B + K_V + d)}{(1-p)Q^3} \qquad (206)
$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$
Q^* = \sqrt{\frac{-2PD(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V)}{h_B\left(p^2-1\right)P + h_V(q(\theta_3-1)+1)(P(q(-\theta_3)+q-1)+D)}} \qquad (207)
$$

The derivative of $ETC(Q,r)$ with respect to r is

$$\frac{\partial(ETC)}{\partial r} = \frac{Dc_b\bar{B}'(r) + Dc_l\bar{s}'(r) + h_B(1-p)Q(1+\bar{s}'(r))}{(1-p)Q} \tag{208}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial r^2} = \frac{Dc_b\bar{B}''(r) + \bar{s}''(r)(h_B(1-p)Q + c_lD)}{(1-p)Q} \tag{209}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial r^2} > 0$. Therefore, there exists unique value for r^* given as

$$\begin{aligned}
F(r) &= \left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r) - B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right) + \left(c_lD + h_BQ(1-p)\right)}\right) \\
r^* &= F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r) - B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right) + \left(c_lD + h_BQ(1-p)\right)}\right)
\end{aligned} \tag{210}$$

Similar to the previous case, the rest of the models, r^* equation will be the same since the integrated cost function is changing only by constant.

For the model where q is deterministic and θ_3 is stochastic with beta distribution, first we will calculate supplier's cost. With θ_3 given as random variable, it has an upper (U_2) and lower (L_2) bounds with the probability distribution function of g_2 and shape parameters are $\alpha_{\theta_3} = 3$ and $\beta_{\theta_3} = 1$. The expected value of the total inventory of nondefective items during t_1 will be

$$\begin{aligned}
E(\Delta_1) &= \int_{L_2}^{U_2} \frac{((1-q)P-D)Q_V^2}{2P^2} g_2 d\theta_3 & \left(g_2 = \frac{(\theta_3 - L_2)^{\alpha_{\theta_3}-1}(U_2-\theta_3)^{\beta_{\theta_3}-1}}{B(\alpha_{\theta_3},\beta_{\theta_3})(U_2-L_2)^{\alpha_{\theta_3}+\beta_{\theta_3}-1}}\right) \\
&= \frac{((1-q)P-D)Q_V^2}{2P^2} \int_{L_2}^{U_2} \left(\frac{(\theta_3-L_2)^2}{B(3,1)(U_2-L_2)^3}\right) d\theta_3 & \left(B(\alpha,\beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}\right) \\
&= \frac{((1-q)P-D)Q_V^2}{2P^2}
\end{aligned} \tag{211}$$

And the inventory during t_2 is

$$E(\Delta_2) = \int_{L_2}^{U_2} \left(\frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2} \right) g_2 d\theta_3$$
$$= \frac{qQ_V^2 \left((3U_2+L_2)((L_2-5)q+5)+6qU_2^2\right)}{20P}$$
$$- \frac{qQ_V^2 D\left((3U_2+L_2)(L_2q+5)+6qU_2^2\right)}{20P^2} \quad (212)$$

During t_3, the inventory becomes

$$E(\Delta_3) = \int_{L_2}^{U_2} \frac{1}{2} D \left(\frac{(P-D)q\theta_3 Q_V + ((1-q)P-D)Q_V}{PD} \right)^2 g_2 d\theta_3$$
$$= \frac{Q_V^2 \left(q^2 \left(L_2^2 + 3(L_2-5)U_2 - 5L_2 + 6_{\theta_3}^2 + 10\right) + 5q(L_2+3U_2-4) + 10\right)}{20D}$$
$$- \frac{Q^2 \left(q^2 \left(2L_2^2 + L_2(6U_2-5) + 3U_2(4U_2-5)\right) + 10q(L_2+3U_2-2) + 20\right)}{20P}$$
$$+ \frac{Q^2 D \left(q\left(3U_2(L_2q+5) + L_2(L_2q+5) + 6qU_2^2\right) + 10\right)}{20P^2} \quad (213)$$

The expected value of the total inventory of defective items during t_1 will be

$$E(\Gamma_1) = \int_{L_2}^{U_2} \frac{q\theta_3 Q_V^2}{2P} g_2 d\theta_3$$
$$= \frac{qQ_V^2(L_2+3U_2)}{8P} \quad (214)$$

and during t_2

$$E(\Gamma_2) = \int_{L_2}^{U_2} \frac{q^2 Q_V^2 \theta_3^2}{2P} g_2 d\theta_3$$
$$= \frac{q^2 Q_V^2 (L_2^2 + 3L_2U_2 + 6U_2^2)}{20P} \quad (215)$$

The expected annual cost is

$$ETC_V(Q_V) = K_V + h_V(E(\Delta_1) + E(\Delta_2) + E(\Delta_3) + E(\Gamma_1) + E(\Gamma_2))$$
$$+ \frac{\eta Q}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2} E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right] + c_P Q + c_r q \mu_{\theta_3} Q \quad (216)$$

where $\mu_{theta_3} = \frac{\alpha_{\theta_3}}{\alpha_{\theta_3}+\beta_{\theta_3}} = \frac{3}{4}$ and $E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right]$ $\left(=\mu_{\ln(\theta_3/\theta_{3_0})}\right)$ is derived as follows:

$$\begin{aligned} E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right] &= E\left[\ln(\theta_3) - \ln(\theta_{3_0})\right] \\ &= E[\ln(\theta_3)] - E[\ln(\theta_{3_0})] \end{aligned} \tag{217}$$

Here,

$$E[\ln(\theta_{3_0})] = \ln(\theta_{3_0}) \tag{218}$$

since θ_{3_0} is constant. However,

$$E[\ln(\theta_3)] \neq \ln[E(\theta_3)] \tag{219}$$

Therefore, we need to find $E(\ln(\theta_3))$ which is a function of θ_3.

$$\begin{aligned} E(\ln\theta_3) &= \int_{-\infty}^{\infty} \ln\theta_3 \cdot g_2 d\theta_3 \\ &= \int_{L_2}^{U_2} \ln\theta_3 \frac{3(\theta_3 - L_2)^2}{(U_2 - L_2)^3} d\theta_3 \\ &= \frac{6L_2^3\ln(L_2) + (U_2 - L_2)\left(11L_2^2 - 7L_2U_2 + 2U_2^2\right)}{6(L_2 - U_2)^3} \\ &- \frac{U_2\left(3L_2^2 - 3L_2U_2 + U_2^2\right)\ln(U_2)}{(L_2 - U_2)^3} \end{aligned} \tag{220}$$

Finally,

$$\begin{aligned} E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right] &= \frac{6L_2^3\ln(L_2) + (U_2 - L_2)\left(11L_2^2 - 7L_2U_2 + 2U_2^2\right)}{6(L_2 - U_2)^3} \\ &- \frac{U_2\left(3L_2^2 - 3L_2U_2 + U_2^2\right)\ln(U_2)}{(L_2 - U_2)^3} - \ln(\theta_{3_0}) \end{aligned} \tag{221}$$

The supplier's cost function per cycle will be

$$
\begin{aligned}
ETC_V(Q) = K_V + h_V & \left(\frac{Q^2 q^2 \left(L_2^2 + 3(L_2 - 5)U_2 - 5L_2 + 6U_2^2 + 10\right)}{20D} \right) \\
& + h_V \left(\frac{Q^2 \left(2P\left(5q(L_2 + 3U_2 - 4) + 10\right) - 5D(q(L_2 + 3U_2 - 4) + 4)\right)}{40PD} \right) \\
& + \frac{\eta Q}{\delta_1} \ln \left(\frac{q_0}{q} \right) + \frac{\eta Q}{\delta_2} \mu_{\ln(\theta_3/\theta_{3_0})} + c_P Q + c_r q Q \mu_{\theta_3}
\end{aligned} \tag{222}
$$

and the buyer's total average cost with stochastic demand in a supplier's cycle is

$$
\begin{aligned}
ETC_B(Q,r) = K_B + h_B & \left(\frac{1}{2} \frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r)) \frac{Q(1-p)}{D} \right) \\
& + F + cQ + d + c_b \bar{B}(r) + c_l \bar{s}(r)
\end{aligned} \tag{223}
$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_B + h_B \left(\frac{1}{2} \frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r)) \frac{Q(1-p)}{D} \right) \\
& \quad + h_V \left(\frac{Q^2 q^2 \left(L_2^2 + 3(L_2 - 5)U_2 - 5L_2 + 6U_2^2 + 10\right)}{20D} \right) + K_V + F \\
& \quad + h_V \left(\frac{Q^2 \left(2P\left(5q(L_2 + 3U_2 - 4) + 10\right) - 5D(q(L_2 + 3U_2 - 4) + 4)\right)}{40PD} \right) \\
& \quad + cQ + c_P Q + c_r q Q \mu_{\theta_3} + \frac{\eta Q}{\delta_1} \ln \left(\frac{q_0}{q} \right) + \frac{\eta Q}{\delta_2} \mu_{\ln(\theta_3/\theta_{3_0})} \\
& \quad + d + c_b \bar{B}(r) + c_l \bar{s}(r)
\end{aligned} \tag{224}
$$

The expected total annual cost is

$$
\begin{aligned}
ETC(Q,r) = \Bigg(& K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
& + h_V\left(\frac{Q^2q^2\left(L_2^2+3(L_2-5)U_2-5L_2+6U_2^2+10\right)}{20D}\right) + K_V + F \\
& + h_V\left(\frac{Q^2\left(2P\left(5q(L_2+3U_2-4)+10\right)-5D(q(L_2+3U_2-4)+4)\right)}{40PD}\right) \\
& + cQ + c_PQ + c_rqQ\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
& + d + c_b\bar{B}(r) + c_l\bar{s}(r)\Bigg)\frac{D}{Q(1-p)}
\end{aligned}
\tag{225}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} = & \frac{D\left(8P(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)+h_VQ^2(q(L_2+3U_2-4)+4)\right)}{8(p-1)PQ^2} \\
& + \frac{10h_B\left(p^2-1\right)-h_V\left(q^2\left(L_2^2+3(L_2-5)U_2-5L_2+6U_2^2+10\right)\right)}{20(p-1)} \\
& - \frac{h_V\left(q(L_2+3U_2-4)+2\right)}{4(p-1)}
\end{aligned}
\tag{226}
$$

Taking the second derivative, we have

$$
\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{(1-p)Q^3} \tag{227}
$$

All parameters are positive, and $\frac{d^2(ETC)}{dQ^2} > 0$. Therefore, there exists unique value for Q^* given as

$$
Q^* = \sqrt{\frac{-40PD(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{20h_B\left(p^2-1\right)P-2h_VP\left(q^2C+5A+10\right)+5h_VD(A+4)}} \tag{228}
$$

where $C = (L_2^2+3(L_2-5)U_2-5L_2+6U_2^2+10)$ and $A = (L_2+3U_2-4)$ and the unique value for r^* given as

$$
r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right) \tag{229}
$$

For the next model, q is stochastic with standard uniform distribution and θ_3 is deterministic. With q given as random variable, it has an upper (U_1) and lower (L_1) bounds with the probability distribution function of g_1 and shape parameters are $\alpha_q = 1$ and $\beta_q = 4$. The expected value of the total inventory of nondefective items during t_1 will be

$$\begin{aligned}
E(\Delta_1) &= \int_{L_1}^{U_1} \frac{((1-q)P-D)Q_V^2}{2P^2} g_1 dq & \left(g_1 = \frac{(q-L_1)^{\alpha_q-1}(U_1-q)^{\beta_q-1}}{B(\alpha_q,\beta_q)(U_1-L_1)^{\alpha_q+\beta_q-1}}\right) \\
&= \frac{((1-q)P-D)Q_V^2}{2P^2} \int_{L_1}^{U_1} \left(\frac{(U_1-q)^3}{B(1,4)(U_1-L_1)^4}\right) dq & \left(B(\alpha,\beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}\right) \\
&= \frac{Q_V^2(P(5-4L_1-U_1)-5D)}{10P^2}
\end{aligned} \tag{230}$$

And the inventory during t_2 is

$$\begin{aligned}
E(\Delta_2) &= \int_{L_1}^{U_1} \left(\frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2}\right) g_1 dq \\
&= \frac{Q_V^2\theta_3\left(10L_1^2(P(\theta_3-2)-\theta_3 D) + 4L_1(P((\theta_3-2)U_1+6) - D(\theta_3 U_1+6))\right)}{30P^2} \\
&+ \frac{Q_V^2\theta_3\left(U_1(P((\theta_3-2)U_1+6) - D(\theta_3 U_1+6))\right)}{30P^2}
\end{aligned} \tag{231}$$

During t_3, the inventory becomes

$$\begin{aligned}
E(\Delta_3) &= \int_{L_1}^{U_1} \frac{1}{2} D \left(\frac{(P-D)q\theta_3 Q_V + ((1-q)P-D)Q_V}{PD}\right)^2 g_1 dq \\
&= \frac{Q_V^2\left(10L_1^2(\theta_3-1)^2 + (4L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+6)+15\right)}{30D} \\
&- \frac{Q_V^2\left(10L_1^2(\theta_3-1)\theta_3 + (4L_1+U_1)(\theta_3((\theta_3-1)U_1+6)-3)+15\right)}{15P} \\
&+ \frac{Q_V^2 D\left(\theta_3\left(10L_1^2\theta_3 + 4L_1(\theta_3 U_1+6) + U_1(\theta_3 U_1+6)\right)+15\right)}{30P^2}
\end{aligned} \tag{232}$$

The expected value of the total inventory of defective items during t_1 will be

$$\begin{aligned}
E(\Gamma_1) &= \int_{L_1}^{U_1} \frac{q\theta_3 Q_V^2}{2P} g_1 dq \\
&= \frac{2L_1 Q_V^2\theta_3}{5P} + \frac{Q_V^2\theta_3 U_1}{10P}
\end{aligned} \tag{233}$$

and during t_2

$$E(\Gamma_2) = \int_{L_2}^{U_2} \frac{q^2 Q_V^2 \theta_3^2}{2P} g_2 d\theta_3$$
$$= \frac{Q_V^2 \theta_3^2 \left(10L_1^2 + 4L_1U_1 + U_1^2\right)}{30P} \tag{234}$$

The expected annual cost is

$$ETC_V(Q_V) = K_V + h_V(E(\Delta_1) + E(\Delta_2) + E(\Delta_3) + E(\Gamma_1) + E(\Gamma_2))$$
$$+ \frac{\eta Q}{\delta_1} E\left[\ln\left(\frac{q_0}{q}\right)\right] + \frac{\eta Q}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_P Q + c_r \mu_q \theta_3 Q \tag{235}$$

where $\mu_q = \frac{\alpha_q}{\alpha_q + \beta_q} = \frac{1}{5}$ and $E\left[\ln\left(\frac{q_0}{q}\right)\right] \left(= \mu_{\ln(q_0/q)}\right)$ is derived as follows:

$$E\left[\ln\left(\frac{q_0}{q}\right)\right] = E\Big[\ln(q_0) - \ln(q)\Big]$$
$$= E[\ln(q_0)] - E[\ln(q)] \tag{236}$$

Here,

$$E[\ln(q_0)] = \ln(q_0) \tag{237}$$

since q_0 is constant. However,

$$E[\ln(q)] \neq \ln[E(q)] \tag{238}$$

Therefore, we need to find $E(\ln(q))$ which is a function of q.

$$E(\ln q) = \int_{-\infty}^{\infty} \ln q \cdot g_1 dq$$
$$= \int_{L_1}^{U_1} \ln q \frac{4(U_1 - q)^3}{(U_1 - L_1)^4} dq$$
$$= \frac{L_1(L_1 - 2U_1)\left(L_1^2 - 2L_1U_1 + 2U_1^2\right)\ln(L_1)}{(L_1 - U_1)^4}$$
$$- \frac{\left(3L_1^3 - 13L_1^2U_1 + 23L_1U_1^2 - 25U_1^3\right)}{12(L_1 - U_1)^3} + \frac{U_1^4 \ln(U_1)}{(L_1 - U_1)^4} \tag{239}$$

Finally,

$$E\left[\ln\left(\frac{q_0}{q}\right)\right] = \ln(q_0) - \frac{L_1(L_1-2U_1)\left(L_1^2-2L_1U_1+2U_1^2\right)\ln(L_1)}{(L_1-U_1)^4}$$
$$-\frac{\left(3L_1^3-13L_1^2U_1+23L_1U_1^2-25U_1^3\right)}{12(L_1-U_1)^3}+\frac{U_1^4\ln(U_1)}{(L_1-U_1)^4} \tag{240}$$

Therefore, supplier's cost function per cycle is

$$ETC_V(Q) = K_V + h_V\left(\frac{Q^2\left(10L_1^2(\theta_3-1)^2+4L_1(\theta_3-1)((\theta_3-1)U_1+6)\right)}{30D}\right)$$
$$+h_V\left(\frac{Q^2\left(P\left((\theta_3-1)U_1((\theta_3-1)U_1+6)+15\right)-3(\theta_3-1)D(4L_1+U_1)-15D\right)}{30PD}\right)$$
$$+c_PQ+c_rQ\theta_3\mu_q+\frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)}+\frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \tag{241}$$

and the buyer's expected total cost with stochastic demand in the supplier's cycle is

$$ETC_B(Q,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)$$
$$+F+cQ+d+c_b\bar{B}(r)+c_l\bar{s}(r) \tag{242}$$

The expected total cost for integrated system is,

$$ETC(Q,r) = ETC_V(Q)+ETC_B(Q,r)$$
$$ETC(Q,r) = K_V+K_B+F+cQ+c_PQ+c_rQ\theta_3\mu_q+\frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)}+\frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)$$
$$+h_V\left(\frac{Q^2\left(10L_1^2(\theta_3-1)^2+4L_1(\theta_3-1)((\theta_3-1)U_1+6)\right)}{30D}\right)$$
$$+h_V\left(\frac{Q^2\left(P\left((\theta_3-1)U_1((\theta_3-1)U_1+6)+15\right)-3(\theta_3-1)D(4L_1+U_1)-15D\right)}{30PD}\right)$$
$$+h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)$$
$$+d+c_b\bar{B}(r)+c_l\bar{s}(r) \tag{243}$$

The expected total annual cost is

$$ETC(Q,r) = \Bigg(K_V + K_B + F + cQ + c_P Q + c_r Q\theta_3\mu_q + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)$$
$$+ h_V\left(\frac{Q^2\left(10L_1^2(\theta_3-1)^2 + 4L_1(\theta_3-1)((\theta_3-1)U_1+6)\right)}{30D}\right)$$
$$+ h_V\left(\frac{Q^2\left(P\left((\theta_3-1)U_1((\theta_3-1)U_1+6)+15\right) - 3(\theta_3-1)D(4L_1+U_1) - 15D\right)}{30PD}\right)$$
$$+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)$$
$$+ d + c_b\bar{B}(r) + c_l\bar{s}(r)\Bigg)\frac{D}{Q(1-p)} \tag{244}$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$\frac{\partial(ETC)}{\partial Q} = \frac{30PD(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V) + 15h_B\left(p^2-1\right)PQ^2}{30(p-1)PQ^2}$$
$$-\frac{h_V\left(10L_1^2(\theta_3-1)^2 + (4L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+6)+15\right)}{30(p-1)}$$
$$+\frac{h_V\left((\theta_3-1)D(4L_1+U_1)+5D\right)}{10(p-1)P} \tag{245}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{(1-p)Q^3} \tag{246}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-30PD(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{15h_B(p^2-1)P - h_VP(10L_1^2B^2+(4L_1+U_1)B(BU_1+6)+15) + 3h_VD(4BL_1+U_1B+5)}} \tag{247}$$

where $B = (\theta_3 - 1)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r) - B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right) + \left(c_lD + h_BQ(1-p)\right)}\right) \tag{248}$$

Our final model is where q and θ_3 are both stochastic with beta distribution. With q and θ_3 given as random variables, they have an upper (U_1,U_2) and lower (L_1,L_2)

bounds with the probability distribution function of g_1, g_2 and shape parameters are $\alpha_q = 19, \alpha_{\theta_3} = 1$ and $\beta_q = 24, \beta_{\theta_3} = 9$. The expected value of the total inventory of nondefective items during t_1 will be

$$\begin{aligned} E(\Delta_1) &= \int_{L_2}^{U_2}\int_{L_1}^{U_1} \frac{((1-q)P-D)Q_V^2}{2P^2} g_1 g_2 dq d\theta_3 \\ &= -\frac{Q_V^2(P(-5+4L_1+U_1)+5D)}{10P^2} \end{aligned} \tag{249}$$

And the inventory during t_2 is

$$\begin{aligned} E(\Delta_2) = &\int_{L_2}^{U_2}\int_{L_1}^{U_1} \left(\frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2}\right) g_1 g_2 dq d\theta_3 \\ = &\frac{Q_V^2\left(L_2^2\left(10L_1^2+4L_1U_1+U_1^2\right)(P-D)\right)}{300P^2} \\ &+ \frac{Q_V^2 L_2\left(10L_1^2(3U_2-5)+4L_1(U_1(3U_2-5)+15)+U_1(U_1(3U_2-5)+15)\right)}{300P} \\ &- \frac{Q_V^2\left(L_2 D\left(10L_1^2U_2+4L_1(U_1U_2+5)+U_1(U_1U_2+5)\right)\right)}{100P^2} \\ &+ \frac{Q_V^2 U_2\left(10L_1^2(2U_2-5)+4L_1(U_1(2U_2-5)+15)+U_1(U_1(2U_2-5)+15)\right)}{100P} \\ &- \frac{Q_V^2\left(U_2 D\left(20L_1^2U_2+L_1(8U_1U_2+60)+U_1(2U_1U_2+15)\right)\right)}{100P^2} \end{aligned} \tag{250}$$

