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Purchasing and remanufacturing decisions with different quality returned material and finished goods

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ABSTRACT

To transition from a linear to a circular value chain, an effective refurbishment policy is crucial to preserve material value and functionality at the end of a product's life-cycle. This study examines a refurbisher that processes first- and second-quality returned materials to produce first- and second-quality products in a make-to-stock system. The refurbisher makes purchasing decisions for the returned materials and determines whether to refurbish or remain idle, and if refurbishing, how to convert them into finished goods of varying quality. There are five refurbishment decisions (converting first-quality to first- or second-quality, second-quality to first- or second-quality, or no production) and two purchasing decisions for the materials. With production and arrival times modelled as exponential random variables, the optimal control problem is formulated as a Markovian Decision Process, using a long-run average profit criterion to identify optimal decisions. A linear programming approach is employed for numerical optimisation. Results show that the most profitable option based solely on sales prices, purchasing, and conversion costs may not be optimal. Instead, the optimal policy is influenced by per-unit profit differences, returned material availability, demand rates, and production times across various refurbishment scenarios.

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Circular economy; refurbishment; remanufacutring; Markov decision processes; linear programming

1. Introduction

Transitioning from a linear to a circular value chain represents a fundamental shift in sustainable resource management, necessitating efficient handling of products at the end of their life-cycle to retain their material value and functionality (World Bank 2022). The circular approach involves retaining a product's material value and functionality by incorporating returned products in various value chain stages–resource extraction, production, distribution, consumption, and disposal (Reike, Vermeulen, and Witjes 2018). Depending on the stage where the returned product flow enters a value chain, a return loop can be short (from consumption back to distribution), medium (from consumption to production), or long (from consumption to resource extraction). Accordingly, processes aiming to retain a product's material value and functionality are referred to as reusing/reselling and repairing for the short loop, refurbishing, remanufacturing, and repurposing for the medium loop, or recycling and recovering for the long loop (Ellen MacArthur Foundation 2021).

This paper concentrates on decisions within the medium loop of a circular value chain, specifically examining refurbishing, remanufacturing, and repurposing processes. Refurbishing involves restoring products to good working conditions by replacing components or enhancing appearance, while remanufacturing aims to bring products to an as-new state through meticulous reengineering. Repurposing involves utilising returned products or their components in creating new products with different functionalities (Ellen MacArthur Foundation 2021).

Making the right refurbishing and purchasing decisions in circular value chains is more complicated due to the differences in the quality of returned material flows and refurbished products, different demand patterns for refurbished products, and differences in the purchasing and refurbishing costs, among others (King et al. 2006; Lahane, Kant, and Shankar 2020). Considering these multifaceted factors, this study delves into optimising the decisions surrounding returned material purchasing and refurbishing.

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The motivation for this study is a project that involves a small-to-medium-sized enterprise (SME) specialising in returned battery cells. Our study presents the optimal control policy for both the refurbishment process for these cells and the procurement strategies to source them. The refurbisher's operations involve converting both first- and second-quality cells into packs of varying quality, tailoring these packs for a range of applications, from high-demand uses such as forklifts to less demanding roles like power supplies for mobile phone base stations. This SME is a recent entrant into the market and thus lacks the market power to set its own sales prices, positioning it as a price taker in the competitive battery market. Crucially, the enterprise functions as a third-party refurbisher, meaning it processes batteries produced by various manufacturers, not just its own. This leads to a key aspect of its business model which is the independence of its supply from its demand. By pooling returned materials from multiple sources, the company ensures a consistent and ample flow of items for refurbishment. Although individual return rates are typically slow in such business environments, the aggregate supply from diverse origins suffices to meet the demand for refurbished products. This independence between supply and demand ensures a steady availability of materials, allowing the refurbishment process to function effectively without being constrained by the return rates from any single source.

A third-party testing process is employed to estimate the remaining capacity of the returned battery cells. Based on the results of this assessment, the cells are then categorised and offered to the manufacturer as either first- or second-quality packs. The decision to classify battery cells into two distinct quality categories–first- and second-quality–is driven by practical considerations of product heterogeneity. The quality of the returned cells can significantly vary depending on their usage and previous lifecycle. Testing is necessary to estimate the remaining capacity of each cell, enabling classification into quality categories that match the specifications required for different applications. For example, batteries designated for use in forklifts may demand higher performance and reliability compared to those used in less demanding applications such as stationary power supply for mobile phone base stations. By differentiating the products into quality categories, the company can operate its refurbishing process more efficiently, allocate resources appropriately, and enhance value recovery. A more comprehensive description of the refurbisher and its business model can be found on the company's website (https://libattion.com/story/) and the Auto-Twin Project's website (https://www.auto-twin-project.eu/use-cases).