During t_3, the inventory becomes

$$
\begin{aligned}
E(\Delta_3) &= \int_{L_2}^{U_2}\int_{L_1}^{U_1}\frac{1}{2}D\left(\frac{(P-D)q\theta_3 Q_V+((1-q)P-D)Q_V}{PD}\right)^2 g_1g_2dqd\theta_3 \\
&= \frac{Q_V^2\left(L_1^2\left(L_2^2+L_2(3U_2-5)+6U_2^2-15U_2+10\right)\right)}{30D} \\
&+ \frac{Q_V^2\left(L_1\left(L_2^2U_1+L_2(U_1(3U_2-5)+15)+U_1\left(6U_2^2-15U_2+10\right)+45U_2-60\right)\right)}{75D} \\
&+ \frac{Q_V^2\left(U_1^2\left(L_2^2+L_2(3U_2-5)+6U_2^2-15U_2+10\right)+15U_1(L_2+3U_2-4)+150\right)}{300D} \\
&- \frac{Q_V^2D\left(L_1^2\left(2L_2^2+L_2(6U_2-5)+3U_2(4U_2-5)\right)\right)}{30PD} \\
&- \frac{Q_V^2D\left(L_1\left(2L_2^2U_1+L_2(U_1(6U_2-5)+30)+3\left(4U_1U_2^2-5(U_1-6)U_2-20\right)\right)\right)}{75PD} \\
&- \frac{Q_V^2D\left(U_1^2\left(2L_2^2+L_2(6U_2-5)+3U_2(4U_2-5)\right)+30U_1(L_2+3U_2-2)+300\right)}{300PD} \\
&+ \frac{Q_V^2D\left(L_2^2\left(10L_1^2+4L_1U_1+U_1^2\right)+3L_2\left(10L_1^2U_2+4L_1(U_1U_2+5)+U_1(U_1U_2+5)\right)\right)}{300P^2} \\
&+ \frac{Q_V^2D\left(3\left(20L_1^2U_2^2+4L_1U_2(2U_1U_2+15)+2U_1^2U_2^2+15U_1U_2+50\right)\right)}{300P^2}
\end{aligned} \tag{251}
$$

The expected value of the total inventory of defective items during t_1 will be

$$
\begin{aligned}
E(\Gamma_1) &= \int_{L_2}^{U_2}\int_{L_1}^{U_1}\frac{q\theta_3 Q_V^2}{2P}g_1g_2dqd\theta_3 \\
&= \frac{Q_V^2(4L_1+U_1)(L_2+3U_2)}{40P}
\end{aligned} \tag{252}
$$

and during t_2

$$
\begin{aligned}
E(\Gamma_2) &= \int_{L_2}^{U_2}\int_{L_2}^{U_2}\frac{q^2Q_V^2\theta_3^2}{2P}g_1g_2dqd\theta_3 \\
&= \frac{Q_V^2(10L_1^2+4L_1U_1+U_1^2)(L_2^2+3L_2U_2+6U_2^2)}{300P}
\end{aligned} \tag{253}
$$

The expected annual cost is

$$
\begin{aligned}
ETC_V(Q) &= K_V+h_V(E(\Delta_1)+E(\Delta_2)+E(\Delta_3)+E(\Gamma_1)+E(\Gamma_2)) \\
&+ \frac{\eta Q}{\delta_1}E\left[\ln\left(\frac{q_0}{q}\right)\right]+\frac{\eta Q}{\delta_2}E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right]+c_PQ+c_r\mu_q\mu_{\theta_3}Q
\end{aligned} \tag{254}
$$

where $\mu_q=\frac{\alpha_q}{\alpha_q+\beta_q}=\frac{1}{5},\mu_{\theta_3}=\frac{\alpha_{\theta_3}}{\alpha_{\theta_3}+\beta_{\theta_3}}=\frac{3}{4}$ and $E\left[\ln\left(\frac{q_0}{q}\right)\right],E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right]$ are derived in previous models.

Supplier's cost function per cycle is

$$
\begin{aligned}
ETC_V(Q) = {} & K_V + c_P Q + c_r Q \mu_q \mu_{\theta_3} + \frac{\eta Q}{\delta_1} \mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2} \mu_{\ln(\theta_3/\theta_{3_0})} \\
& + h_V \left(\frac{Q^2 \left(L_1^2 \left(L_2^2 + 3(L_2 - 5)U_2 - 5L_2 + 6U_2^2 + 10 \right) \right)}{30D} \right) \\
& + h_V \left(\frac{Q^2 \left(U_1^2 \left(L_2^2 + 3(L_2 - 5)U_2 - 5L_2 + 6U_2^2 + 10 \right) \right)}{300D} \right) \\
& + h_V \left(\frac{Q^2 \left(L_1 \left(3(L_2 - 5)U_1 U_2 + L_2((L_2 - 5)U_1 + 15) \right) \right)}{75D} \right) \\
& + h_V \left(\frac{Q^2 \left(4L_1 \left(6U_1 U_2^2 + 5(2U_1 + 9U_2 - 12) \right) + 15U_1(L_2 + 3U_2 - 4) + 150 \right)}{300D} \right) \\
& - h_V \left(\frac{Q^2 \left((L_2 + 3U_2 - 4)(4L_1 + U_1) + 20 \right)}{40P} \right) \qquad (255)
\end{aligned}
$$

and the buyer's expected total cost with deterministic demand in a supplier's cycle is

$$
\begin{aligned}
ETC_B(Q,r) = {} & K_B + h_B \left(\frac{1}{2} \frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r)) \frac{Q(1-p)}{D} \right) \\
& + F + cQ + d + c_b \bar{B}(r) + c_l \bar{s}(r) \qquad (256)
\end{aligned}
$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) = {} & ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) = {} & K_B + K_V + F + cQ + c_P Q + c_r Q \mu_q \mu_{\theta_3} + \frac{\eta Q}{\delta_1} \mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2} \mu_{\ln(\theta_3/\theta_{3_0})} \\
& + h_B \left(\frac{1}{2} \frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r)) \frac{Q(1-p)}{D} \right) \\
& + h_V \left(\frac{Q^2 \left(4L_1 \left(6U_1 U_2^2 + 5(2U_1 + 9U_2 - 12) \right) + 15U_1(L_2 + 3U_2 - 4) + 150 \right)}{300D} \right) \\
& + h_V \left(\frac{Q^2 \left(L_1^2 \left(L_2^2 + 3(L_2 - 5)U_2 - 5L_2 + 6U_2^2 + 10 \right) \right)}{30D} \right) + d + c_b \bar{B}(r) + c_l \bar{s}(r) \\
& + h_V \left(\frac{Q^2 \left(U_1^2 \left(L_2^2 + 3(L_2 - 5)U_2 - 5L_2 + 6U_2^2 + 10 \right) \right)}{300D} \right) \\
& + h_V \left(\frac{Q^2 \left(L_1 \left(3(L_2 - 5)U_1 U_2 + L_2((L_2 - 5)U_1 + 15) \right) \right)}{75D} \right) \\
& - h_V \left(\frac{Q^2 \left((L_2 + 3U_2 - 4)(4L_1 + U_1) + 20 \right)}{40P} \right) \qquad (257)
\end{aligned}
$$

The expected total annual cost becomes

$$
\begin{aligned}
ETC(Q) = \Bigg(& K_B + K_V + F + cQ + c_P Q + c_r Q\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
& + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
& + h_V\left(\frac{Q^2\left(4L_1\left(6U_1U_2^2+5(2U_1+9U_2-12)\right)+15U_1(L_2+3U_2-4)+150\right)}{300D}\right) \\
& + h_V\left(\frac{Q^2\left(L_1^2\left(L_2^2+3(L_2-5)U_2-5L_2+6U_2^2+10\right)\right)}{30D}\right) + d + c_b\bar{B}(r) + c_l\bar{s}(r) \\
& + h_V\left(\frac{Q^2\left(U_1^2\left(L_2^2+3(L_2-5)U_2-5L_2+6U_2^2+10\right)\right)}{300D}\right) \\
& + h_V\left(\frac{Q^2\left(L_1\left(3(L_2-5)U_1U_2+L_2((L_2-5)U_1+15)\right)\right)}{75D}\right) \\
& - h_V\left(\frac{Q^2\left((L_2+3U_2-4)(4L_1+U_1)+20\right)}{40P}\right)\Bigg)\frac{D}{Q(1-p)}
\end{aligned} \tag{258}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} = & \frac{600PD(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{600(p-1)PQ^2} \\
& + \frac{h_B\left(p^2-1\right)}{2(p-1)} - \frac{h_V\left(L_1^2\left(L_2^2+3(L_2-5)U_2-5L_2+6U_2^2+10\right)\right)}{30(p-1)} \\
& - \frac{h_V\left(U_1^2\left(L_2^2+3(L_2-5)U_2-5L_2+6U_2^2+10\right)\right)}{300(p-1)} \\
& - \frac{h_V\left(L_1\left(3(L_2-5)U_1U_2+L_2((L_2-5)U_1+15)+6U_1U_2^2+5(2U_1+9U_2-12)\right)\right)}{75(p-1)} \\
& - \frac{h_V\left(U_1(L_2+3U_2-4)+10\right)+15D(L_2+3U_2-4)(4L_1+U_1)+20D}{20(p-1)}
\end{aligned} \tag{259}
$$

Taking the second derivative, we have

$$
\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{(1-p)Q^3} \tag{260}
$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$
Q^* = \sqrt{\frac{-600PD(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{300h_B\left(p^2-1\right)P - h_V\left(2P(C+E)-15D(L_2+3U_2-4)(4L_1+U_1)-300D\right)}} \tag{261}
$$

where $C = (L_2^2+3(L_2-5)U_2-5L_2+6U_2^2+10)(10L_1^2+U_1^2)$ and $E = (4L_1(3(L_2-5)U_1U_2+L_2((L_2-5)U_1+15)+6U_1U_2^2+5(2U_1+9U_2-12))+15U_1(L_2+3U_2-4)+150)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right) \tag{262}$$

3.2.4. *Integrated Models with partial backordering when q and θ_3 follow triangular distribution*

After beta distribution, in this section we will examine the four models when q and θ_3 follow triangular distribution. Let $[L_1, m_1, U_1]$ are the parameters of triangular distribution for defective rates and $[L_2, m_2, U_2]$ for reworking rates where $[L_1, L_2]$ are the lower limits and $[U_1, U_2]$ are the upper limits, respectively. Additionally, $[m_1, m_2]$ are the mode of the triangular distribution. Since these parameters are proportions, they satisfy the condition of $0 \leq L_1 \leq m_1 \leq U_1 \leq 1$ and $0 \leq L_2 \leq m_2 \leq U_2 \leq 1$. As the first model, supplier's total cost per cycle for deterministic q and θ_3 is

$$\begin{aligned} TC_V(Q) = K_V + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) \\ + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_PQ + c_rQq\theta_3 \end{aligned} \tag{263}$$

and the buyer's total average cost with stochastic demand and partial backordering is

$$\begin{aligned} TC_B(Q,r) = K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\ + F + cQ + d + c_b\bar{B}(r) + c_l\bar{s}(r) \end{aligned} \tag{264}$$

The total cost for integrated system,

$$\begin{aligned} TC(Q,r) &= TC_V(Q) + TC_B(Q,r) \\ TC(Q,r) &= K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + c_PQ \\ &+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) + c_rQq\theta_3 \\ &+ d + c_b\bar{B}(r) + c_l\bar{s}(r) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + cQ \end{aligned} \tag{265}$$

We have cycle time $T = \frac{Q(1-p)}{D}$, so the expected total annual cost would be

$$\begin{aligned} ETC(Q,r) = \Bigg(& K_V + K_B + F + h_V \left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD} \right) + c_P Q \\ & + h_B \left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D} \right) + c_r Q q \theta_3 \\ & + d + c_b \bar{B}(r) + c_l \bar{s}(r) + \frac{Q\eta}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + cQ \Bigg) \frac{D}{Q(1-p)} \end{aligned} \tag{266}$$

To minimise $ETC(Q,r)$, we take first derivative with respect to Q and r,

$$\begin{aligned} \frac{\partial(ETC)}{\partial Q} = & \frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)+h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2} \\ & - \frac{h_V(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2(p-1)P} \end{aligned} \tag{267}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+K_B+K_V+d)}{(1-p)Q^3} \tag{268}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-2PD(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{h_B\left(p^2-1\right)P+h_V(q(\theta_3-1)+1)(P(q(1-\theta_3)-1)+D)}} \tag{269}$$

The derivative of $ETC(Q,r)$ with respect to r is

$$\frac{\partial(ETC)}{\partial r} = \frac{Dc_b\bar{B}'(r)+Dc_l\bar{s}'(r)+h_B(1-p)Q(1+\bar{s}'(r))}{(1-p)Q} \tag{270}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial r^2} = \frac{Dc_b\bar{B}''(r)+\bar{s}''(r)(h_B(1-p)Q+c_lD)}{(1-p)Q} \tag{271}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial r^2} > 0$. Therefore, there exists unique value for r^* given as

$$
\begin{aligned}
F(r) &= \left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right) \\
r^* &= F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right)
\end{aligned} \tag{272}
$$

Similar to the previous case, the rest of the models, r^* equation will be the same since the integrated cost function is changing only by constant.

For the model where q is deterministic and θ_3 is stochastic with triangular distribution, first we will calculate supplier's cost. With θ_3 given as random variable, it has an upper (U_2) and lower (L_2) bounds with the probability distribution function of g_2 and mode is m_2. That is,

$$
g_2 = \begin{cases} \frac{2(\theta_3-L_2)}{(U_2-L_2)(m_2-L_2)} & L_2 \leq \theta_3 \leq m_2 \\ \\ \frac{2(U_2-\theta_3)}{(U_2-L_2)(U_2-m_2)} & m_2 \leq \theta_3 \leq U_2 \end{cases}
$$

The expected value of the total inventory of nondefective items during t_1 will be

$$
\begin{aligned}
E(\Delta_1) &= \int_{L_2}^{U_2} \frac{((1-q)P-D)Q_V^2}{2P^2} g_2 d\theta_3 \\
&= \frac{((1-q)P-D)Q_V^2}{2P^2}\left(\int_{L_2}^{m_2} \frac{2(\theta_3-L_2)}{(U_2-L_2)(m_2-L_2)}d\theta_3 + \int_{m_2}^{U_2} \frac{2(U_2-\theta_3)}{(U_2-L_2)(U_2-m_2)}d\theta_3\right) \\
&= \frac{((1-q)P-D)Q_V^2}{2P^2}
\end{aligned} \tag{273}
$$

And the inventory during t_2 is

$$
\begin{aligned}
E(\Delta_2) &= \int_{L_2}^{U_2} \left(\frac{(P-D)q^2\theta_3^2Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3Q_V^2}{P^2}\right) g_2 d\theta_3 \\
&= \frac{qQ_V^2\left(L_2^2q + L_2(q(m_2+U_2-4)+4) + q\left(m_2^2+(m_2+U_2)(U_2-4)\right) + 4(m_2+U_2)\right)}{12P} \\
&- \frac{qQ_V^2D\left(L_2^2q + L_2(q(m_2+U_2)+4) + q\left(m_2^2+m_2U_2+U_2^2\right) + 4(m_2+U_2)\right)}{12P^2}
\end{aligned} \tag{274}
$$

During t_3, the inventory becomes

$$\begin{aligned}
E(\Delta_3) &= \int_{L_2}^{U_2} \frac{1}{2} D \left(\frac{(P-D)q\theta_3 Q_V + ((1-q)P-D)Q_V}{PD} \right)^2 g_2 d\theta_3 \\
&= \frac{Q_V^2 q^2 \left(L_2^2 + L_2(m_2+U_2-4) + m_2^2 + (m_2+U_2)(U_2-4) + 6 \right)}{12D} \\
&- \frac{Q_V^2 q^2 \left(L_2^2 + L_2(m_2+U_2-2) + (m_2-2)(U_2+m_2) + U_2^2 \right)}{6P} \\
&+ \frac{Q_V^2 \left(2q(L_2+m_2+U_2-3)+3 \right)}{6D} - \frac{Q_V^2 \left(q(2L_2+2m_2+2U_2-3)+3 \right)}{3P} \\
&+ \frac{Q_V^2 D \left(q \left(L_2^2 q + L_2(q(m_2+U_2)+4) + q\left(m_2^2 + m_2 U_2 + U_2^2 \right) + 4(m_2+U_2) \right) + 6 \right)}{12P^2}
\end{aligned} \tag{275}$$

The expected value of the total inventory of defective items during t_1 will be

$$\begin{aligned}
E(\Gamma_1) &= \int_{L_2}^{U_2} \frac{q\theta_3 Q_V^2}{2P} g_2 d\theta_3 \\
&= \frac{qQ_V^2 (L_2+m_2+U_2)}{6P}
\end{aligned} \tag{276}$$

and during t_2

$$\begin{aligned}
E(\Gamma_2) &= \int_{L_2}^{U_2} \frac{q^2 Q_V^2 \theta_3^2}{2P} g_2 d\theta_3 \\
&= \frac{q^2 Q_V^2 \left(L_2^2 + L_2(m_2+U_2) + m_2^2 + m_2 U_2 + U_2^2 \right)}{12P}
\end{aligned} \tag{277}$$

The expected annual cost is

$$\begin{aligned}
ETC_V(Q_V) &= K_V + h_V(E(\Delta_1) + E(\Delta_2) + E(\Delta_3) + E(\Gamma_1) + E(\Gamma_2)) \\
&+ \frac{\eta Q}{\delta_1} \ln\left(\frac{q_0}{q} \right) + \frac{\eta Q}{\delta_2} E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}} \right) \right] + c_P Q + c_r q \mu_{\theta_3} Q
\end{aligned} \tag{278}$$

where $\mu_{\theta_3} = \frac{L_2+m_2+U_2}{3} = \frac{3}{4}$ and $E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}} \right) \right] \left(= \mu_{\ln(\theta_3/\theta_{3_0})} \right)$ is derived as follows:

$$\begin{aligned}
E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}} \right) \right] &= E\left[\ln(\theta_3) - \ln(\theta_{3_0}) \right] \\
&= E[\ln(\theta_3)] - E[\ln(\theta_{3_0})]
\end{aligned} \tag{279}$$

Here,

$$E[\ln(\theta_{3_0})] = \ln(\theta_{3_0}) \tag{280}$$

since θ_{3_0} is constant. However,

$$E[\ln(\theta_3)] \neq \ln[E(\theta_3)] \tag{281}$$

Therefore, we need to find $E(\ln(\theta_3))$ which is a function of θ_3.

$$\begin{aligned}
E(\ln\theta_3) &= \int_{-\infty}^{\infty} \ln\theta_3 \cdot g_2 d\theta_3 \\
&= \int_{L_2}^{m_2} \ln\theta_3 \frac{2(\theta_3 - L_2)}{(U_2 - L_2)(m_2 - L_2)} d\theta_3 + \int_{m_2}^{U_2} \ln\theta_3 \frac{2(U_2 - \theta_3)}{(U_2 - L_2)(U_2 - m_2)} d\theta_3 \\
&= \frac{L_2^2 \ln(L_2)(m_2 - U_2) - (L_2 - U_2)\left(3(L_2 - m_2)(m_2 - U_2) + 2m_2^2 \ln(m_2)\right)}{(L_2 - m_2)(L_2 - U_2)(m_2 - U_2)} \\
&+ \frac{U_2^2 \ln(U_2)}{(L_2 - U_2)(m_2 - U_2)}
\end{aligned} \tag{282}$$

Finally,

$$\begin{aligned}
E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right] &= \frac{L_2^2 \ln(L_2)(m_2 - U_2) - (L_2 - U_2)\left(3(L_2 - m_2)(m_2 - U_2) + 2m_2^2 \ln(m_2)\right)}{(L_2 - m_2)(L_2 - U_2)(m_2 - U_2)} \\
&+ \frac{U_2^2 \ln(U_2)}{(L_2 - U_2)(m_2 - U_2)} - \ln(\theta_{3_0})
\end{aligned} \tag{283}$$

The supplier's cost function per cycle will be

$$\begin{aligned}
ETC_V(Q) = K_V &+ h_V \left(\frac{Q^2 \left(P\left(4q(L_2 + m_2 + U_2 - 3) + 6\right) - 2D(q(L_2 + m_2 + U_2 - 3) + 3)\right)}{12PD} \right) \\
&+ h_V \left(\frac{Q^2 q^2 \left(L_2^2 + L_2(m_2 + U_2 - 4) + m_2^2 + (m_2 + U_2)(U_2 - 4) + 6\right)}{12D} \right) \\
&+ \frac{\eta Q}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2} \mu_{\ln(\theta_3/\theta_{3_0})} + c_P Q + c_r q Q \mu_{\theta_3}
\end{aligned} \tag{284}$$

and the buyer's total average cost with stochastic demand in a supplier's cycle is

$$\begin{aligned}
ETC_B(Q, r) = K_B &+ h_B \left(\frac{1}{2} \frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r)) \frac{Q(1-p)}{D} \right) \\
&+ F + cQ + d + c_b \bar{B}(r) + c_l \bar{s}(r)
\end{aligned} \tag{285}$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(P\left(4q(L_2+m_2+U_2-3)+6\right)-2D(q(L_2+m_2+U_2-3)+3)\right)}{12PD}\right) \\
&\quad + h_V\left(\frac{Q^2q^2\left(L_2^2+L_2(m_2+U_2-4)+m_2^2+(m_2+U_2)(U_2-4)+6\right)}{12D}\right) \\
&\quad + K_V + F + cQ + c_PQ + c_rqQ\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&\quad + d + c_b\bar{B}(r) + c_l\bar{s}(r)
\end{aligned}
\tag{286}
$$

The expected total annual cost is

$$
\begin{aligned}
ETC(Q,r) = \Bigg(& K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right) \\
& + h_V\left(\frac{Q^2\left(P\left(4q(L_2+m_2+U_2-3)+6\right)-2D(q(L_2+m_2+U_2-3)+3)\right)}{12PD}\right) \\
& + h_V\left(\frac{Q^2q^2\left(L_2^2+L_2(m_2+U_2-4)+m_2^2+(m_2+U_2)(U_2-4)+6\right)}{12D}\right) \\
& + K_V + F + cQ + c_PQ + c_rqQ\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
& + d + c_b\bar{B}(r) + c_l\bar{s}(r)\Bigg)\frac{D}{Q(1-p)}
\end{aligned}
\tag{287}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} &= \frac{12PD(c_b\bar{B}+d+F+c_l\bar{s}+K_B+K_V)+6h_B\left(p^2-1\right)PQ^2}{12(p-1)PQ^2} \\
&\quad - \frac{h_V\left(2q(L_2+m_2+U_2-3)+3\right)-D(q(L_2+m_2+U_2-3)+3)}{6(p-1)} \\
&\quad - \frac{h_Vq^2\left(L_2^2+L_2(m_2+U_2-4)+m_2^2+(m_2+U_2)(U_2-4)+6\right)}{12(p-1)}
\end{aligned}
\tag{288}
$$

Taking the second derivative, we have

$$
\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{(1-p)Q^3} \tag{289}
$$

All parameters are positive, and $\frac{d^2(ETC)}{dQ^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-12PD(c_b\bar{B}+d+F+c_l\bar{s}+K_B+K_V)}{6h_B\left(p^2-1\right)P - h_V P\left(q^2A+4qC+6\right)+2h_V D(qC+3)}} \tag{290}$$

where $A = \left(L_2^2+L_2(m_2+U_2-4)+m_2^2+(U_2-4)(m_2+U_2)+6\right)$ and $C = (L_2+m_2+U_2-3)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1-\frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right) \tag{291}$$

For the next model, q is stochastic with standard uniform distribution and θ_3 is deterministic. With q given as random variable, it has an upper (U_1) and lower (L_1) bounds with the probability distribution function of g_1 and mode m_1. That is,

$$g_1 = \begin{cases} \frac{2(q-L_1)}{(U_1-L_1)(m_1-L_1)} & L_1 \le q \le m_1 \\ \\ \frac{2(U_1-q)}{(U_1-L_1)(U_1-m_1)} & m_1 \le q \le U_1 \end{cases}$$

The expected value of the total inventory of nondefective items during t_1 will be

$$\begin{aligned} E(\Delta_1) &= \int_{L_1}^{U_1} \frac{((1-q)P-D)Q_V^2}{2P^2} g_1 dq \\ &= \int_{L_1}^{m_1} \frac{((1-q)P-D)Q_V^2}{2P^2} \cdot \frac{2(q-L_1)}{(U_1-L_1)(m_1-L_1)} dq \\ &+ \int_{m_1}^{U_1} \frac{((1-q)P-D)Q_V^2}{2P^2} \cdot \frac{2(U_1-q)}{(U_1-L_1)(U_1-m_1)} dq \\ &= -\frac{Q^2(P(L_1+m_1+U_1-3)+3D)}{6P^2} \end{aligned} \tag{292}$$

And the inventory during t_2 is

$$\begin{aligned}
E(\Delta_2) &= \int_{L_1}^{U_1} \left(\frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2} \right) g_1 dq \\
&= \frac{Q_V^2\theta_3 \left(L_1^2(\theta_3-2)+L_1((m_1+U_1)(\theta_3-2)+4)+(\theta_3-2)\left(m_1^2+m_1U_1+U_1^2\right)\right)}{12P} \\
&- \frac{Q_V^2\theta_3 D\left(L_1^2\theta_3+L_1(\theta_3(m_1+U_1)+4)+\theta_3\left(m_1^2+m_1U_1+U_1^2\right)+4(m_1+U_1)\right)}{12P^2} \\
&+ \frac{Q_V^2\theta_3(m_1+U_1)}{3P}
\end{aligned} \tag{293}$$

During t_3, the inventory becomes

$$\begin{aligned}
E(\Delta_3) &= \int_{L_1}^{U_1} \frac{1}{2}D\left(\frac{(P-D)q\theta_3 Q_V+((1-q)P-D)Q_V}{PD}\right)^2 g_1 dq \\
&= \frac{Q_V^2\left(L_1^2(\theta_3-1)^2+L_1(\theta_3-1)(m_1+U_1)(\theta_3-1)+4)+m_1^2(\theta_3-1)^2\right)}{12D} \\
&+ \frac{Q_V^2\left(m_1(\theta_3-1)((\theta_3-1)U_1+4)+(\theta_3-1)U_1((\theta_3-1)U_1+4)+6\right)}{12D} \\
&- \frac{Q_V^2\left(L_1^2(\theta_3-1)\theta_3+L_1\left(\theta_3^2(m_1+U_1)-\theta_3(m_1+U_1-4)-2\right)\right)}{6P} \\
&- \frac{Q_V^2\left(\theta_3\left(m_1^2(\theta_3-1)+(m_1+U_1)((\theta_3-1)U_1+4)\right)-2(m_1+U_1-3)\right)}{6P} \\
&+ \frac{Q_V^2 D\left(\theta_3\left(L_1^2\theta_3+L_1(\theta_3(m_1+U_1)+4)+\theta_3\left(m_1^2+m_1U_1+U_1^2\right)+4(m_1+U_1)\right)+6\right)}{12P^2}
\end{aligned} \tag{294}$$

The expected value of the total inventory of defective items during t_1 will be

$$\begin{aligned}
E(\Gamma_1) &= \int_{L_1}^{U_1} \frac{q\theta_3 Q_V^2}{2P} g_1 dq \\
&= \frac{Q_V^2\theta_3(L_1+m_1+U_1)}{6P}
\end{aligned} \tag{295}$$

and during t_2

$$\begin{aligned}
E(\Gamma_2) &= \int_{L_2}^{U_2} \frac{q^2 Q_V^2\theta_3^2}{2P} g_2 d\theta_3 \\
&= \frac{Q_V^2\theta_3^2\left(L_1^2+(L_1+m_1)(m_1+U_1)+U_1^2\right)}{12P}
\end{aligned} \tag{296}$$

The expected annual cost is

$$ETC_V(Q_V) = K_V + h_V(E(\Delta_1) + E(\Delta_2) + E(\Delta_3) + E(\Gamma_1) + E(\Gamma_2)) + \frac{\eta Q}{\delta_1} E\left[\ln\left(\frac{q_0}{q}\right)\right] + \frac{\eta Q}{\delta_2} \ln\left(\frac{\theta_3}{\theta_{3_0}}\right) + c_P Q + c_r \mu_q \theta_3 Q \tag{297}$$

where $\mu_q = \frac{L_1+m_1+U_1}{3} = \frac{1}{5}$ and $E\left[\ln\left(\frac{q_0}{q}\right)\right] \left(= \mu_{\ln(q_0/q)}\right)$ is derived as follows:

$$E\left[\ln\left(\frac{q_0}{q}\right)\right] = E\left[\ln(q_0) - \ln(q)\right] = E[\ln(q_0)] - E[\ln(q)] \tag{298}$$

Here,

$$E[\ln(q_0)] = \ln(q_0) \tag{299}$$

since q_0 is constant. However,

$$E[\ln(q)] \neq \ln[E(q)] \tag{300}$$

Therefore, we need to find $E(\ln(q))$ which is a function of q.

$$\begin{aligned} E(\ln q) &= \int_{-\infty}^{\infty} \ln q \cdot g_1 dq \\ &= \int_{L_1}^{m_1} \ln q \cdot \frac{2(q-L_1)}{(U_1-L_1)(m_1-L_1)} dq + \int_{m_1}^{U_1} \ln q \cdot \frac{2(U_1-q)}{(U_1-L_1)(U_1-m_1)} dq \\ &= \frac{2L_1^2 \ln(L_1)(m_1-U_1) - (L_1-U_1)\left(3(L_1-m_1)(m_1-U_1) + 2m_1^2 \ln(m_1)\right)}{2(L_1-m_1)(L_1-U_1)(m_1-U_1)} \\ &+ \frac{U_1^2 \ln(U_1)}{(L_1-U_1)(m_1-U_1)} \end{aligned} \tag{301}$$

Finally,

$$E\left[\ln\left(\frac{q_0}{q}\right)\right] = \ln(q_0) + \frac{U_1^2\ln(U_1)}{(L_1-U_1)(m_1-U_1)} - \frac{2L_1^2\ln(L_1)(m_1-U_1)-(L_1-U_1)\left(3(L_1-m_1)(m_1-U_1)+2m_1^2\ln(m_1)\right)}{2(L_1-m_1)(L_1-U_1)(m_1-U_1)} \tag{302}$$

Therefore, supplier's cost function per cycle is

$$\begin{aligned} ETC_V(Q) = K_V &+ h_V\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2+L_1(\theta_3-1)((m_1+U_1)(\theta_3-1)+4)\right)}{12D}\right)\\ &+ h_V\left(\frac{Q^2\left(m_1^2(\theta_3-1)^2+(m_1+U_1)(\theta_3-1)((\theta_3-1)U_1+4)+6\right)}{12D}\right)\\ &- h_V\left(\frac{Q^2\left((\theta_3-1)(L_1+m_1+U_1)+3\right)}{6P}\right)\\ &+ c_PQ + c_rQ\theta_3\mu_q + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \end{aligned} \tag{303}$$

and the buyer's expected total cost with stochastic demand in the supplier's cycle is

$$\begin{aligned} ETC_B(Q,r) = K_B &+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)\\ &+ F + cQ + d + c_b\bar{B}(r) + c_l\bar{s}(r) \end{aligned} \tag{304}$$

The expected total cost for integrated system is,

$$\begin{aligned} ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r)\\ ETC(Q,r) &= K_V + K_B + F + cQ + c_PQ + c_rQ\theta_3\mu_q + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\\ &\quad + h_V\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2+L_1(\theta_3-1)((m_1+U_1)(\theta_3-1)+4)\right)}{12D}\right)\\ &\quad + h_V\left(\frac{Q^2\left(m_1^2(\theta_3-1)^2+(m_1+U_1)(\theta_3-1)((\theta_3-1)U_1+4)+6\right)}{12D}\right)\\ &\quad - h_V\left(\frac{Q^2\left((\theta_3-1)(L_1+m_1+U_1)+3\right)}{6P}\right)\\ &\quad + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)\\ &\quad + d + c_b\bar{B}(r) + c_l\bar{s}(r) \end{aligned} \tag{305}$$

The expected total annual cost is

$$ETC(Q,r) = \left(K_V + K_B + F + cQ + c_P Q + c_r Q\theta_3\mu_q + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right.$$
$$+ h_V\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2 + L_1(\theta_3-1)((m_1+U_1)(\theta_3-1)+4)\right)}{12D}\right)$$
$$+ h_V\left(\frac{Q^2\left(m_1^2(\theta_3-1)^2 + (m_1+U_1)(\theta_3-1)((\theta_3-1)U_1+4)+6\right)}{12D}\right) \tag{306}$$

$$- h_V\left(\frac{Q^2\left((\theta_3-1)(L_1+m_1+U_1)+3\right)}{6P}\right)$$
$$+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)$$
$$\left. + d + c_b\bar{B}(r) + c_l\bar{s}(r)\right)\frac{D}{Q(1-p)} \tag{307}$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$\frac{\partial(ETC)}{\partial Q} = \frac{2D(c_b\bar{B} + d + F + c_l\bar{s} + K_B + K_V) + h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2}$$
$$- \frac{h_V\left(L_1^2(\theta_3-1)^2 + L_1(\theta_3-1)((m_1+U_1)(\theta_3-1)+4) + m_1^2(\theta_3-1)^2\right)}{12(p-1)}$$
$$- \frac{h_V\left((m_1+U_1)(\theta_3-1)((\theta_3-1)U_1+4)+6\right)}{12(p-1)}$$
$$+ \frac{h_V D\left((\theta_3-1)(L_1+m_1+U_1)+3\right)}{6(p-1)P} \tag{308}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V)}{(1-p)Q^3} \tag{309}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{12PD(c_b\bar{B} + d + F + c_l\bar{s} + K_B + K_V)}{P\left(h_V B - 6h_B\left(p^2-1\right)\right) - h_V\left(D(6 - 2(\theta_3-1)(L_1+m_1+U_1))\right)}} \tag{310}$$

where $B = \left(L_1^2(\theta_3-1)^2 + L_1(\theta_3-1)((\theta_3-1)(m_1+U_1+4)) + m_1^2(\theta_3-1)^2\right)$

$+(((\theta_3-1)U_1+4)(m_1(\theta_3-1)(m_1+U_1))+6)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1-\frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right) \tag{311}$$

Our final model is where q and θ_3 are both stochastic with triangular distribution. With q and θ_3 given as random variables, they have an upper (U_1,U_2) and lower (L_1,L_2) bounds with the probability distribution function of g_1, g_2 and modes are $m_1 = 0.2, m_2 = 0.75$. The expected value of the total inventory of nondefective items during t_1 will be

$$\begin{aligned} E(\Delta_1) &= \int_{L_2}^{U_2}\int_{L_1}^{U_1} \frac{((1-q)P-D)Q_V^2}{2P^2} g_1 g_2 dq d\theta_3 \\ &= \frac{Q_V^2(L_2-m_2)(m_1-L_1)(P(L_1+2m_1-3)+3D)}{6P^2(L_2-U_2)(L_1-U_1)} \\ &+ \frac{Q_V^2(U_1-m_1)(m_2-U_2)(P(2m_1+U_1-3)+3D)}{6P^2(L_2-U_2)(L_1-U_1)} \end{aligned} \tag{312}$$

And the inventory during t_2 is

$$\begin{aligned} E(\Delta_2) = &\int_{L_2}^{U_2}\int_{L_1}^{U_1} \left(\frac{(P-D)q^2\theta_3^2 Q_V^2}{2P^2} + \frac{((1-q)P-D)q\theta_3 Q_V^2}{P^2}\right) g_1 g_2 dq d\theta_3 \\ = &\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(L_2^2\left(L_1^2+2L_1m_1+3m_1^2\right)(P-D)\right)}{72P^2(L_2-U_2)(L_1-U_1)} \\ + &\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(2L_2P\left(L_1^2(m_2-2)+2L_1(m_1(m_2-2)+2)\right)\right)}{72P^2(L_2-U_2)(L_1-U_1)} \\ + &\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(2L_2P(m_1(3m_1(m_2-2)+8))\right)}{72P^2(L_2-U_2)(L_1-U_1)} \\ - &\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(2L_2D\left(L_1^2m_2+2L_1(m_1m_2+2)+m_1(3m_1m_2+8)\right)\right)}{72P^2(L_2-U_2)(L_1-U_1)} \end{aligned}$$

$$
\begin{aligned}
&+\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(m_2\left(L_1^2(3m_2-8)+2L_1(m_1(3m_{\theta_3}-8)+8)\right)\right)}{72P(L_2-U_2)(L_1-U_1)}\\
&+\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(m_2\left(m_1(9m_1m_2-24m_1+32)\right)\right)}{72P(L_2-U_2)(L_1-U_1)}\\
&-\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(m_2D\left(3L_1^2m_2+2L_1(3m_2m_2+8)+m_1(9m_1m_2+32)\right)\right)}{72P^2(L_2-U_2)(L_1-U_1)}\\
&+\frac{Q_V^2(m_1-U_1)(m_2-U_2)\left(m_1^2\left(3m_2^2+2m_2(U_2-4)+(U_2-4)U_2\right)\right)}{24P(L_2-U_2)(L_1-U_1)}\\
&+\frac{Q_V^2(m_1-U_1)(m_2-U_2)\left(2m_1\left(3m_2^2U_1+2m_2((U_1+U_2)(U_2-4)+8)\right)\right)}{72P(L_2-U_2)(L_1-U_1)}\\
&+\frac{Q_V^2(m_1-U_1)(m_2-U_2)\left(P\left(U_1\left(3m_2^2U_1+(2m_2+U_2)(U_1(U_2-4)+8)\right)\right)\right)}{72P^2(L_2-U_2)(L_1-U_1)}\\
&-\frac{Q_V^2\left((m_1-U_1)(m_2-U_2)\left(D\left(m_1^2\left(3m_2^2+2m_2U_2+U_2^2\right)\right)\right)\right)}{24P^2(L_2-U_2)(L_1-U_1)}\\
&-\frac{Q_V^2(m_1-U_1)(m_2-U_2)((D(2m_q+U_1)(3m_2^2U_1+(2m_2+U_2)(U_1U_2+8)))}{72P^2(L_2-U_2)(L_1-U_1)}
\end{aligned}
\tag{313}
$$

During t_3, the inventory becomes

$$
\begin{aligned}
E(\Delta_3)&=\int_{L_2}^{U_2}\int_{L_1}^{U_1}\frac{1}{2}D\left(\frac{(P-D)q\theta_3Q_V+((1-q)P-D)Q_V}{PD}\right)^2 g_1g_2\,dq\,d\theta_3\\
&=\frac{Q_V^2(L_2-m_2)(L_1-m_1)(L_1^2+3m_1^2)\left(L_2^2+2L_2(m_2-2)+m_2(3m_2-8)+6\right)}{72D(L_2-U_2)(L_1-U_1)}\\
&+\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(L_1\left((L_2-4)m_1(2m_{\theta_3}+L_2)+4L_2+3m_1m_2^2\right)\right)}{36D(L_2-U_2)(L_1-U_1)}\\
&+\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(L_1(6m_1+8m_2-12)+8m_1(L_2+2m_2-3)+18\right)}{36D(L_2-U_2)(L_1-U_1)}\\
&-\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(L_1^2\left(L_2^2+2L_2(m_2-1)+m_2(3m_2-4)\right)\right)}{36P(L_2-U_2)(L_1-U_1)}\\
&-\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(L_1\left((L_2-2)m_1(2m_{\theta_3}+L_2)+4L_2\right)\right)}{36P(L_2-U_2)(L_1-U_1)}\\
&-\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(L_1\left(3m_1m_2^2+8m_2-6\right)\right)}{18P(L_2-U_2)(L_1-U_1)}\\
&-\frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(m_1^2\left((L_2-2)(2m_{\theta_3}+L_2)+3m_2^2\right)\right)}{12P(L_2-U_2)(L_1-U_1)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{Q_V^2(L_2-m_2)(L_1-m_1)\,(2m_1(2L_2+4m_2-3)+9)}{9P(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(D\left((2m_2+L_2)\left((L_2L_1+8)(2m_1+L_1)+3L_2m_1^2\right)\right)\right)}{72P^2(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2(L_2-m_2)(L_1-m_1)\left(D\left(m_2^2\left(L_1^2+2L_1m_1+3m_1^2\right)+12\right)\right)}{24P^2(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2(m_1-U_1)(m_2-U_2)\left(m_1^2\left(3m_2^2+(U_2-4)(2m_2+U_2)+6\right)\right)}{24D(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2(m_1-U_1)(m_2-U_2)(m_1(3m_2^2U_1+2m_2(U_1(U_2-4)+8)+U_1((U_2-4)U_2+6)))}{36D(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2(m_1-U_1)(m_2-U_2)\left(16m_1(U_2-3)+U_1^2\left(3m_2^2+(U_2-4)(2m_2+U_2)+6\right)\right)}{72D(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2(m_1-U_1)(m_2-U_2)\,(2U_1(2m_2+U_2-3)+9)}{18D(L_2-U_2)(L_1-U_1)}\\
&- \frac{Q_V^2(m_1-U_1)(m_2-U_2)\left(m_1^2\left(3m_2^2+(U_2-2)(2m_2+U_2)\right)\right)}{12P(L_2-U_2)(L_1-U_1)}\\
&- \frac{Q_V^2(m_1-U_1)(m_2-U_2)\left((2m_1+U_1)\left(3m_2^2U_1+(U_1(U_2-2)+8(2m_2+U_2)-12\right)\right)}{36P(L_2-U_2)(L_1-U_1)}\\
&- \frac{12Q_V^2(m_1-U_1)(m_2-U_2)+D^2\left(m_1^2\left(3m_2^2+2m_2U_2+U_2^2\right)\right)}{12P(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2(m_1-U_1)(m_2-U_2)\left(Dm_1\left(3m_2^2U_1+(2m_2+U_2)(U_1U_2+8)\right)\right)}{36P^2(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2(m_1-U_1)(m_2-U_2)\left(D\left(U_1\left(3m_2^2U_1+(2m_2+U_2)(U_1U_2+8)\right)+36\right)\right)}{72P^2(L_2-U_2)(L_1-U_1)} \qquad (314)
\end{aligned}
$$