The refurbishment model considered here has broader applicability to other returned products beyond battery cells. Many industries, particularly those dealing with electronic devices, face similar challenges related to product heterogeneity, quality assessment, and decisions regarding the conversion or procurement of returned products. Examples encompass a variety of products, including but not limited to refurbished electronic devices, tires, and OEM parts, which are systematically categorised based on their condition and performance attributes (see, e.g. Chen 2021; Lebreton and Tuma 2006; Liu and Papier 2022; Zhang, Liu, and Niu 2020). The framework developed in this study can potentially be adapted to other contexts where products exhibit varying quality levels and where the refurbishment process must respond to both supply-side and demand-side uncertainties. The optimal decisions of when to purchase returned battery packs of different quality levels and how to assemble them into various quality products not only significantly enhance profitability but may also help improve environmental benefits by optimising the use of returned materials. Consequently, while the current study is motivated by the battery refurbishment industry, the insights derived can inform decision-making in other sectors facing analogous operational dynamics.

In this paper, we consider a refurbisher that refurbishes returned materials to produce and sell a product in two different quality levels: first- and second-quality finished goods. At the end of their life, the returned products are collected by a third party and grouped as first- or second-quality returned materials after a functional test. The first- and second-quality returned materials are offered to the refurbisher. When a returned material of a given quality arrives, depending on its input and finished goods inventory levels, the refurbisher decides whether to purchase the returned material. The refurbisher also determines whether to refurbish or stay idle and what quality returned materials to convert to what quality of finished goods if it decides to refurbish. Considering all combinations, there are five different refurbishing decisions (refurbishing first-quality returned materials to a first-quality finished good, first to second, second to first, second to second, and do not produce) and two purchasing decisions (for the first- and second-quality returned materials). Note that each decision yields a different per-unit cost. When the production and arrival times are modelled as exponential random variables, the optimal control problem to determine these refurbishing decisions that maximise the expected profit in the long-run is modelled as a Markovian Decision Process, and the optimal policy is determined by using a linear programming approach. Our numerical experiments reveal that using the most profitable refurbishment option based on the sales price of the products, the purchasing cost of returned materials, and the conversion costs is not necessarily the optimal refurbishing decision for the refurbisher. The optimal refurbishment policy is driven by the per-unit profits for different refurbishment options and also by returned material, demand rates, and production times for different conversion options. Furthermore, in all numerical instances examined within this paper, it is observed that the optimal purchasing and conversion decisions adhere to state-dependent threshold policies.

The primary contribution of this study lies in presenting a detailed analytical model for analyzing optimal returned material purchasing and refurbishing decisions in a circular value chain with varying qualities of returned materials and products and showing the effects of returned material and demand rates, production times, prices and costs on the refurbishment decisions, profitability, and some other performance indicators. The organisation of the remaining part of this paper is as follows: The pertinent literature is reviewed in Section 2. Section 3 presents the model and its assumptions. The optimal control problem is defined and solved using the LP approach in Section 4. The numerical results are given in Section 5. Finally, conclusions are provided in Section 6.

2. Literature review

Integrating refurbishment and product recovery into linear production systems has garnered substantial attention in recent decades from both practitioners and researchers. The utilisation of refurbishment options and product recovery strategies is now acknowledged as a lucrative and sustainable strategy for many companies. Motivations for embracing these strategies encompass economic, legislative, and environmental considerations. The design and control of such systems become more intricate due to variations in the quality of returned material flows and refurbished products, distinct demand patterns for refurbished products, and disparities in purchasing and refurbishing costs, among other factors. Comprehensive systematic reviews of such systems and the potential challenges associated with their operational and strategic decisions have been presented by Dekker et al. (2004), Srivastava (2007), and Khan et al. (2021). In this review, we concentrate exclusively on the closest studies and their contributions concerning the model proposed in this work. Specifically, our focus is directed toward reviewing the pertinent studies that present stochastic models based on Markov Decision Processes to examine decisions associated with remanufacturing strategies for systems operating in a make-to-stock fashion.