The expected value of the total inventory of defective items during t_1 will be

$$
\begin{aligned}
E(\Gamma_1) &= \int_{L_2}^{U_2}\int_{L_1}^{U_1} \frac{q\theta_3 Q_V^2}{2P} g_1 g_2\, dq\, d\theta_3\\
&= \frac{Q_V^2((L_2-m_2)(L_2+2m_2)(L_1-m_1)(L_1+2m_1))}{18P(L_2-U_2)(L_1-U_1)}\\
&+ \frac{Q_V^2((m_1-U_1)(2m_1+U_1)(m_2-U_2)(2m_2+U_2))}{18P(L_2-U_2)(L_1-U_1)} \qquad (315)
\end{aligned}
$$

and during t_2

$$E(\Gamma_2) = \int_{L_2}^{U_2}\int_{L_2}^{U_2} \frac{q^2 Q_V^2 \theta_3^2}{2P} g_1 g_2 dq d\theta_3$$
$$= \frac{Q_V^2\left((L_2-m_2)\left(L_2^2+2L_2m_2+3m_2^2\right)(L_1-m_1)\left(L_1^2+2L_1m_1+3m_1^2\right)\right)}{72P(L_2-U_2)(L_1-U_1)}$$
$$+\frac{Q_V^2(m_1-U_1)\left(3m_1^2+2m_1U_1+U_1^2\right)(m_2-U_2)\left(3m_2^2+2m_2U_2+U_2^2\right)}{72P(L_2-U_2)(L_1-U_1)} \quad (316)$$

The expected annual cost is

$$ETC_V(Q) = K_V + h_V(E(\Delta_1)+E(\Delta_2)+E(\Delta_3)+E(\Gamma_1)+E(\Gamma_2))$$
$$+\frac{\eta Q}{\delta_1}E\left[\ln\left(\frac{q_0}{q}\right)\right]+\frac{\eta Q}{\delta_2}E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right]+c_PQ+c_r\mu_q\mu_{\theta_3}Q \quad (317)$$

where $\mu_q = \frac{L_1+m_1+U_1}{3} = \frac{1}{5}, \mu_{\theta_3} = \frac{L_2+m_2+U_2}{3} = \frac{3}{4}$ and $E\left[\ln\left(\frac{q_0}{q}\right)\right], E\left[\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\right]$ are derived in previous models. Supplier's cost function per cycle is

$$ETC_V(Q) = K_V + c_PQ + c_rQ\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})}$$
$$+h_V\left(\frac{Q^2\left((L_1-m_1)\left(L_1^2+2m_1L_1+3m_1^2\right)PL_2^3\right)}{72P(L_1-U_1)(L_2-U_2)D}+\frac{Q^2\left(U_1U_2(9-m_1^2m_2+2m_1m_2)\right)}{18(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2(L_1-m_1)\left((m_2-4)(L_1^2+2L_1m_13m_q^2)+8(L_1+2m_q)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2(L_1-m_1)\left((m_2-4)(m_2L_1^2+8L_1+2m_qL_1m_2)+6L_1(L_1^3+2m_q)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2(L_1-m_1)\left(16m_1(m_2-3)+3m_1^2((m_2-4)m_2+6)+36\right)}{72(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(4D(L_1-m_1)((L_1+2m_1)(3-m_2)-9)L_2-m_2^2PU_1^3(3m_{\theta_3}+8)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(U_1^3(U_2^3-6m_2)+m_1U_1^2(U_2^3-3m_2^3)-m_1^3U_2^3(3m_q+U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(m_2U_1^2(8m_1m_2-16m_2-6m_1+24+U_1^3)\right)}{72(L_1-U_1)(L_2-U_2)D}-\frac{Q^2(L_1-m_1)(L_1+2m_1)L_2^2)}{18P(L_1-U_1)(L_2-U_2)}\right)$$

$$
\begin{aligned}
&+h_V\left(\frac{Q^2\left(U_2^2(m_1m_2U_1^2-4U_1^3-4m_1U_1^22+8U_1^2+12m_1^3-16m_1^2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(m_1(8U_1U_2^2-3m_1^2m_2U_2^2-4m_1U_1U_2^2+m_1m_2U_1U_2^2+18m_1^2m_2^3)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(m_1m_2(18-12m_1^2m_2+16m_1m_2+9m_1^2-24m_1)\right)}{18(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(L_1^3m_2((8-3m_2)m_2-6)-m_1m_2^2U_1(3m_1m_2+8m_1-16U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(m_2U_1(24m_1-6m_1^2-36+m_2U_1^2U_2-4U_1^2U_2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(U_1^2U_2(6U_1+m_1m_2^2+6m_1-4m_1m_2+8m_2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(U_2(16m_1^2-8U_1^2-6m_1^3-m_1^3m_2^2-12m_1+4m_1^3m_2)\right)}{24(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(U_2m_q(-16m_1m_2+6m_1U_1+m_1m_2^2U_1-24U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&-h_V\left(\frac{Q^2m_1^2\left(4m_2^2-(U_2+6)m_2-(U_2-3)U_2\right)}{9P(L_1-U_1)(L_2-U_2)D}\right)\\
&-h_V\left(\frac{Q^2\left((U_1(U_2+3)+18)m_2-2U_1m_2^2+(U_1(U_2-3)-9)U_2\right)m_1}{18P(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(U_1(m_2-U_2)(U_1(2m_2+U_2-3)+9)\right)}{18P(L_1-U_1)(L_2-U_2)}\right)\\
&+h_V\left(\frac{Q^2\left(L_1^2m_2((-16m_2+m_1((8-3m_2)m_2-6)+24)P+4(2m_2-3)D)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(L_1m_2\left(4(m_1(2m_2-3)+9)D-\left((m_2(3m_2-8)+6)m_1^2+8(2m_2-3)m_1+36\right)P\right)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)
\end{aligned}
\tag{318}
$$

and the buyer's expected total cost with deterministic demand in a supplier's cycle is

$$
\begin{aligned}
ETC_B(Q,r)=K_B+h_B&\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)\\
&+F+cQ+d+c_b\bar{B}(r)+c_l\bar{s}(r)
\end{aligned}
\tag{319}
$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_B + K_V + F + cQ + c_P Q + c_r Q \mu_q \mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&\quad + c_l \bar{s}(r) + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r))\frac{Q(1-p)}{D}\right) \\
&\quad + h_V\left(\frac{Q^2((L_1-m_1)(L_1^2+2m_1L_1+3m_1^2)PL_2^3)}{72P(L_1-U_1)(L_2-U_2)D} + \frac{Q^2(U_1U_2(9-m_1^2m_2+2m_1m_2))}{18(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + d + c_b\bar{B}(r) + h_V\left(\frac{Q^2(L_1-m_1)\left((m_2-4)(L_1^2+2L_1m_1 3m_q^2)+8(L_1+2m_q)\right)}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2(L_1-m_1)\left((m_2-4)(m_2L_1^2+8L_1+2m_qL_1m_2)+6L_1(L_1^3+2m_q)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2(L_1-m_1)\left(16m_1(m_2-3)+3m_1^2((m_2-4)m_2+6)+36\right)}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(4D(L_1-m_1)\left((L_1+2m_1)(3-m_2)-9\right)L_2 - m_2^2PU_1^3(3m_{\theta_3}+8)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(U_1^3(U_2^3-6m_2)+m_1U_1^2(U_2^3-3m_2^3)-m_1^3U_2^3(3m_q+U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2(m_2U_1^2(8m_1m_2-16m_2-6m_1+24+U_1^3))}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad - h_V\left(\frac{Q^2(L_1-m_1)(L_1+2m_1)L_2^2)}{18P(L_1-U_1)(L_2-U_2)}\right) \\
&\quad + h_V\left(\frac{Q^2\left(U_2^2(m_1m_2U_1^2-4U_1^3-4m_1U_1^2 2+8U_1^2+12m_1^3-16m_1^2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(m_1(8U_1U_2^2-3m_1^2m_2U_2^2-4m_1U_1U_2^2+m_1m_2U_1U_2^2+18m_1^2m_2^3)\right)}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(m_1m_2(18-12m_1^2m_2+16m_1m_2+9m_1^2-24m_1)\right)}{18(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(L_1^3m_2((8-3m_2)m_2-6)-m_1m_2^2U_1(3m_1m_2+8m_1-16U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(m_2U_1(24m_1-6m_1^2-36+m_2U_1^2U_2-4U_1^2U_2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(U_1^2U_2(6U_1+m_1m_2^2+6m_1-4m_1m_2+8m_2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(U_2(16m_1^2-8U_1^2-6m_1^3-m_1^3m_2^2-12m_1+4m_1^3m_2)\right)}{24(L_1-U_1)(L_2-U_2)D}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+h_V\left(\frac{Q^2\left(U_2m_q(-16m_1m_2+6m_1U_1+m_1m_2^2U_1-24U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&-h_V\left(\frac{Q^2m_1^2\left(4m_2^2-(U_2+6)m_2-(U_2-3)U_2\right)}{9P(L_1-U_1)(L_2-U_2)D}\right)\\
&-h_V\left(\frac{Q^2\left((U_1(U_2+3)+18)m_2-2U_1m_2^2+(U_1(U_2-3)-9)U_2\right)m_1}{18P(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(U_1(m_2-U_2)(U_1(2m_2+U_2-3)+9)\right)}{18P(L_1-U_1)(L_2-U_2)}\right)\\
&+h_V\left(\frac{Q^2\left(L_1^2m_2((-16m_2+m_1((8-3m_2)m_2-6)+24)P+4(2m_2-3)D)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(L_1m_2\left(4(m_1(2m_2-3)+9)D-\left((m_2(3m_2-8)+6)m_1^2+8(2m_2-3)m_1+36\right)P\right)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)
\end{aligned}
\tag{320}
$$

The expected total annual cost becomes

$$
\begin{aligned}
ETC(Q,r)=\Bigg(&K_B+K_V+F+cQ+c_PQ+c_rQ\mu_q\mu_{\theta_3}+\frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)}+\frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})}\\
&+c_l\bar{s}(r)+h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)\\
&+h_V\left(\frac{Q^2((L_1-m_1)(L_1^2+2m_1L_1+3m_1^2)PL_2^3)}{72P(L_1-U_1)(L_2-U_2)D}+\frac{Q^2(U_1U_2(9-m_1^2m_2+2m_1m_2))}{18(L_1-U_1)(L_2-U_2)D}\right)\\
&+d+c_b\bar{B}(r)+h_V\left(\frac{Q^2(L_1-m_1)\left((m_2-4)(L_1^2+2L_1m_13m_q^2)+8(L_1+2m_q)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2(L_1-m_1)\left((m_2-4)(m_2L_1^2+8L_1+2m_qL_1m_2)+6L_1(L_1^3+2m_q)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2(L_1-m_1)\left(16m_1(m_2-3)+3m_1^2((m_2-4)m_2+6)+36\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(4D(L_1-m_1)\left((L_1+2m_1)(3-m_2)-9\right)L_2-m_2^2PU_1^3(3m_{\theta_3}+8)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(U_1^3(U_2^3-6m_2)+m_1U_1^2(U_2^3-3m_2^3)-m_1^3U_2^3(3m_q+U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2(m_2U_1^2(8m_1m_2-16m_2-6m_1+24+U_1^3))}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&-h_V\left(\frac{Q^2(L_1-m_1)(L_1+2m_1)L_2^2)}{18P(L_1-U_1)(L_2-U_2)}\right)\\
&+h_V\left(\frac{Q^2\left(U_2^2(m_1m_2U_1^2-4U_1^3-4m_1U_1^22+8U_1^2+12m_1^3-16m_1^2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)\\
&+h_V\left(\frac{Q^2\left(m_1(8U_1U_2^2-3m_1^2m_2U_2^2-4m_1U_1U_2^2+m_1m_2U_1U_2^2+18m_1^2m_2^3)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)
\end{aligned}
$$

$$+h_V\left(\frac{Q^2\left(m_1m_2(18-12m_1^2m_2+16m_1m_2+9m_1^2-24m_1)\right)}{18(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(L_1^3m_2((8-3m_2)m_2-6)-m_1m_2^2U_1(3m_1m_2+8m_1-16U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(m_2U_1(24m_1-6m_1^2-36+m_2U_1^2U_2-4U_1^2U_2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(U_1^2U_2(6U_1+m_1m_2^2+6m_1-4m_1m_2+8m_2)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(U_2(16m_1^2-8U_1^2-6m_1^3-m_1^3m_2^2-12m_1+4m_1^3m_2)\right)}{24(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(U_2m_q(-16m_1m_2+6m_1U_1+m_1m_2^2U_1-24U_1)\right)}{72(L_1-U_1)(L_2-U_2)D}\right)$$
$$-h_V\left(\frac{Q^2m_1^2\left(4m_2^2-(U_2+6)m_2-(U_2-3)U_2\right)}{9P(L_1-U_1)(L_2-U_2)D}\right)$$
$$-h_V\left(\frac{Q^2\left((U_1(U_2+3)+18)m_2-2U_1m_2^2+(U_1(U_2-3)-9)U_2\right)m_1}{18P(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(L_1^2m_2((-16m_2+m_1((8-3m_2)m_2-6)+24)P+4(2m_2-3)D)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(L_1m_2\left(4(m_1(2m_2-3)+9)D-\left((m_2(3m_2-8)+6)m_1^2+8(2m_2-3)m_1+36\right)P\right)\right)}{72P(L_1-U_1)(L_2-U_2)D}\right)$$
$$+h_V\left(\frac{Q^2\left(U_1(m_2-U_2)(U_1(2m_2+U_2-3)+9)\right)}{18P(L_1-U_1)(L_2-U_2)}\right)\Bigg)\Bigg)\frac{D}{Q(1-p)} \tag{321}$$

To minimise $ETC(Q,r)$, we take first derivative with respect to Q

$$\frac{\partial(ETC)}{\partial Q}=\frac{L_2\left(72PD(L_1-U_1)(c_b\bar{B}+d+F+c_l\bar{s}+K_B+K_V)+36h_B\left(p^2-1\right)PQ^2(L_1-U_1)\right)}{72(p-1)PQ^2(L_2-U_2)(L_1-U_1)}$$
$$+\frac{L_2\left(-h_VQ^2(L_1-m_1)\left(P\left(L_1^2((m_2-4)m_2+6)\right)\right)\right)}{72(p-1)PQ^2(L_2-U_2)(L_1-U_1)}$$
$$+\frac{L_2\left(-h_VQ^2(L_1-m_1)\left(P\left(+2L_1(m_1((m_2-4)m_2+6)+4(m_2-3)\right)\right)\right)\right)}{72(p-1)PQ^2(L_2-U_2)(L_1-U_1)}$$
$$+\frac{L_2\left(-h_VQ^2(L_1-m_1)\left(P\left(+m_1(3m_1((m_2-4)m_2+6)+16(m_2-3))+36\right)\right)\right)}{72(p-1)PQ^2(L_2-U_2)(L_1-U_1)}$$
$$+\frac{L_2\left(-h_VQ^2(L_1-m_1)\left(-4(m_2-3)D(L_1+2m_1)-36D\right)\right)}{72(p-1)PQ^2(L_2-U_2)(L_1-U_1)}$$
$$-\frac{36U_2(L_1-U_1)\left(2D(c_b\bar{B}+d+F+c_l\bar{s}+K_B+K_V)+h_B\left(p^2-1\right)Q^2\right)}{72(p-1)Q^2(L_2-U_2)(L_1-U_1)}$$

$$
\begin{aligned}
&-\frac{L_2^3 h_V (L_1-m_1)\left(L_1^2+2L_1m_1+3m_1^2\right)}{72(p-1)(L_2-U_2)(L_1-U_1)}-\frac{h_V\left(L_1m_2\left(4D(2m_1m_2-3m_1+9)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&-\frac{L_2^2 h_V (L_1-m_1)\left(P\left(L_1^2(m_2-4)+2L_1(m_1(m_2-4)+4)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&-\frac{L_2^2 h_V (L_1-m_1)\left(P\left(+m_1(3m_1(m_2-4)+16)\right)-4D(L_1+2m_1)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(L_1^3m_2(m_2(3m_2-8)+6)P+L_1^2m_2\left(P\left(3m_1m_2^2-8(m_1-2)m_2+6(m_1-4)\right)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(L_1^2m_2\left(4(3-2m_2)D\right)+L_1m_2\left(P\left(m_1\left(3m_1m_2^2-8(m_1-2)m_2+6(m_1-4)\right)+36\right)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(3m_1^3P\left(-6m_2^3+m_2^2(U_2+16)+m_2(U_2-6)(U_2+2)+U_2((U_2-4)U_2+6)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(m_1^2\left(8D\left(4m_2^2-m_2(U_2+6)-(U_2-3)U_2\right)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(m_1^2\left(P\left(3m_2^3U_1-m_2^2(U_1(U_2+8)+64)\right)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(m_1^2\left(P\left(m_2(U_1(6-(U_2-4)U_2)+16(U_2+6))\right)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(m_1^2\left(P\left(U_2\left(-U_1U_2^2+4(U_1+4)U_2-6(U_1+8)\right)\right)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(4m_1D\left(-2m_2^2U_1+m_2(U_1(U_2+3)+18)+U_2(U_1(U_2-3)-9)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(m_1P\left(3m_2^3U_1^2-m_2^2U_1(U_1(U_2+8)-16)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(m_1P\left(-m_2(U_1(U_1((U_2-4)U_2-6)+8(U_2+3))+72)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(m_1P\left(U_2(36-U_1(U_1((U_2-4)U_2+6)+8(U_2-3)))\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(U_1(m_2-U_2)\left(P\left(U_1^2\left(3m_2^2+2m_2(U_2-4)+(U_2-4)U_2+6\right)\right)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)}\\
&+\frac{h_V\left(U_1(m_2-U_2)\left(P\left(+8U_1(2m_2+U_2-3)+36\right)-4D(U_1(2m_2+U_2-3)+9)\right)\right)}{72(p-1)P(L_2-U_2)(L_1-U_1)} \qquad (322)
\end{aligned}
$$

Taking the second derivative, we have

$$
\frac{\partial^2(ETC)}{\partial Q^2}=\frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{(1-p)Q^3} \qquad (323)
$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2}>0$. Therefore, there exists unique value for Q^* given as

$$
Q^*=\sqrt{\frac{-576000PD(L_2-U_2)(L_1-U_1)(c_b\bar{B}+d+F+c_l\bar{s}+K_B+K_V)}{288000h_B\left(p^2-1\right)P(L_2-U_2)(L_1-U_1)-Ch_V}} \qquad (324)
$$

where $C = (P(12L_2(-1+5L_1)(5L_1(95L_1-442)+3897)-45L_1(5L_1(45L_1-311)+4489))$
$+P\left(64L_2^3(-1+5L_1)(5L_1(5L_1+2)+3)-16L_2^2(-1+5L_1)(5L_1(65L_1-134)-281)\right)$
$+P\left(125U_1^3(4U_2-3)(8U_2(2U_2-5)+27)+25U_1^2(4U_2-3)(8U_2(2U_2+75)-933)\right)$
$+P\left(5U_1(4U_2-3)(8U_2(2U_2+75)+13467)-16U_2^2(12U_2+281)-46764U_2+75366\right)$
$-160D\left(8L_2^2(-1+5L_1)(5L_1+2)-18L_2(-1+5L_1)(5L_1-18)\right))$
$-160D\left(504+8(5U_1-1)(5U_1+2)U_2^2-18(5U_1-18)(5U_1-1)U_2)\right)$
$-1440(5L_1(5L_1-29)+5U_1(5U_1-29))$ and due to computational length, $m_2 = 0.75$ and $m_1 = 0.2$. The unique value for r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r)+\bar{s}(r)-r)-B_p(r)\right)\left(D(c_l-c_b)+h_BQ(1-p)\right)+\left(c_lD+h_BQ(1-p)\right)}\right) \tag{325}$$

3.2.5. Integrated Models with partial backordering when demand has exponential distribution

When there is partial backordering, demand can follow exponential distribution. The difference comes from probability density function ($f(x)$) in Eq.21 - 23 in **3.1.2.3**. Since there is no structural changes, this model will be run in R studio and shown in Illustrative Example with Table 7.

3.2.6. Integrated Models with complete backorder

With stochastic demand, we will analyse the complete backordering case at buyer's side. As the first model, supplier's total cost per cycle for deterministic q and θ_3 is

$$\begin{aligned} TC_V(Q) = K_V + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + c_PQ + c_rQq\theta_3 \\ + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \end{aligned} \tag{326}$$

and the buyer's total average cost with stochastic demand and partial backordering is

$$\begin{aligned} TC_B(Q,r) = K_B + F + cQ + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) \\ + d + c_b\bar{B}(r) \end{aligned} \tag{327}$$

The total cost for integrated system is,

$$\begin{aligned}
TC(Q,r) &= TC_V(Q) + TC_B(Q,r) \\
TC(Q,r) &= K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) \\
&\quad + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) + d + c_b\bar{B}(r) \\
&\quad + c_PQ + c_rQq\theta_3 + cQ + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \qquad (328)
\end{aligned}$$

We have cycle time $T = \frac{Q(1-p)}{D}$, so the expected total annual cost would be

$$\begin{aligned}
ETC(Q,r) &= \Bigg(K_V + K_B + F + h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) \\
&\quad + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) + c_rQq\theta_3 \\
&\quad + c_PQ + cQ + d + c_b\bar{B}(r) + \frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right) + \frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\Bigg)\frac{D}{Q(1-p)}
\end{aligned} \qquad (329)$$

To minimise $ETC(Q,r)$, we take first derivative with respect to Q and r,

$$\begin{aligned}
\frac{\partial(ETC)}{\partial Q} &= \frac{2c_bD\bar{B}(r) + 2PD(d+F+K_B+K_V) + h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2} \\
&\quad - \frac{h_VQ^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2(p-1)PQ^2}
\end{aligned} \qquad (330)$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D\left(d+F+K_B+K_V+c_b\bar{B}(r)\right)}{(1-p)Q^3} \qquad (331)$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-2PD\left(c_b\bar{B}(r)+d+F+K_B+K_V\right)}{h_B\left(p^2-1\right)P - h_V(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}}. \qquad (332)$$

The derivative of $ETC(Q,r)$ with respect to r is

$$\frac{\partial(ETC)}{\partial r} = \frac{Dc_b\bar{B}'(r) + h_B Q(1-p)}{(1-p)Q} \tag{333}$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial r^2} = \frac{Dc_b\bar{B}''(r)}{(1-p)Q} \tag{334}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial r^2} > 0$. Therefore, there exists unique value for r^* given as

$$\begin{aligned} F(r) &= \left(1 - \frac{h_B(1-p)Q}{Dc_b}\right) \\ r^* &= F^{-1}\left(1 - \frac{h_B(1-p)Q}{Dc_b}\right) \end{aligned} \tag{335}$$

One more time for the rest of the models, r^* equation will be the same since the integrated cost function is changing only by constant. In the model where q is deterministic and θ_3 is stochastic with standard uniform distribution, the supplier's cost function per cycle will be

$$\begin{aligned} ETC_V(Q) = {} & K_V - h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) \\ & + h_V\left(\frac{Q^2\left(q^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+3q(L_2+U_2-2)+3\right)}{6D}\right) \\ & + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} + c_P Q + c_r q Q \mu_{\theta_3} \end{aligned} \tag{336}$$

and the buyer's total average cost with stochastic demand in a supplier's cycle is

$$\begin{aligned} ETC_B(Q,r) = {} & K_B + F + cQ + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) \\ & + d + c_b\bar{B}(r) \end{aligned} \tag{337}$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_B + K_V + F + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) \\
&\quad + h_V\left(\frac{Q^2\left(q^2\left(L_2^2 + (L_2+U_2)(U_2-3)+3\right) + 3q(L_2+U_2-2)+3\right)}{6D}\right) \\
&\quad - h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + cQ + c_PQ + c_rqQ\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) \\
&\quad + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} + d + c_b\bar{B}(r)
\end{aligned} \tag{338}
$$

The expected total annual cost is

$$
\begin{aligned}
ETC(Q,r) = &\left(K_B + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right)\right. \\
&+ h_V\left(\frac{Q^2(q^2(L_2^2 + (L_2+U_2)(U_2-3)+3) + 3q(L_2+U_2-2)+3)}{6D}\right) \\
&- h_V\left(\frac{Q^2(q(L_2+U_2-2)+2)}{4P}\right) + cQ + c_PQ + c_rqQ\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{q}\right) \\
&\left. + K_V + F + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} + d + c_b\bar{B}(r)\right)\frac{D}{Q(1-p)}
\end{aligned} \tag{339}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} = &\frac{D\left(4P(c_b\bar{B}(r) + d + F + K_B + K_V) + h_VQ^2(q(L_2+U_2-2)+2)\right)}{4(p-1)PQ^2} \\
&- \frac{h_V\left(q^2\left(L_2^2 + (L_2+U_2)(U_2-3)+3\right) + 3q(L_2+U_2-2)+3\right)}{6(p-1)} \\
&+ \frac{h_B(p+1)}{2}
\end{aligned} \tag{340}
$$

Taking the second derivative, we have

$$
\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D\left(c_b\bar{B}(r) + d + F + K_B + K_V\right)}{(1-p)Q^3} \tag{341}
$$

All parameters are positive, and $\frac{d^2(ETC)}{dQ^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-12PD\left(c_b\bar{B}(r) + d + F + K_B + K_V\right)}{6h_B\left(p^2-1\right)P - 2h_V P\left(q^2C + 3qA + 3\right) + 3h_V D(qA+2)}} \tag{342}$$

where $C = (L_2^2 + L_2(U_2-3) + (U_2-3)U_2 + 3)$ and $A = (U_2 + L_2 - 2)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(p-1)Q}{Dc_b}\right) \tag{343}$$

For the next model, q is stochastic with standard uniform distribution and θ_3 is deterministic. Therefore, supplier's cost function per cycle is

$$\begin{aligned} ETC_V(Q) = K_V + h_V &\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2 + (L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6D}\right) \\ &+ h_V\left(\frac{Q^2\left(-3(\theta_3-1)D(L_1+U_1) - 6D\right)}{12PD}\right) + c_P Q + c_r Q\theta_3\mu_q \\ &+ \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \end{aligned} \tag{344}$$

and the buyer's expected total cost with stochastic demand in the supplier's cycle is

$$\begin{aligned} ETC_B(Q,r) = K_B + F + cQ + h_B &\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) \\ &+ d + c_b\bar{B}(r) \end{aligned} \tag{345}$$

The expected total cost for integrated system is,

$$
\begin{aligned}
ETC(Q,r) &= ETC_V(Q) + ETC_B(Q,r) \\
ETC(Q,r) &= K_V + K_B + F + cQ + c_P Q + c_r Q\theta_3\mu_q \\
&\quad + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} \\
&\quad + h_V\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2 + (L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6D}\right) \\
&\quad + h_V\left(\frac{Q^2(-3(\theta_3-1)D(L_1+U_1)-6D)}{12PD}\right) + d + c_b\bar{B}(r) + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)
\end{aligned}
\tag{346}
$$

The expected total annual cost is

$$
\begin{aligned}
ETC(Q,r) = \Bigg(& K_V + K_B + F + cQ + c_P Q + c_r Q\theta_3\mu_q \\
&+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} \\
&+ h_V\left(\frac{Q^2\left(L_1^2(\theta_3-1)^2 + (L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6D}\right) \\
&+ h_V\left(\frac{Q^2(-3(\theta_3-1)D(L_1+U_1)-6D)}{12PD}\right) \\
&+ d + c_b\bar{B}(r) + \frac{\eta Q}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)\Bigg)\frac{D}{Q(1-p)}
\end{aligned}
\tag{347}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} &= \frac{2D(c_b\bar{B}(r)+d+F+K_B+K_V) + h_B\left(p^2-1\right)Q^2}{2(p-1)Q^2} \\
&\quad - \frac{h_V\left(L_1^2(\theta_3-1)^2+(L_1+U_1)(\theta_3-1)((\theta_3-1)U_1+3)+3\right)}{6(p-1)} \\
&\quad + \frac{h_V D((\theta_3-1)(L_1+U_1)+2)}{4(p-1)P}
\end{aligned}
\tag{348}
$$

Taking the second derivative, we have

$$
\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+d+F+K_B+K_V)}{(1-p)Q^3} \tag{349}
$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2} > 0$. Therefore, there exists unique value for Q^* given as

$$Q^* = \sqrt{\frac{-12PD\left(c_b\bar{B}(r)+d+F+K_B+K_V\right)}{6h_B\left(p^2-1\right)P-h_V\left(2P\left(L_1^2B^2+(L_1+U_1)B(BU_1+3)+3\right)-3BD(L_1+U_1)-6D\right)}} \tag{350}$$

where $B=(\theta_3-1)$ and the unique value for r^* given as

$$r^* = F^{-1}\left(1-\frac{h_B(p-1)Q}{Dc_b}\right) \tag{351}$$

As our final model, where q and θ_3 are both stochastic with standard uniform distribution, supplier's cost function per cycle is

$$\begin{aligned}
ETC_V(Q) = {} & K_V + c_PQ + c_rQ\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
& + h_V\left(\frac{Q^2L_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D}\right) \\
& + h_V\left(\frac{Q^2L_1\left(2L_2^2U_1+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36D}\right) \\
& + h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36D}\right) \\
& - h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right)
\end{aligned} \tag{352}$$

and the buyer's expected total cost with deterministic demand in a supplier's cycle is

$$\begin{aligned}
ETC_B(Q,r) = {} & K_B + F + cQ + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) \\
& + d + c_b\bar{B}(r)
\end{aligned} \tag{353}$$

The expected total cost for integrated system is,

$$
\begin{aligned}
&ETC(Q,r) = ETC_V(Q) + ETC_B(Q,r) \\
&ETC(Q,r) = K_B + K_V + F + cQ + c_P Q + c_r Q\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&\qquad + d + c_b\bar{B}(r) + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) \\
&\qquad + h_V\left(\frac{Q^2L_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D}\right) \\
&\qquad + h_V\left(\frac{Q^2L_1\left(2L_2^2U_1+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36D}\right) \\
&\qquad + h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36D}\right) \\
&\qquad - h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right)
\end{aligned}
\tag{354}
$$

The expected total annual cost becomes

$$
\begin{aligned}
ETC(Q) = &\Bigg(K_B + K_V + F + cQ + c_P Q + c_r Q\mu_q\mu_{\theta_3} + \frac{\eta Q}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q}{\delta_2}\mu_{\ln(\theta_3/\theta_{3_0})} \\
&+ d + c_b\bar{B}(r) + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2p(1-p)}{D} + (r-\mu_L)\frac{Q(1-p)}{D}\right) \\
&+ h_V\left(\frac{Q^2L_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18D}\right) \\
&+ h_V\left(\frac{Q^2L_1\left(2L_2^2U_1+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36D}\right) \\
&+ h_V\left(\frac{Q^2\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36D}\right) \\
&- h_V\left(\frac{Q^2\left((L_2+U_2-2)(L_1+U_1)+4\right)}{8P}\right)\Bigg)\frac{D}{Q(1-p)}
\end{aligned}
\tag{355}
$$

To minimise $ETC(Q)$, we take first derivative with respect to Q

$$
\begin{aligned}
\frac{\partial(ETC)}{\partial Q} = &\frac{(D(c_b\bar{B}(r)+d+F+K_B+K_V))}{(p-1)Q^2} \\
&- \frac{h_VL_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)}{18(p-1)} \\
&- \frac{h_VL_1\left(2L_2^2U_1+L_2(2U_1(U_2-3)+9)+2U_1((U_2-3)U_2+3)+9(U_2-2)\right)}{36(p-1)} \\
&- \frac{h_V\left(2U_1^2\left(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3\right)+9U_1(L_2+U_2-2)+18\right)}{36(p-1)} \\
&+ \frac{h_VQ^2\left(D(L_2+U_2-2)(L_1+U_1)+4D\right)+4h_B\left(p^2-1\right)PQ^2}{8(p-1)PQ^2}
\end{aligned}
\tag{356}
$$

Taking the second derivative, we have

$$\frac{\partial^2(ETC)}{\partial Q^2}=\frac{2D\left(c_b\bar{B}(r)+d+F+K_B+K_V\right)}{(1-p)Q^3} \tag{357}$$

All parameters are positive, and $\frac{\partial^2(ETC)}{\partial Q^2}>0$. Therefore, there exists unique value for Q^* given as

$$Q^*=\sqrt{\frac{-72PD\left(c_b\bar{B}(r)+d+F+K_B+K_V\right)}{36h_B\left(p^2-1\right)P-h_V\left(2P\left(2L_1^2C+\left(L_1+U_1\right)\left(2U_1C+9A\right)+18\right)-9D(AB-4)\right)}} \tag{358}$$

where $C=(L_2^2+L_2(U_2-3)+(U_2-3)U_2+3)$, $B=(L_1+U_1)$ and $A=(U_2+L_2-2)$ and the unique value for r^* given as

$$r^*=F^{-1}\left(1-\frac{h_B(p-1)Q}{Dc_b}\right) \tag{359}$$

CHAPTER 4: EXPERIMENTAL RESULTS

4.1. *Illustrative Example, Sensitivity Analysis for the Buyer and Case Figures for the Supplier*

4.1.1. *Illustrative Example*

In this section, figures and tables show the change of cost according to different cases. The numerical values for the calculations are as follows: expected demand rate $D = 50,000$, deterministic defective rate $q = 0.2005$, deterministic reworkable rate $\theta_3 = 0.75$, for stochastic q lower and upper bounds $L_1 = 0.001$ and $U_1 = 0.4$, for stochastic θ_3 lower and upper bounds $L_2 = 0.5$ and $U_2 = 1$. The rest of the parameter values; $q_0 = 0.4$, $\theta_{3_0} = 0.5$ $F = 25$, $K_B = 100$, $h_B = 5$, $c = 0.5$, $p = 0.006$, $K_V = 300$, $h_V = 2$, $P = 160,000$, $\eta = 0.01$, $\delta = 0.9$, $\delta_2 = 0.2$, $c_P = 5$, $c_r = 2$, $c_l = 10$, $c_b = 8$, $d = 25$, $\alpha = 0.7$, $b = 0.01$, $\tau = 4$, $\alpha_q = 1$, $\beta_q = 4$, $\alpha_{\theta_3} = 3$, $\beta_{\theta_3} = 1$,$\mu_L = 1300$, and $\sigma = 80$. When $q = 0.2005$ and $\theta_3 = 0.75$,it means with investment, the defective rate decreased from 0.4 to 0.2005, and rework rate increased from 0.5 to 0.75. Other than Table 7, all models follow normal distribution when demand is stochastic.

For deterministic demand case, Table 3 shows that supplier's optimal lot size Q_V^* and integrated Q^* slightly decreased with different scenarios on q and θ_3 compared to their deterministic case (Case I). Here, q and θ_3 follow uniform distribution, therefore their expected values are $\mu_q = 0.2005$ and $\mu_{\theta_3} = 0.75$. For the cost perspective, supplier's individual cost $ETC_V(Q_V^*)$ is at its lowest when rework rate is stochastic and defective rate is deterministic and at its highest when defective rate is stochastic and rework rate is deterministic. Due to its convex nature, supplier has higher cost if it produces buyer's quantity ($ETC_V(Q_B^*)$ overall. Integrated cost has its highest value when both parameters are stochastic. The last column shows which case is the most profitable by percentage difference and it is clear that other than first three cases are better than the last one.

In deterministic demand case buyer's independent lot size is not changing due to its cost function structure. Buyer's reorder point on deterministic demand case with and without backordering is fixed and calculated with the formula $L*D$. This is making sense, since he knows exactly how much demand it will have. It is also true for integrated model reorder point because it is still provided by buyer.

For stochastic demand case (normal distribution), Q_V^* is still decreasing slightly. Buyer's optimal order quantity (Q_B^* and r_B^*) are higher than deterministic demand case. Integrated order quantity and reorder point (Q^* and r^*) are also higher than previous demand case. Supplier's biggest cost happens when both parameters are stochastic if it decides to produce buyer's order quantity. Integrated cost has the similar pattern and is highest when q and θ_3 are both stochastic. The first case, where both parameters are deterministic gives the best savings when integrated policy is adopted.

When there is partial backordering (Table 4), -time-sensitive customers-, buyer's order quantity Q_B^* is more than lost sales case ($Q_B^* = 1760$), however reorder point is smaller. Even though it looks like a contradiction, it might be the case that lower reorder point helps to wait longer because now order quantity is higher. Buyer's total cost and integrated cost are higher than lost-sales case and it can be explained by the higher order quantity and inclusion of backorder cost as well. Since rework rate is less than 1, not all defective items are reworkable. Supplier's individual cost ($ETC_V(Q_V^*)$), its cost when buyer's order quantity is considered ($ETC_V(Q_B^*)$) and integrated cost show similar behaviour and have their highest value when defective rate is stochastic and rework rate is deterministic. Moreover, their lowest value happens when defective rate is deterministic and rework rate is stochastic. This shows the stochasticity for defective rate has more impact than rework rate. The buyer's lot size Q_B^* and reorder point r_B^* do not change since its individual equation does not depend on q and θ_3. Finally, if the supplier adopts integrated policy, it will produce smaller lot size overall. Overall, it is clear that the integrated model is beneficial to both supplier and buyer, according to the costs under stochastic demand case with partial backordering. Related to these, the last column shows third case (q is deterministic and θ_3 is stochastic) has the highest savings on integrated approach.

To see the behaviour change with different distributions on q and θ_3, Table 5 shows optimal values with their cost under beta distribution. Here, shape parameters (α_q, α_{θ_3}, β_q, β_{θ_3}) values are chosen specifically to have same expected values ($E(q) = 0.2$ and

Table 3. Table for the all cases of demand, q, θ_3 with the optimal order size and their independent and integrated costs with complete lost-sales.

Demand	q	θ_3	Q_V^*	Q_B^*	r_B^*	Q^*	r^*	$ETC_V(Q_V^*)$	$ETC_V(Q_B^*)$	$ETC_B(Q_B^*)$	$ETC_V(Q_B^*)+ETC_B(Q_B^*)$	$ETC(Q^*)$	$\%ETC$
Deterministic	Det. 0.2005	Det. 0.75	4978	1581	1370	2617	1370	$ 274,107	$ 277,151	$ 33,104	$ 310,255	$ 308,137	0.6827 %
	Stoch. (0.001-0.4)	Det. 0.75	4974	1581	1370	2616	1370	$ 274,275	$ 277,153	$ 33,104	$ 310,257	$ 308,139	0.6827 %
	Det. 0.2005	Stoch. (0.5-1)	4974	1581	1370	2616	1370	$ 274,063	$ 277,153	$ 33,104	$ 310,257	$ 308,139	0.6827 %
	Stoch. (0.001-0.4)	Stoch. (0.5-1)	4970	1581	1370	2616	1370	$ 274,233	$ 277,154	$ 33,104	$ 310,258	$ 308,142	0.6820 %
Stochastic	Det. 0.2005	Det. 0.75	4978	1609	1472	2640	1456	$ 274,107	$ 278,402	$ 34,108	$ 312,510	$ 310,471	0.6525 %
	Stoch. (0.001-0.4)	Det. 0.75	4974	1609	1472	2639	1456	$ 274,275	$ 278,403	$ 34,108	$ 312,511	$ 310,473	0.6521 %
	Det. 0.2005	Stoch. (0.5-1)	4974	1609	1472	2639	1456	$ 274,063	$ 278,403	$ 34,108	$ 312,511	$ 310,473	0.6521 %
	Stoch. (0.001-0.4)	Stoch. (0.5-1)	4970	1609	1472	2639	1456	$ 274,233	$ 278,405	$ 34,108	$ 312,513	$ 310,476	0.6518 %

$E(\theta_3) = 0.75$) as in the standard uniform distribution (Table 4). The goal is to compare both tables and see the effect of distributions. With this logic, integrated order quantity Q^* values are slightly lower in Table 5. First case is same for both distributions, so there is no change. The main difference is when the rework rate is stochastic with same lower and upper values and expected values, total cost is higher under beta distribution even though Q^* is lower. Same pattern can be seen for Q_V^* and other cost values as well. Only when defective rate is stochastic, the cost values are lower than standard uniform distribution. Therefore, it might be more cost effective to use standard uniform distribution when defective rate is stochastic and rework rate is known and constant. When we look at Table 5, the behaviours of Q_V^* and $ETC_V(Q_V^*)$ are similar with Table 4. Comparison of second and forth case shows decrease on Q_V^* with increased total cost $ETC_V(Q_V^*)$. Two cases have same expected values on defective rates and rework rate, however stochasticity on rework rate cause cost increase. On the other hand, first and second case show slightly decreased cost on $ETC_V(Q_V^*)$. This can be explained by distribution effect on defective rate. Last column shows the difference between total cost for integrated model and individual cost sums and unlike Table 4, first case has the highest difference. So, when defective rate and rework rate both deterministic, the integrated model results in greater savings.

As the third distribution example, Table 6 shows the changes of optimal values when q and θ_3 follows triangular distribution. While the behaviour of Q_V^*, Q^* and corresponding costs have the similar behaviour as previous ones, forth case (where q and θ_3 are both stochastic) has the highest value (among uniform, beta and triangular distributions). Other than the first and second case, integrated costs ($ETC(Q^*)$) and ($ETC_V(Q_V^*)$) are lower than beta distribution. However, third and forth cases results higher supplier cost than uniform with lower integrated cost. It shows the effect of distribution even though we have same expected values. The last column shows the forth case where both parameters are stochastic has the best savings for integrated model.

In Table 7, we can see that when demand follows exponential distribution, buyer's order quantity and reorder point as well as integrated order quantity and reorder point are higher than normal distribution (Table 4). Since demand is coming from buyer's

Table 4. Table for stochastic demand, q, θ_3 following standard uniform distribution with the optimal order size and their independent and integrated costs with partial backlogging

Demand	q	θ_3	Q_V^*	Q_B^*	r_B^*	Q^*	r^*	$ETC_V(Q_V^*)$	$ETC_V(Q_B^*)$	$ETC_B(Q_B^*)$	$ETC_V(Q_B^*)+ETC_B(Q_B^*)$	$ETC(Q^*)$	$\%ETC$
Stochastic	Det. 0.2005	Det. 0.75	4978	1760	1465	2715	1451	\$ 274,107	\$ 277,689	\$ 34,831	\$ 312,520	\$ 310,914	0.5139 %
	Stoch. (0.001-0.4)	Det. 0.75	4974	1760	1465	2715	1451	\$ 274,275	\$ 277,855	\$ 34,831	\$ 312,686	\$ 311,081	0.5133 %
	Det. 0.2005	Stoch. (0.5-1)	4974	1760	1465	2715	1451	\$ 274,063	\$ 277,643	\$ 34,831	\$ 312,474	\$ 310,868	0.5140 %
	Stoch. (0.001-0.4)	Stoch. (0.5-1)	4970	1760	1465	2714	1451	\$ 274,233	\$ 277,809	\$ 34,831	\$ 312,640	\$ 311,036	0.5131 %

Table 5. Table for stochastic demand, q, θ_3 following beta distribution with the optimal order size and their independent and integrated costs with partial backlogging

Demand	q	θ_3	Q_V^*	Q_B^*	r_B^*	Q^*	r^*	$ETC_V(Q_V^*)$	$ETC_V(Q_B^*)$	$ETC_B(Q_B^*)$	$ETC_V(Q_B^*)+ETC_B(Q_B^*)$	$ETC(Q^*)$	$\%ETC$
Stochastic	Det. 0.2005	Det. 0.75	4978	1760	1465	2715	1451	$ 274,107	$ 277,689	$ 34,831	$ 312,520	$ 310,914	0.5139 %
	Stoch. (0.001-0.4)	Det. 0.75	4978	1760	1465	2715	1451	$ 274,107	$ 277,689	$ 34,831	$ 312,520	$ 310,914	0.5139 %
	Det. 0.2005	Stoch. (0.5-1)	4718	1760	1465	2685	1452	$ 274,680	$ 278,133	$ 34,831	$ 312,964	$ 311,396	0.5010 %
	Stoch. (0.001-0.4)	Stoch. (0.5-1)	4729	1760	1465	2687	1451	$ 274,593	$ 277,947	$ 34,831	$ 312,778	$ 311,242	0.4911 %

equation, supplier's individual equations did not change. If supplier would follow buyer's order quantity, he would have lower cost because Q_B^* value is higher with exponential distribution (instead of producing 1760 items, he would produce 3428). Buyer has higher cost due to higher volume of order quantity in Table 7. However, overall total cost is increased for every case therefore it is not helpful for business. The last column shows that difference between integrated total cost and sum of individual costs is really small and the best at third case where q is deterministic and θ_3 is stochastic. This is same behaviour as in Table 4.

Table 8 shows when all shortage is backordered. Compared to partial backordering, supplier's individual order quantity and cost stays the same while buyer's individual order quantity increases and reorder point and its cost decrease. Moreover, if supplier produces buyer's order quantity, the cost ($ETC_V(Q_B^*)$) will be slightly lower. Integrated optimal order quantity, reorder point and cost follow the same pattern as buyer's . In this scenario. the first case where both q and θ_3 are deterministic has the highest savings under integrated policy. Overall, it is clear that this model is pushing order placement further when there is backordering (due to lower reorder point).

In Table 9, the optimal values for Q and r with their corresponding costs are compared under different distributions for q and θ_3. The optimal order quantity Q^* follows different behaviour through different distributions denotes each distribution's impact. It is interesting to see that how close the first three cases between uniform and triangular distributions. In triangular distribution, when both parameters are stochastic, Q^* is increasing more than it is in standard uniform distribution. Moreover, the optimal reorder point r^* drops more in triangular distribution at the last case. This may indicate that triangular distribution has more advantageous when defective and rework rates are stochastic. The last point of comparison between two is their integrated total cost values. When q and θ_3 follow triangular distribution, total cost is less than standard uniform distribution and saving more money than both standard uniform distribution for the second and forth cases. While beta distribution is also an option with its lower optimal order quantity, it has higher costs than uniform and triangular distributions for second and forth cases. Moreover, it is saving the least amount of money among three distributions.

Table 6. Table for stochastic demand, q, θ_3 following triangular distribution with the optimal order size and their independent and integrated costs with partial backlogging

Demand	q	θ_3	Q_V^*	Q_B^*	r_B^*	Q^*	r^*	$ETC_V(Q_V^*)$	$ETC_V(Q_B^*)$	$ETC_B(Q_B^*)$	$ETC_V(Q_B^*)+ETC_B(Q_B^*)$	$ETC(Q^*)$	$\%ETC$
Stochastic	Det. 0.2005	Det. 0.75	4978	1760	1465	2715	1451	$ 274,107	$ 277,689	$ 34,831	$ 312,520	$ 310,914	0.5139 %
	Stoch. (0.001-0.4)	Det. 0.75	4976	1760	1465	2715	1451	$ 274,134	$ 277,715	$ 34,831	$ 312,546	$ 310,941	0.5135 %
	Det. 0.2005	Stoch. (0.5-1)	4976	1760	1465	2715	1451	$ 274,085	$ 277,666	$ 34,831	$ 312,497	$ 310,891	0.5139 %
	Stoch. (0.001-0.4)	Stoch. (0.5-1)	6986	1760	1465	2857	1449	$ 272,365	$ 276,894	$ 34,831	$ 311,725	$ 310,080	0.5277 %

Table 7. Table for exponentially distributed demand, q, θ_3 with the optimal order size and their independent and integrated costs with partial backlogging

Demand	q	θ_3	Q_V^*	Q_B^*	r_B^*	Q^*	r^*	$ETC_V(Q_V^*)$	$ETC_V(Q_B^*)$	$ETC_B(Q_B^*)$	$ETC_V(Q_B^*)+ETC_B(Q_B^*)$	$ETC(Q^*)$	$\%ETC$
Stochastic	Det. 0.2005	Det. 0.75	4978	3428	4212	3907	4045	\$ 274,107	\$ 274,533	\$ 57,033	\$ 331,566	\$ 331,413	0.0461 %
	Stoch. (0.001-0.4)	Det. 0.75	4974	3428	4212	3906	4045	\$ 274,275	\$ 274,701	\$ 57,033	\$ 331,734	\$ 331,581	0.0461 %
	Det. 0.2005	Stoch. (0.5-1)	4974	3428	4212	3906	4045	\$ 274,063	\$ 274,488	\$ 57,033	\$ 331,521	\$ 331,368	0.0462 %
	Stoch. (0.001-0.4)	Stoch. (0.5-1)	4970	3428	4212	3905	4045	\$ 274,233	\$ 274,656	\$ 57,033	\$ 331,689	\$ 331,537	0.0458 %

For independent and integrated cost comparisons, we can see from Table 3 to Figure 24 that integrated policy is more beneficial than policy is made solely from the buyer's perspective. Figures show that there is a decreasing trend till the optimal value that is cost is decreasing while lot size is increasing.

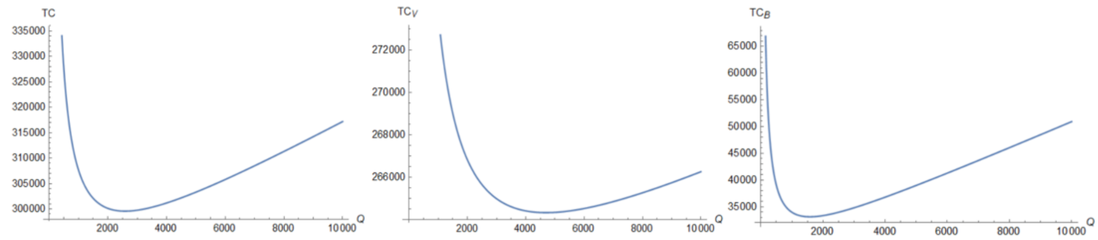

Figure 9. Total cost for the supplier and buyer individually (left) and integrated (right) for deterministic demand, q and θ_3 with complete lost-sales

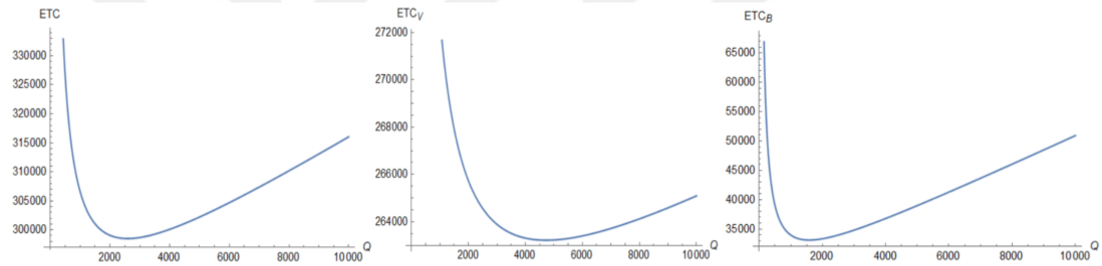

Figure 10. Total cost for the supplier and buyer individually (left) and integrated (right) for deterministic demand and q, with stochastic θ_3 with complete lost-sales

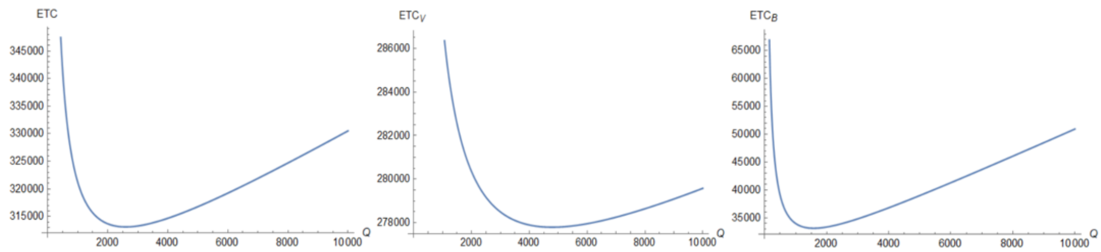

Figure 11. Total cost for the supplier and buyer individually (left) and integrated (right) for deterministic demand and θ_3, with stochastic q with complete lost-sales

Table 8. Table for stochastic demand, q, θ_3 with the optimal order size and their independent and integrated costs with complete backlogging.

Demand	q	θ_3	Q_V^*	Q_B^*	r_B^*	Q^*	r^*	$ETC_V(Q_V^*)$	$ETC_V(Q_B^*)$	$ETC_B(Q_B^*)$	$ETC_V(Q_B^*)+ETC_B(Q_B^*)$	$ETC(Q^*)$	$\%ETC$
Stochastic	Det. 0.2005	Det. 0.75	4978	1762	1461	2717	1446	$ 274,107	$ 277,681	$ 34,820	$ 312,501	$ 310,902	0.5117 %
	Stoch. (0.001-0.4)	Det. 0.75	4974	1762	1461	2717	1446	$ 274,275	$ 277,847	$ 34,820	$ 312,667	$ 311,069	0.5111 %
	Det. 0.2005	Stoch. (0.5-1)	4974	1762	1461	2717	1446	$ 274,063	$ 277,634	$ 34,820	$ 312,454	$ 310,856	0.5114 %
	Stoch. (0.001-0.4)	Stoch. (0.5-1)	4970	1762	1461	2716	1446	$ 274,233	$ 277,871	$ 34,820	$ 312,621	$ 311,024	0.5108 %

Table 9. Table for comparison of Q^* and r^* for cases of q and θ_3 with partial backlogging

		q and θ_3 follow standard uniform distribution				q and θ_3 follow beta distribution				q and θ_3 follow triangular distribution			
q	θ_3	Q^*	r^*	$ETC(Q^*)$	$\%ETC$	Q^*	r^*	$ETC(Q^*)$	$\%ETC$	Q^*	r^*	$ETC(Q^*)$	$\%ETC$
Det. 0.2005	Det. 0.75	2715	1451	$ 310,914	0.5139 %	2715	1451	$ 310,914	0.5139 %	2715	1451	$ 310,914	0.5139 %
Stoch. (0.001-0.4)	Det. 0.75	2715	1451	$ 311,081	0.5133 %	2694	1451	$ 310,803	0.4994 %	2715	1451	$ 310,941	0.5135 %
Det. 0.2005	Stoc. (0.5-1)	2715	1451	$ 310,868	0.5140 %	2685	1451	$ 311,396	0.5010 %	2715	1451	$ 310,891	0.5139 %
Stoch. (0.001-0.4)	Stoc. (0.5-1)	2714	1451	$ 311,036	0.5131 %	2687	1451	$ 311,242	0.4911 %	2857	1449	$ 310,080	0.5277 %

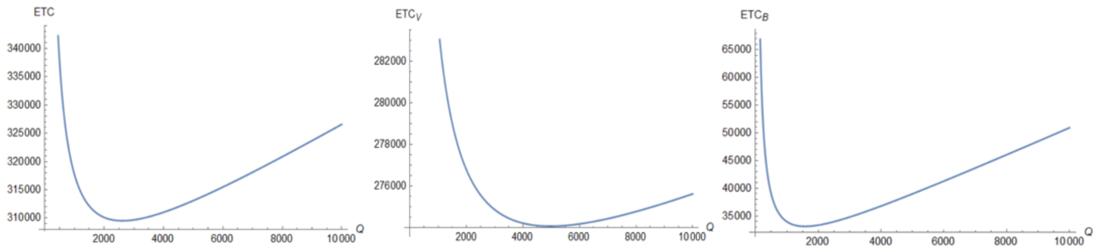

Figure 12. Total cost for the supplier and buyer individually (left) and integrated (right) for deterministic demand and stochastic θ_3 with stochastic q with complete lost-sales

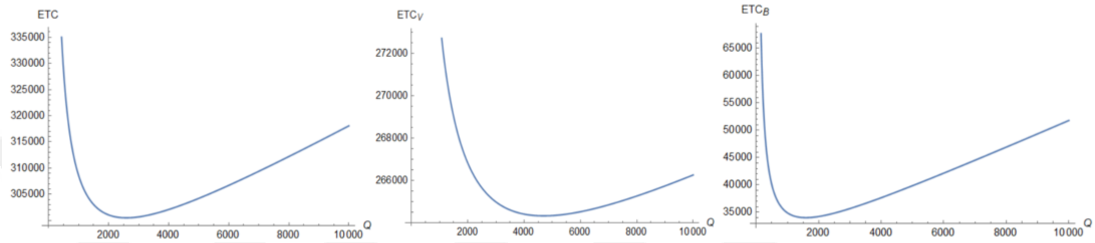

Figure 13. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and deterministic θ_3 and q with complete lost-sales

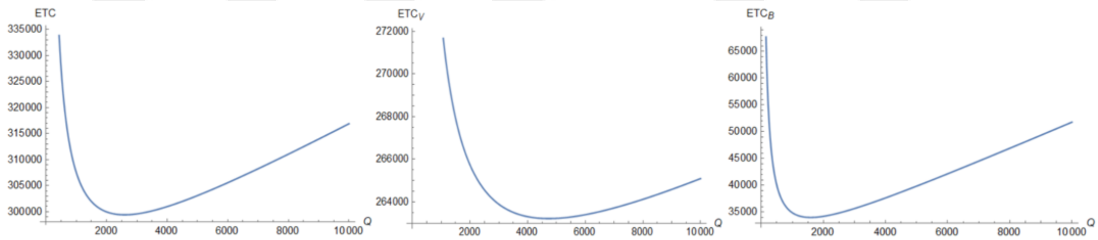

Figure 14. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and deterministic q and stochastic θ_3 with complete lost-sales

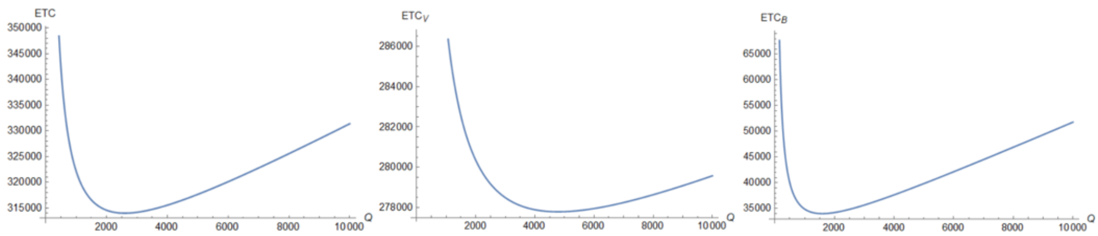

Figure 15. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and stochastic q and deterministic θ_3 with complete lost-sales

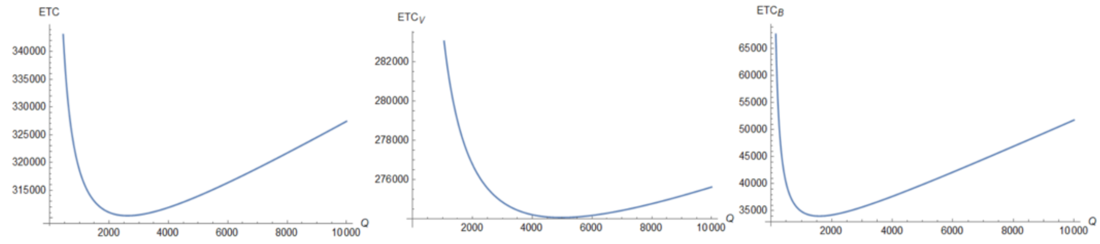

Figure 16. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and stochastic q and θ_3 with complete lost-sales

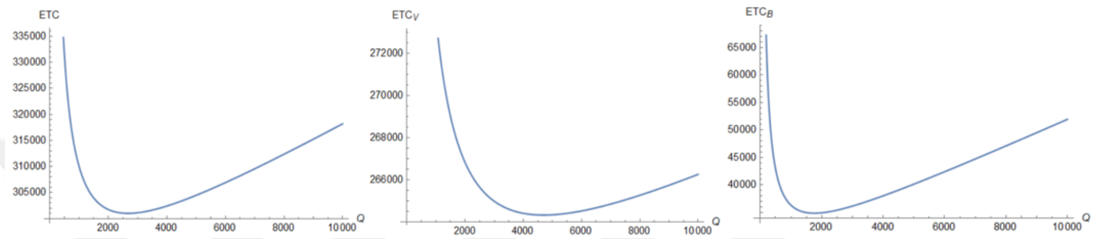

Figure 17. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and deterministic q and θ_3 with partial backordering

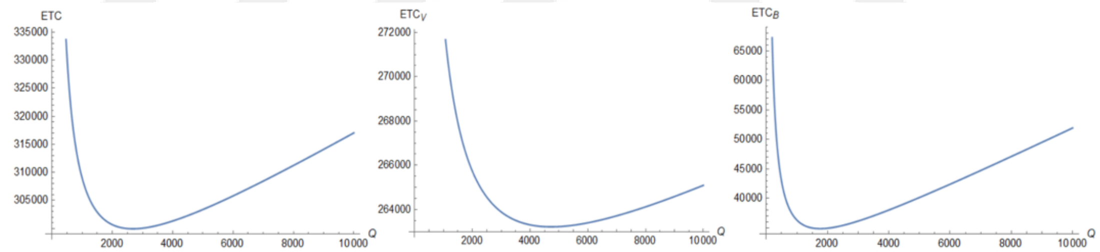

Figure 18. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and deterministic q and stochastic θ_3 with partial backordering

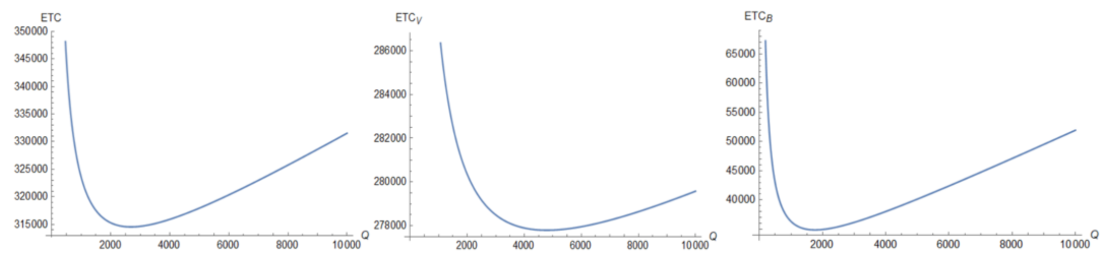

Figure 19. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and stochastic q and deterministic θ_3 with partial backordering

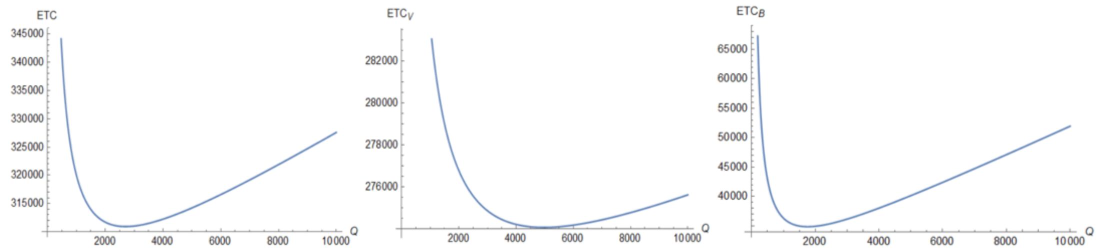

Figure 20. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and stochastic q and θ_3 with partial backordering

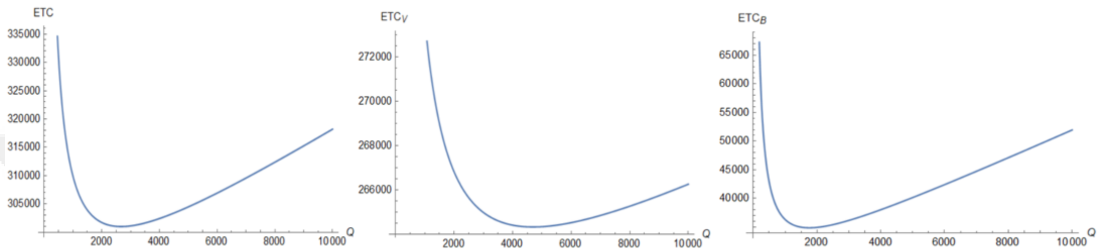

Figure 21. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and deterministic q and θ_3 with complete backordering

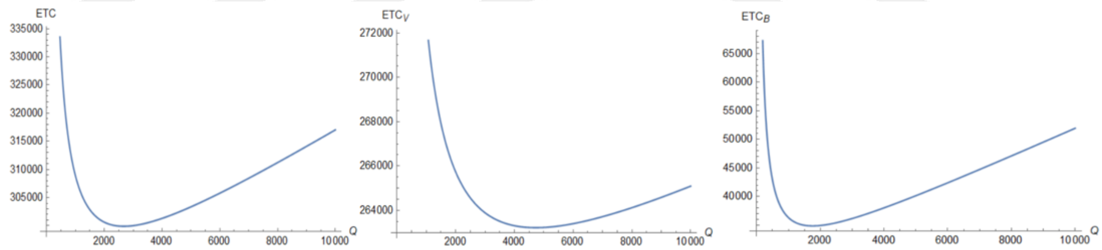

Figure 22. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and deterministic q and stochastic θ_3 with complete backordering

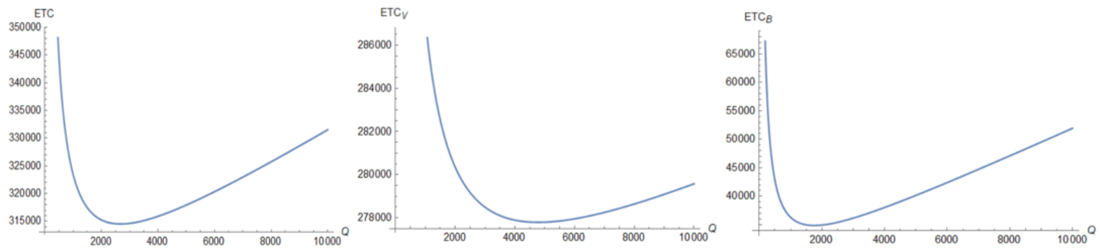

Figure 23. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and stochastic q and deterministic θ_3 with complete backordering

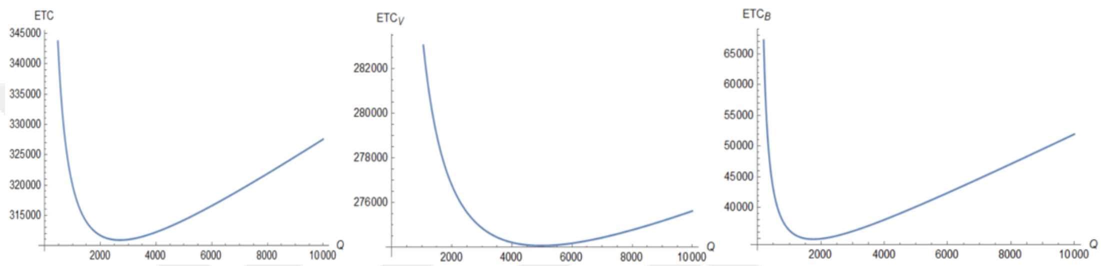

Figure 24. Total cost for the supplier and buyer individually (left) and integrated (right) for stochastic demand, and stochastic q and θ_3 with complete backordering

4.1.2. Sensitivity Analysis for the Buyer's model

We will start to examine the buyer's sensitivity analysis under the stochastic demand and continue with supplier's. The values of parameters are given in Illustrative Example. Table 10 shows that with increasing imperfect rate p, the optimal lot size Q_B^*, reorder point r^*, and naturally expected cost $ETC_B(Q_B^*)$ are increasing. With $\bar{s}(r)$ column, we can see that expected shortage that becomes lost-sales, is really small and decreasing. This is impact of order quantity, such as we place more items due to imperfection and that causes smaller shortage. Since p values are small, the change in the values are small as well.

Table 10. The change in Q_B^*, r^* and $ETC_B(Q_B^*)$ according to change in p for complete lost-sales case

p	Q_B^*	r^*	$\bar{s}(r)$	$ETC_B(Q_B^*)$
0.000	1609.48	1471.87	0.45	$33,909
0.006	1609.33	1472.07	0.45	$34,108
0.016	1609.20	1472.39	0.44	$34,445
0.026	1609.23	1472.71	0.44	$34,788
0.036	1609.42	1473.02	0.43	$35,138
0.046	1609.78	1473.34	0.43	$35,493
0.056	1610.31	1473.66	0.42	$35,856
0.066	1611.00	1473.98	0.42	$36,225
0.076	1611.85	1474.30	0.42	$36,602
0.086	1612.88	1474.62	0.41	$36,985
0.096	1614.07	1474.94	0.41	$37,377

Buyer's analysis when there is time-sensitive customers for partial backordering shows the similar behaviour with previous table (Table 11). Here, $\bar{B}(r)$ as expected backorder amount is also decreasing with increasing imperfect rate. This can be explained with higher order quantity leads to lower shortage therefore lower backorder amount. Even though the behaviours are similar, it is interesting to see that order quantity is much higher and reorder points are lower than previous case.

Table 11. The change in Q_B^*, r^* and $ETC_B(Q_B^*)$ according to change in p for partial backordering case

p	Q_B^*	r^*	$\bar{B}(r)$	$\bar{s}(r)$	$ETC_B(Q_B^*)$
0.000	1760.84	1465.20	0.42	0.16	\$34,631
0.006	1759.81	1465.39	0.41	0.16	\$34,831
0.016	1760.01	1465.72	0.41	0.16	\$35,177
0.026	1760.07	1466.04	0.40	0.16	\$35,528
0.036	1760.32	1466.37	0.40	0.15	\$35,885
0.046	1760.74	1466.69	0.40	0.15	\$36,248
0.056	1761.06	1467.02	0.39	0.15	\$36,617
0.066	1762.13	1467.34	0.39	0.15	\$36,996
0.076	1763.10	1467.67	0.38	0.15	\$37,381
0.086	1764.25	1467.99	0.38	0.15	\$37,773
0.096	1765.59	1468.99	0.37	0.14	\$38,173

For the complete backordering case (Table 12), increase in defective rates is resulting the highest order quantity and lowest reorder points. Total cost is slightly lower than partially backordering case. Additionally, we can see a small decrease is in Q_B^* lasting longer than previous case (same behaviour as complete lost-sales case). As a conclusion, it is clear that we can order more with smaller reorder point at lower cost when there is no risk to lose the customer.

Table 12. The change in Q_B^*, r^* and $ETC_B(Q_B^*)$ according to change in p for complete backordering case

p	Q_B^*	r^*	$\bar{B}(r)$	$ETC_B(Q_B^*)$
0.000	1762.04	1461.09	0.65	\$34,616
0.006	1761.87	1461.29	0.65	\$34,820
0.016	1761.73	1461.63	0.64	\$35,164
0.026	1761.77	1461.98	0.64	\$35,515
0.036	1761.98	1462.32	0.63	\$35,872
0.046	1762.38	1462.66	0.62	\$36,236
0.056	1762.96	1463.00	0.61	\$36,606
0.066	1763.72	1463.33	0.61	\$36,984
0.076	1764.66	1463.67	0.60	\$37,369
0.086	1765.78	1464.01	0.59	\$37,761
0.096	1767.09	1464.35	0.59	\$38,130

At the supplier's side, when defective rate q and reworkable rate θ_3 are both increasing (Table 13), the optimal production lot size Q_V^* is increasing until $q = 0.21$, then starts to decrease. This can be indicator that even though we have more defective items (before or after investing), if we can rework on them mostly then there is no need to produce more items. This values can be considered break even points as well. On the other hand, cost is still increasing due to reworkable items increasing number. In Table 14, case shows that defective rate is increasing while reworking rate is decreasing. Order quantity follows increasing trend completely, since we need to produce more items to supply and TC_V is increasing due to ordering more items and reworking on them. Table 15 and 16 shows when one side is fixed what happens to Q_V^* and TC_V. When we have fixed reworkable rate (Table 15), greater defective rate means bigger lot size. The cost is increasing since we have more items to rework on. On the other hand, when q is fixed (Table 16), with higher reworking rates, the optimal lot size Q_V^* decreases. Additionally, cost is slightly increasing because of increasing reworking rate on fixed amount of defective items. For Table 13- 16, θ_{3_0} to θ_3 and q_0 to q shows the change in those parameters once we invest to improve the process.

Table 13. The change in the optimal production size Q_V^* and the cost $TC_V(Q_V^*)$ when q and θ_3 are both increasing

q_0	q	θ_{3_0}	θ_3	Q_V^*	$TC_V(Q_V^*)$
0.00	0.00	0.50	0.55	4671	$258,215
0.02	0.02	0.46	0.57	4719	$260,078
0.04	0.04	0.47	0.59	4762	$261,185
0.06	0.06	0.49	0.61	4802	$262,375
0.08	0.08	0.50	0.63	4838	$263,647
0.10	0.10	0.52	0.65	4870	$265,002
0.12	0.11	0.54	0.67	4897	$266,439
0.14	0.13	0.55	0.69	4920	$267,959
0.16	0.15	0.57	0.71	4939	$269,562
0.18	0.17	0.58	0.73	4952	$271,246
0.20	0.19	0.60	0.75	4961	$273,014
0.22	0.21	0.62	0.77	4964	$274,863
0.24	0.23	0.63	0.79	4963	$276,796
0.26	0.25	0.65	0.81	4957	$278,811
0.28	0.27	0.66	0.83	4946	$280,908
0.30	0.29	0.68	0.85	4930	$283,088
0.32	0.30	0.70	0.87	4909	$285,350
0.34	0.32	0.71	0.89	4884	$287,695
0.36	0.34	0.73	0.91	4855	$290,122
0.38	0.36	0.74	0.93	4821	$292,632
0.40	0.38	0.76	0.95	4783	$295,224
0.42	0.40	0.78	0.97	4741	$297,899
0.44	0.42	0.79	0.99	4695	$300,656

Table 14. The change in the optimal production size Q_V^* and the cost $TC_V(Q_V^*)$ when q is increasing and θ_3 is decreasing

q_0	q	θ_{3_0}	θ_3	Q_V^*	$TC_V(Q_V^*)$
0.00	0.00	0.80	1.00	4671	$258,542
0.02	0.02	0.78	0.98	4673	$260,927
0.04	0.04	0.77	0.96	4680	$262,715
0.06	0.06	0.75	0.94	4691	$264,420
0.08	0.08	0.74	0.92	4706	$266,042
0.10	0.10	0.72	0.90	4726	$267,582
0.12	0.11	0.70	0.88	4751	$269,040
0.14	0.13	0.69	0.86	4780	$270,415
0.16	0.15	0.67	0.84	4815	$271,708
0.18	0.17	0.66	0.82	4855	$272,918
0.20	0.19	0.64	0.80	4900	$274,045
0.22	0.21	0.62	0.78	4951	$275,090
0.24	0.23	0.61	0.76	5008	$276,053
0.26	0.25	0.59	0.74	5071	$276,933
0.28	0.27	0.58	0.72	5142	$277,730
0.30	0.29	0.56	0.70	5220	$278,445
0.32	0.30	0.54	0.68	5306	$279,078
0.34	0.32	0.53	0.66	5401	$279,627
0.36	0.34	0.51	0.64	5506	$280,094
0.38	0.36	0.50	0.62	5621	$280,479
0.40	0.38	0.48	0.60	5748	$280,781
0.42	0.40	0.46	0.58	5888	$291,000
0.44	0.42	0.45	0.56	6042	$281,136

Table 15. The change in the optimal production size Q_V^* and $TC_V(Q_V^*)$ when q is increasing and θ_3 is fixed

q_0	q	θ_{3_0}	θ_3	Q_V^*	$TC_V(Q_V^*)$
0.00	0.00	0.5	0.7	4671	$258,824
0.02	0.02	0.5	0.7	4704	$260,632
0.04	0.04	0.5	0.7	4737	$261,925
0.06	0.06	0.5	0.7	4771	$263,218
0.08	0.08	0.5	0.7	4806	$264,510
0.10	0.10	0.5	0.7	4841	$265,803
0.12	0.11	0.5	0.7	4876	$267,096
0.14	0.13	0.5	0.7	4912	$268,389
0.16	0.15	0.5	0.7	4948	$269,681
0.18	0.17	0.5	0.7	4985	$270,974
0.20	0.19	0.5	0.7	5023	$272,267
0.22	0.21	0.5	0.7	5061	$273,560
0.24	0.23	0.5	0.7	5100	$274,852
0.26	0.25	0.5	0.7	5139	$276,145
0.28	0.27	0.5	0.7	5179	$277,438
0.30	0.29	0.5	0.7	5220	$278,730
0.32	0.30	0.5	0.7	5261	$280,023
0.34	0.32	0.5	0.7	5303	$281,315
0.36	0.34	0.5	0.7	5346	$282,608
0.38	0.36	0.5	0.7	5389	$283,901
0.40	0.38	0.5	0.7	5434	$285,193
0.42	0.40	0.5	0.7	5478	$286,486
0.44	0.42	0.5	0.7	5524	$287,778

Table 16. The change in the optimal production size Q_V^* and $TC_V(Q_V^*)$ when q is fixed and θ_3 is increasing

q_0	q	θ_{3_0}	θ_3	Q_V^*	$TC_V(Q_V^*)$
0.04	0.02	0.50	0.70	4706	$267,151
0.04	0.02	0.57	0.71	4704	$266,895
0.04	0.02	0.58	0.73	4703	$266,924
0.04	0.02	0.59	0.74	4701	$266,954
0.04	0.02	0.60	0.75	4699	$266,983
0.04	0.02	0.61	0.77	4698	$267,012
0.04	0.02	0.62	0.78	4696	$267,042
0.04	0.02	0.64	0.79	4695	$267,071
0.04	0.02	0.65	0.81	4693	$267,100
0.04	0.02	0.66	0.82	4692	$267,130
0.04	0.02	0.67	0.84	4690	$267,159
0.04	0.02	0.68	0.85	4688	$267,188
0.04	0.02	0.69	0.86	4687	$267,217
0.04	0.02	0.70	0.88	4685	$267,247
0.04	0.02	0.71	0.89	4684	$267,276
0.04	0.02	0.72	0.90	4682	$267,305
0.04	0.02	0.73	0.92	4681	$267,335
0.04	0.02	0.74	0.93	4679	$267,364
0.04	0.02	0.75	0.94	4678	$267,393
0.04	0.02	0.77	0.96	4676	$267,423
0.04	0.02	0.78	0.97	4674	$267,452
0.04	0.02	0.79	0.98	4673	$267,481
0.04	0.02	0.80	1.00	4671	$267,511

4.1.3. Case Figures for the Supplier

As we analysed the four cases above, here we investigate the behaviour of Q^* with respect to q and θ_3. Figure 25 shows the case when q and θ_3 are both deterministic. For each value of fixed q, as θ_3 increases, the optimal production size Q^* decreases. For higher value of q, optimal production size starts to decrease from Q^* value. Additionally, Figure 26 shows the change of q and θ_3 and their effect on Q^* simultaneously.

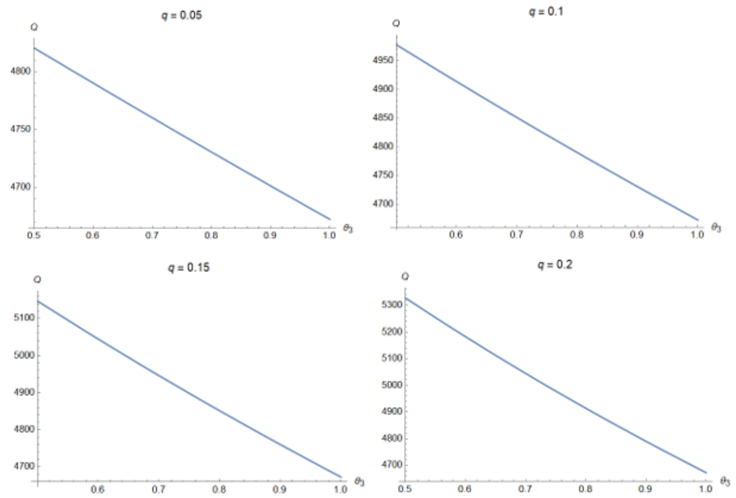

Figure 25. The change in the optimal production size Q^* when q and θ_3 is deterministic

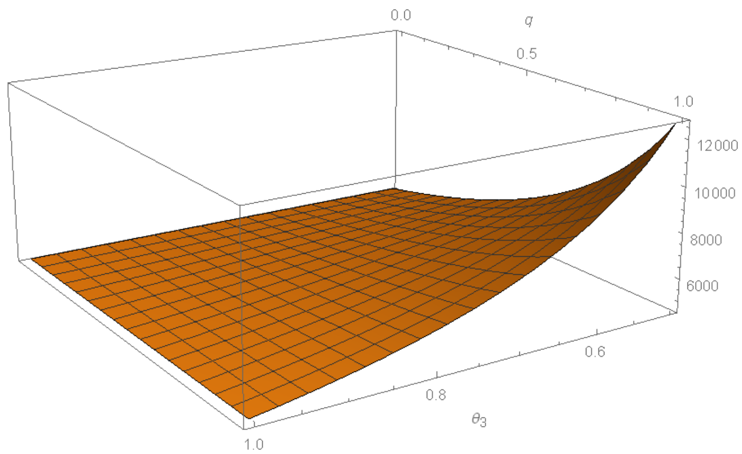

Figure 26. The change in Q^* for q and θ_3

When q is deterministic and θ_3 is stochastic with standard uniform distribution

(Figure 27), we have clear convex behaviour for small U_2 and L_2. As those bounds are increasing, Q^* increases in almost linear trend. As their values are getting bigger with increasing q values, Q^* values are starting to decrease.

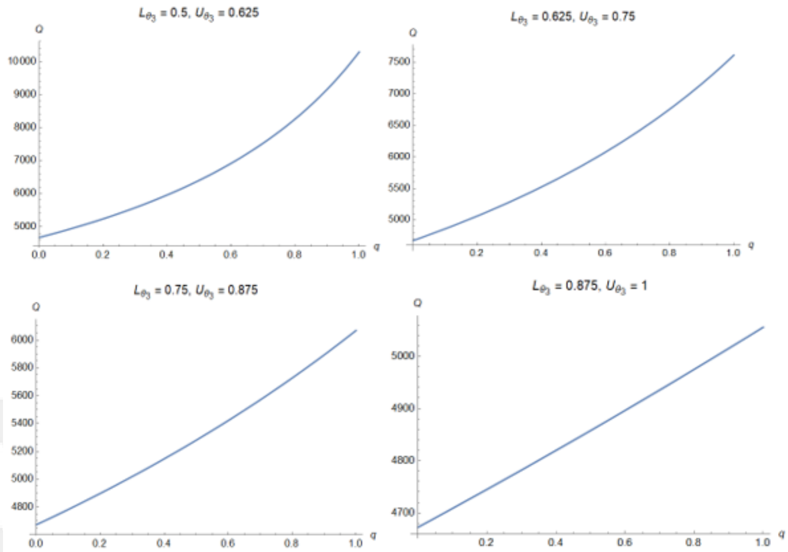

Figure 27. The change in the optimal production size Q^* when q is deterministic and θ_3 is stochastic

For the stochastic q with standard uniform distribution and deterministic θ_3, Figure 28 demonstrates the change of Q^*. It can be easily seen that when we fix the difference between upper and lower bound for q and increase step by step, the optimal production size starts from higher initial values and decreases.

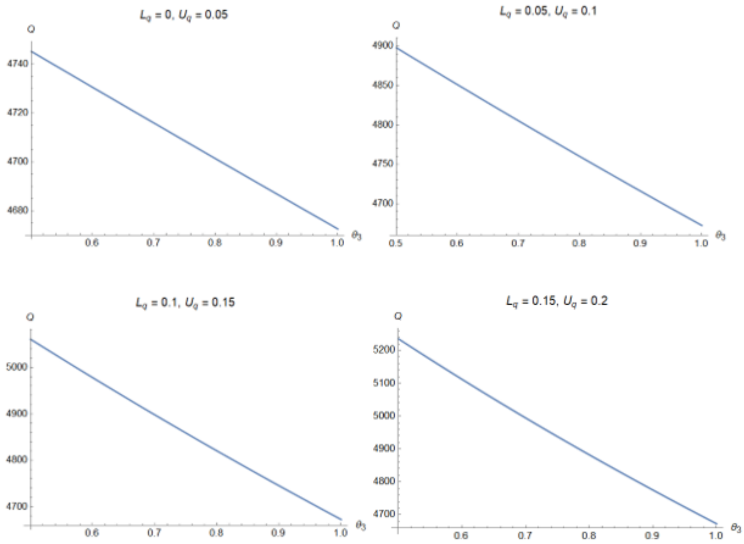

Figure 28. The change in the optimal production size Q^* when q is stochastic and θ_3 is deterministic

In the last case, when we have both q and θ_3 stochastic Figure 29 shows that Q^* has higher values as U_1 and L_1 gets higher. As upper and lower bounds for q and θ_3 are getting larger, Q^* is increasing in almost linear trend.

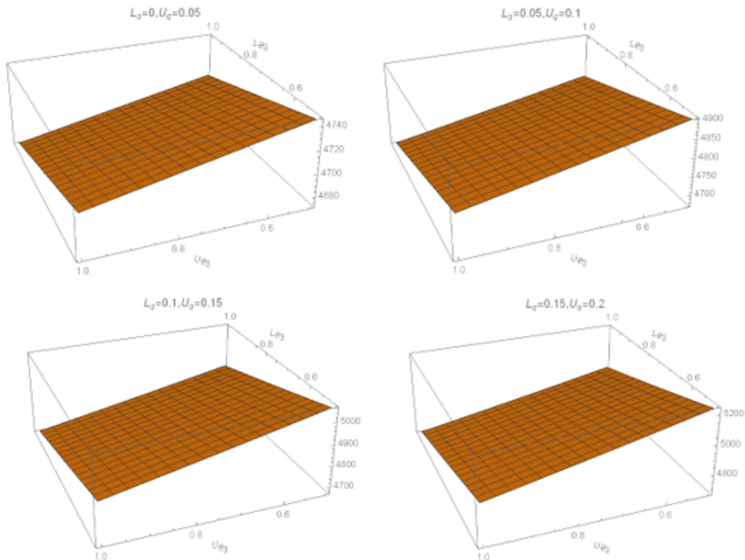

Figure 29. The change in the optimal production size Q^* when q and θ_3 are stochastic

4.2. *Real life scenarios*

In this section, we analysed three scenarios that were mentioned in the motivation. For Scenario I, defective rate q is a random variable and we can rework all defective items therefore $\theta_3 = 1$. Moreover, reworking is perfect, so all items are in perfect condition after the process. In Scenario II, q is random variable and reworkable rate $\theta_3 = 0.75$, that is, we can not rework on all defective units. In this case, at the end of reworking, there will be lower quality items therefore process is not perfect. For the last scenario, we have deterministic defective rate and stochastic reworkable proportion. At the end of reworking, all items are lower quality. All three scenarios are considered in partially backordered scenario since it is including backorder and lost-sales together. Since the logic is still same, only total cost per cycle, optimal order quantity, and reorder point formulas are given without detailed calculations. After the cases, values are shown in the Table 17.

4.2.1. *Scenario I*

Starting with vendor's total cost per cycle where q is random variable with standard uniform distribution, and all defective items are reworkable ($\theta_3 = 1$), we have

$$\begin{aligned} ETC_V(Q_V) = K_V + h_V \frac{Q_V^2}{2D} - h_V \frac{Q_V^2}{2P} + c_P Q_V + c_r Q_V \mu_q \\ + \frac{\eta Q_V}{\delta_1} \mu_{\ln(q_0/q)} + \frac{\eta Q_V}{\delta_2} \ln\left(\frac{1}{\theta_{3_0}}\right) \end{aligned} \tag{1}$$

and its optimal order quantity will be

$$Q_V^* = \sqrt{\frac{2PDK_V}{h_V(P-D)}} \tag{2}$$

The expected total annual cost is

$$ETC(Q,r) = \Bigg(K_V + K_B + F + cQ + c_P Q + c_r Q\mu_q + \frac{\eta Q}{\delta_1}\ln\left(\frac{q_0}{\mu_q}\right) + \frac{\eta Q}{\delta_2}\mu_{\ln(q_0/q)}$$
$$+ h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r))\frac{Q(1-p)}{D}\right)$$
$$+ h_V\frac{Q^2}{2D} - h_V\frac{Q^2}{2P} + d + c_b\bar{B}(r) + c_l\bar{s}(r)\Bigg)\frac{D}{Q(1-p)} \qquad (3)$$

and the optimal order quantity for the system is

$$Q^* = \sqrt{\frac{-2PD(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V)}{h_B(p^2-1)P - h_V(P-D)}} \qquad (4)$$

and the unique value for r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q}{\Big(B_p'(r)(\bar{B}(r) + \bar{s}(r) - r) - B_p(r)\Big)\Big(D(c_l - c_b) + h_B Q(1-p)\Big) + \Big(c_l D + h_B Q(1-p)\Big)}\right) \qquad (5)$$

4.2.2. Scenario II

This time, q is random variable that follows standard uniform distribution, and all defective items are not reworkable ($\theta_3 = 0.75$) and reworking is imperfect. There are new parameters introduced $\theta_l = 0.4$ as the proportion for lower quality items, and h_{V_l} as holding cost for those items ($h_{V_l} < h_V$). The cost for vendor per cycle is

$$ETC_V(Q_V) = K_V + h_V\left(\frac{Q_V^2\left(2\theta_l L_1^2\theta_3^2 + (L_1+U_1)(\theta_3(2\theta_l\theta_3 U_1 + 6\theta_3 - 3) + 3) - 6\right)}{12P}\right)$$
$$+ h_V\left(\frac{Q_V^2\left(L_1((\theta_l - 1)\theta_3 + 1)(\theta_l\theta_3 U_1 - \theta_3 U_1 + U_1 - 3) + ((\theta_l - 1)L_1\theta_3 + L_1)^2\right)}{6D}\right)$$
$$+ h_V\left(\frac{Q_V^2\left(U_1((\theta_l - 1)\theta_3 + 1)(\theta_l\theta_3 U_1 - \theta_3 U_1 + U_1 - 3) + 3\right)}{6D}\right)$$
$$+ h_{V_l} Q_V\theta_3\theta_l\mu_q + c_P Q_V + c_r Q_V\theta_3\mu_q + \frac{\eta Q_V}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q_V}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \qquad (6)$$

Due to time consumption in Wolfram Mathematica, the values for $\theta_3 = 0.75$ and $\theta_l = 0.4$ are inserted for optimal order quantity formula,

$$Q_V^* = \sqrt{\frac{2400K_V PD}{h_V P\left(121L_1^2 + A(11U_1 - 60) + 1200\right) + 30h_V D\left(3L_1^2 + A(3U_1 + 17) - 40\right)}} \tag{7}$$

where $A = (L_1 + U_1)$.

The expected total annual cost is

$$\begin{aligned}
ETC(Q,r) = \Bigg(& K_V + K_B + F + cQ + c_P Q + c_r Q\theta_3\mu_q + \frac{\eta Q_V}{\delta_1}\mu_{\ln(q_0/q)} + \frac{\eta Q_V}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right) \\
& + h_V\left(\frac{Q^2\left(2\theta_l L_1^2\theta_3^2 + (L_1+U_1)(\theta_3(2\theta_l\theta_3 U_1 + 6\theta_3 - 3) + 3) - 6\right)}{12P}\right) \\
& + h_V\left(\frac{Q^2\left(L_1((\theta_l - 1)\theta_3 + 1)(\theta_l\theta_3 U_1 - \theta_3 U_1 + U_1 - 3) + ((\theta_l - 1)L_1\theta_3 + L_1)^2\right)}{6D}\right) \\
& + h_V\left(\frac{Q^2\left(U_1((\theta_l - 1)\theta_3 + 1)(\theta_l\theta_3 U_1 - \theta_3 U_1 + U_1 - 3) + 3\right)}{6D}\right) \\
& + h_{V_l} Q\theta_3\theta_l\mu_q + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r))\frac{Q(1-p)}{D}\right) \\
& + d + c_b\bar{B}(r) + c_l\bar{s}(r)\Bigg)\frac{D}{Q(1-p)}
\end{aligned} \tag{8}$$

and the optimal order quantity

$$Q^* = \sqrt{\frac{-2400PD(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V)}{1200h_B\left(p^2 - 1\right)P - h_V P\left(121L_1^2 + AB + 1200\right) - 30h_V D\left(3L_1^2 + A(3U_1 + 17) - 40\right)}} \tag{9}$$

where $A = (L_1 + U_1)$ and $B = (11(11U_1 - 60))$ with the unique r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q}{\left(B_p'(r)(\bar{B}(r) + \bar{s}(r) - r) - B_p(r)\right)\left(D(c_l - c_b) + h_B Q(1-p)\right) + \left(c_l D + h_B Q(1-p)\right)}\right) \tag{10}$$

4.2.3. Scenario III

As the last case, we have q deterministic and θ_3 as random variable that follow standard uniform distribution and reworking completely results in lower quality items, that is $\theta_l = 1$. Vendor's total expected cost per cycle becomes

$$ETC_V(Q_V) = K_V + h_{V_l} Q_V q \mu_{\theta_3} + c_P Q_V + c_r Q_V q \mu_{\theta_3} + \frac{\eta Q_V}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{\eta Q_V}{\delta_2} \mu_{\ln(\theta_3/\theta_{3_0})}$$
$$+ h_V \left(\frac{Q_V^2 (D(2q^2(L_2^2 + L_2U_2 + U_2^2) + 3q(L_2 + U_2 + 2) - 6) + 6P(q-1)^2)}{12PD} \right) \quad (11)$$

and the optimal quantity is given as

$$Q_V^* = \sqrt{\frac{12K_V PD}{h_V D\left(2q^2\left(L_2^2 + L_2U_2 + U_2^2\right) + 3q(L_2 + U_2 + 2) - 6\right) + 6h_V P(q-1)^2}} \quad (12)$$

Accordingly, the expected total annual cost is

$$\begin{aligned} ETC(Q,r) = \Bigg(& K_V + K_B + F + cQ + c_P Q + c_r Q q \mu_{\theta_3} + h_{V_l} Q q \mu_{\theta_3} + d + c_b \bar{B}(r) + c_l \bar{s}(r) \\ & + h_V \left(\frac{Q^2\left(D\left(2q^2\left(L_2^2 + L_2U_2 + U_2^2\right) + 3q(L_2 + U_2 + 2) - 6\right) + 6P(q-1)^2\right)}{12PD} \right) \\ & + h_B \left(\frac{1}{2}\frac{(Q(1-p))^2}{D} + \frac{Q^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r))\frac{Q(1-p)}{D} \right) \\ & + \frac{\eta Q}{\delta_1} \ln\left(\frac{q_0}{q}\right) + \frac{\eta Q}{\delta_2} \mu_{\ln(\theta_3/\theta_{3_0})} \Bigg) \frac{D}{Q(1-p)} \end{aligned} \quad (13)$$

and the optimal order quantity for integrated model is

$$Q^* = \sqrt{\frac{12PD(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B + K_V)}{h_V D(2q^2(L_2^2 + L_2U_2 + U_2^2) + 3q(L_2 + U_2 + 2) - 6) - 6h_B(p^2 - 1)P + 6h_V P(q-1)^2}} \quad (14)$$

and the unique r^* given as

$$r^* = F^{-1}\left(1 - \frac{h_B(1-p)Q_B}{\left(B_p'(r)(\bar{B}(r) + \bar{s}(r) - r) - B_p(r)\right)\left(D(c_l - c_b) + h_B Q_B(1-p)\right) + \left(c_l D + h_B Q_B(1-p)\right)}\right) \quad (15)$$

4.2.4. Analysis on scenarios

We used the same parameter values with two new ones (θ_l, h_{V_l}) and calculated the total costs, optimal quantities and reorder points. Table 17 shows that when we have chance to rework on all defective items (Scenario I), the integrated optimal quantity is the minimum among all cases. Due to same form of buyer equation, reorder points

(r_B^*, r^*) and buyer's optimal quantity (Q_B^*) has not changed. Ideally, Scenario I has the lowest difference on individual and integrated costs (0.4802%), because of the rework cost. In Scenario II, we have smaller reworkable rate with lower quality items. Because of extra work, the optimal quantities for both integrated and vendor are higher with total costs. Integrated cost is affected by holding cost for those lower quality items along with higher quantity since demand is satisfied only from perfect items. Scenario III shows q as deterministic and θ_3 as stochastic ($\mu_{\theta_3} = 0.75$) with higher integrated order quantity and total cost than previous cases. With known defective rate and stochastic rework rate, we need to produce more especially when all reworked items are considered as lower quality. Finally, this case has the highest difference between sum of individual costs and integrated cost, that means cooperation of buyer and vendor has the most advantage when there is lower defective rate. From the perspective of lot size, it is the highest case, however if the goal is cost efficiency lower quality items is not a problem since we are not reworking on them.

Table 17. Three real life scenarios for the supplier and buyer individually and integrated for stochastic demand with partial backordering

	q	θ_3	θ_l	Q_V^*	Q_B^*	r_B^*	Q^*	r^*	$ETC_V(Q_V^*)$	$ETC_V(Q_B^*)$	$ETC_B(Q_B^*)$	$ETC_V(Q_B^*)+ETC_B(Q_B^*)$	$ETC(Q^*)$	$\%ETC$
Case I	Stoch. (0.001-0.4)	Det. 1	0	4671	1760	1465	2679	1452	\$ 280,435	\$ 283,765	\$ 34,831	\$ 318,596	\$ 317,068	0.4796 %
Case II	Stoch. (0.001-0.4)	Det. 0.75	0.4	5270	1760	1465	2746	1451	\$ 278,474	\$ 282,277	\$ 34,831	\$ 317,108	\$ 315,438	0.5266 %
Case III	Det. 0.2005	Stoch. (0.5-1)	1	5814	1760	1465	2789	1450	\$ 284,532	\$ 288,701	\$ 34,831	\$ 323,572	\$ 321,764	0.5465 %

CHAPTER 5: CONCLUSION

In this thesis, we proposed a new integrated model in a complex supply-chain environment with imperfect production processes and defective items are present. The considered model has investment both in the production process and reworking process as separate functions. This point is fundamental in supply chain since process quality can be better by investing in the production process such as maintenance and repair of machines, and buying new machines for higher performance. Additionally, the control on process quality provides more non-defective items, which means smaller production lot size, less shipments from vendor to buyer and overall higher trust and reliability in business. We also introduced a customer time-sensitivity term for partial backordering, which is a significant extension to the existing literature. By incorporating stochastic demand and other parameters, the our proposed model provided a cost-efficient solution compared to independent decision-making by the buyer. Moreover, our cost function is strictly convex and nonlinear which is the case in some literature (such as Hsu and Hsu (2016) and Al-Salamah (2019)). All the additions in this study fill the gaps that are not considered in the existing literature: for instance, Hsu and Hsu (2016) and Al-Salamah (2019) for stochastic demand; Taleizadeh et al. (2015) and Gutgutia and Jha (2018) for investment; Sarkar et al. (2017) and Al-Salamah (2019) for stochastic defective rate; Hsu and Hsu (2016) and Gutgutia and Jha (2018) for reworking; and Gutgutia and Jha (2018) and Al-Salamah (2019) for time-sensitive customer behaviour.

One of the possible extensions to our study can be the inclusion of an inspection rate and its cost. A future study can be carried out by addition of those terms to avoid poor quality items in stock. Moreover, depending on the relationship with the production rate, one can see whether the inspection rate is sufficient. Another extension is to consider the integration of different quality levels for reworked items. There can be cases where reworked items are not of perfect quality, but these can still be sold at certain prices. This point is also connected to the concept of sustainability and environment-friendly products. Finally, analysing various distribution models for demand is a good analysis to evaluate the behaviour on the costs, the reorder point, and the optimal order quantity.

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Appendix A: Proofs for convexity

Proposition 1. *The $ETC_B(Q_B, r)$ expected annual total cost is convex in (Q_B, r).*

Proof. The total expected annual cost for buyer, Eq. 25, is:

$$ETC_B(Q_B, r) = \left(h_B \left(\frac{1}{2} \frac{(Q_B(1-p))^2}{D} + \frac{Q_B^2 p(1-p)}{D} + (r - \mu_L + \bar{s}(r)) \frac{Q_B(1-p)}{D} \right) \right.$$
$$\left. + K_B + F + cQ_B + d + c_b\bar{B}(r) + c_l\bar{s}(r) \right) \frac{D}{Q_B(1-p)}$$

Taking the first and second partial derivatives of $ETC_B(Q_B, r)$ with respect to Q_B and r, we get

$$\frac{\partial(ETC_B)}{\partial Q_B} = \frac{2D(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B) + h_B\left(p^2 - 1\right)Q_B^2}{2(p-1)Q_B^2}, \tag{A.1}$$

$$\frac{\partial(ETC_B)}{\partial r} = \frac{D(c_b\bar{B}'(r) + c_l\bar{s}'(r)) + h_B(1-p)Q_B(1 + \bar{s}'(r))}{(1-p)Q_B}, \tag{A.2}$$

$$\frac{\partial^2(ETC_B)}{\partial Q_B^2} = \frac{2D(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B)}{(1-p)Q_B^3} > 0, \tag{A.3}$$

$$\frac{\partial^2(ETC_B)}{\partial r^2} = \frac{\bar{s}''(r)(h_B(1-p)Q_B + c_lD) + c_bD\bar{B}''(r)}{(1-p)Q_B} > 0, \tag{A.4}$$

and

$$\frac{\partial^2(ETC_B)}{\partial r \partial Q_B} = \frac{\partial^2(ETC_B)}{\partial Q_B \partial r} = \frac{D\left(c_b\bar{B}'(r) + c_l\bar{s}'(r)\right)}{(p-1)Q_B^2} \tag{A.5}$$

With those equations, we obtain the determinant

$$\begin{vmatrix} \frac{\partial^2(ETC_B)}{\partial Q_B^2} & \frac{\partial^2(ETC_B)}{\partial Q_B \partial r} \\ \frac{\partial^2(ETC_B)}{\partial r \partial Q_B} & \frac{\partial^2(ETC_B)}{\partial r^2} \end{vmatrix}$$

$$= \frac{D\left(2(c_b\bar{B}(r) + d + F + c_l\bar{s}(r) + K_B)\left(\bar{s}''(r)(h_B(1-p)Q + c_lD) + c_bD\bar{B}''(r)\right)\right)}{(p-1)^2Q^4}$$
$$- \frac{D^2\left(c_b\bar{B}'(r) + c_l\bar{s}'(r)\right)^2}{(p-1)^2Q^4} \tag{A.6}$$

This term is non-negative when

$$2(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B)\left(\bar{s}''(r)(h_B(1-p)Q+c_lD)+c_bD\bar{B}''(r)\right) \geq D\left(c_b\bar{B}'(r)+c_l\bar{s}'(r)\right)^2 \tag{A.7}$$

with $\bar{s}(r)>0$, $\bar{B}(r)>0$, and all positive parameters. Therefore, $ETC_B(Q_B,r)$ is convex function in (Q_B,r) when above condition is satisfied.

Proposition 2. *$TC_V(Q_V)$, annual total cost of supplier is strictly convex in Q_V.*

Proof. The total annual cost for the supplier, Eq. 62, is:

$$TC_V(Q_V)=\Bigg(K_V+h_V\left(\frac{Q_V^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right) + \frac{Q_V\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right)+\frac{Q_V\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)+c_PQ_V+c_rQ_Vq\theta_3\Bigg)\frac{D}{Q_V(1-p)} \tag{A.8}$$

Taking the first and second partial derivatives of TC_V with respect to Q_V, we get

$$\frac{d(TC_V)}{dQ_V}=\frac{2PDK_V-h_VQ_V^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2(p-1)PQ_V^2} \tag{A.9}$$

$$\frac{d^2(TC_V)}{dQ_V^2}=\frac{2K_VD}{(1-p)Q_V^3} \tag{A.10}$$

since all parameters are positive, $\frac{d^2(TC_V)}{dQ_V^2}>0$. Therefore, $TC_V(Q_V)$ is strictly convex in Q_V. All other cases can be shown with similar calculation therefore they are skipped.

Proposition 3. *ETC(Q,r) expected annual total cost is strictly convex in (Q,r).*

Proof. The integrated total expected annual cost in Eq. 170 is:

$$ETC(Q,r)=\Bigg(K_V+K_B+F+h_V\left(\frac{Q^2(q(\theta_3-1)+1)(P(q(\theta_3-1)+1)-D)}{2PD}\right)+c_PQ + h_B\left(\frac{1}{2}\frac{(Q(1-p))^2}{D}+\frac{Q^2p(1-p)}{D}+(r-\mu_L+\bar{s}(r))\frac{Q(1-p)}{D}\right)+c_rQq\theta_3 + d+c_b\bar{B}(r)+c_l\bar{s}(r)+\frac{Q\eta}{\delta_1}\ln\left(\frac{q_0}{q}\right)+\frac{Q\eta}{\delta_2}\ln\left(\frac{\theta_3}{\theta_{3_0}}\right)+cQ\Bigg)\frac{D}{Q(1-p)} \tag{A.11}$$

Taking the first and second partial derivatives of $ETC(Q,r)$ with respect to Q and r, we get

$$\frac{\partial(ETC)}{\partial Q} = \frac{D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)}{(p-1)Q^2} + \frac{h_B\left(p^2-1\right)}{2(p-1)} - \frac{h_V(q(\theta_3-1)+1)(Pq(\theta_3-1)+P-D)}{2(p-1)P}, \tag{A.12}$$

$$\frac{\partial(ETC)}{\partial r} = \frac{Dc_b\bar{B}'(r)+Dc_l\bar{s}'(r)+h_B(1-p)Q(1+\bar{s}'(r))}{(1-p)Q}, \tag{A.13}$$

$$\frac{\partial^2(ETC)}{\partial Q^2} = \frac{2D(c_b\bar{B}(r)+K_B+K_V+d)}{(1-p)Q^3} > 0, \tag{A.14}$$

$$\frac{\partial^2(ETC)}{\partial r^2} = \frac{Dc_b\bar{B}''(r)+\bar{s}''(r)(h_B(1-p)Q+c_lD)}{(1-p)Q} > 0, \tag{A.15}$$

and

$$\frac{\partial^2(ETC)}{\partial r\partial Q} = \frac{\partial^2(ETC)}{\partial Q\partial r} = \frac{h_B\left(\bar{s}'(r)+1\right)}{Q} - \frac{D\left(c_b\bar{B}'(r)+c_l\bar{s}'(r)+\frac{h_B(1-p)Q(\bar{s}'(r)+1)}{D}\right)}{(1-p)Q^2} \tag{A.16}$$

With those equations, we obtain the determinant

$$\begin{vmatrix} \frac{\partial^2(ETC)}{\partial Q^2} & \frac{\partial^2(ETC)}{\partial Q\partial r} \\ \frac{\partial^2(ETC)}{\partial r\partial Q} & \frac{\partial^2(ETC)}{\partial r^2} \end{vmatrix}$$

$$= \frac{2D(c_b\bar{B}(r)+d+F+c_l\bar{s}(r)+K_B+K_V)\left(c_bD\bar{B}''(r)+\bar{s}''(r)(c_lD-h_BpQ+h_BQ)\right)}{(p-1)^2Q^4} + \frac{h_B^2}{Q^2} + \frac{\bar{s}'(r)\left(2h_B^2(p-1)Q-c_lD(h_B+Q)\right)+h_B\bar{s}'(r)^2(h_B(p-1)Q-c_lD)}{(p-1)Q^3} + \frac{c_bD\bar{B}'(r)\left(h_B\bar{s}'(r)+h_B+Q\right)}{(1-p)Q^3} \tag{A.17}$$

because $\bar{s}(r) > 0$ and $\bar{B}(r) > 0$ with all positive parameters. Therefore, $ETC(Q,r)$ is convex function in (Q,r).

VITA

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