In the literature, numerous studies explore integrated remanufacturing and disposal decisions for systems operating in a make-to-stock fashion, considering diverse system characteristics and dynamics in the context of product reuse. As early works in this literature stream, Ching, Li, and Xue (2007) and Flapper, Gayon, and Vercraene (2012) examine a single-product hybrid system operating in a make-to-stock fashion, where both manufacturing and remanufacturing operations are executed in the same facility. Ching, Li, and Xue (2007) employ a matrix-geometric method to evaluate the system performance for a given policy, considering continuous review with Markovian assumptions. On the other hand, Flapper, Gayon, and Vercraene (2012) adopt Markov decision processes to investigate the optimal joint manufacturing and remanufacturing policy, aiming to minimise backorder, holding, remanufacturing, and manufacturing costs per unit of time over an infinite horizon. Kim, Saghafian, and Van Oyen (2013) and Vercraene, Gayon, and Flapper (2014) address a similar problem to that of Flapper, Gayon, and Vercraene (2012) and extend the relevant work to include a disposal decision without considering the quality of returns. In the same context, Gayon, Vercraene, and Flapper (2017) embrace a new disposal mechanism that offers two options for handling returns. In their model, returns can be disposed of upon arrival or after becoming serviceable products. The authors prove that the optimal production control policy is a threshold-type policy with three parameters. Fathi, Zandi, and Jouini (2015) and Farahani, Otieno, and Omwando (2020) also work on optimal disposition policies for remanufacturing systems. Different from previous studies, they consider diverse quality grades of product returns and a limited storage capacity for recoverable items.

Furthermore, studies have been expanding by considering integrated remanufacturing and disposal decisions in various system configurations such as tandem lines (Vercraene and Gayon 2013) or make-to-order systems (Nadar et al. 2023). Specifically, Vercraene and Gayon (2013) focus on a n-stage production-inventory system with Poisson returns at each stage and investigate under which circumstances the performances of three types of heuristic policies (fixed buffer, base-stock, and Kanban) generate reasonable solutions concerning the optimal policy. Nadar et al. (2023) study the used-item acquisition and disposition problem for a single-product remanufacturing system operating in a make-to-order fashion with multiple unknown quality conditions for acquired cores, random procurement lead times, and lost sales.

In recent years, this literature has also been broadening through investigations that contemplate diverse phenomena within remanufacturing settings, such as advanced return information, setup time requirements in changes between manufacturing and remanufacturing modes, dependency between demand and return, and substitution between new and remanufactured products. Specifically, Flapper, Gayon, and Lim (2014) study the optimal production control of a make-to-stock system in which consumers announce their intention to return a product in advance and make these returns after a stochastic return lead time. With this study, the authors provide insight into the potential value of using imperfect advance information on returns. Polotski, Kenne, and Gharbi (2015, 2017) propose a general structure of the optimal control policy for a hybrid system that requires setups to switch between manufacturing and remanufacturing modes. Polotski, Kenne, and Gharbi (2019) examine a similar problem to those of Polotski, Kenne, and Gharbi (2015, 2017); but, different than their previous work, the authors assume that no setup times are required to switch between manufacturing and remanufacturing modes and incorporate an additional decision regarding preventive maintenance planning into the model. Chen (2021) addresses a dynamic production control problem in remanufacturing systems considering the dependency relation of returns with demand and propose a time-efficient performance evaluation model to determine the optimal policy and the average cost of the system. Liu and Papier (2022) present an MDP model of a make-to-stock system with regulated two-way substitutions between new and remanufactured products and show that allowing for controlled two-way substitutions significantly improves the manufacturer's profitability.

Moreover, in the extant literature, dynamic production control strategies for remanufacturing systems often coexist with other strategies, such as dynamic disposal, pricing, and acquisition. Gayon and Dallery (2007) address the problem of finding dynamic production and pricing strategies for a make-to-stock production system having uncontrolled product returns and conduct a numerical study on the potential benefits of dynamic pricing compared to static pricing in such a setting. Gao et al. (2015) explore joint production and pricing policies for a firm selling a single type of final product that is either manufactured from new parts or remanufactured from returned products. Yan et al. (2017) extend the work of Gao et al. (2015) by considering customers' differentiation based on their willingness to buy new or remanufactured products. In their study, the firm either adopts a make-to-order or make-to-stock production strategy for the new product. The authors demonstrate that the base-stock type of production policy is optimal when the firm operates in a make-to-stock fashion with an additive demand model.

Based on the literature review, our work sets itself apart from the existing body of research by examining integrated purchasing and remanufacturing decisions for a system operating in a make-to-stock manner, building upon the foundation laid in our earlier conference work (Karabağ, Karaesmen, and Tan 2024). More specifically, to the best of our knowledge, this is the first study to analyze a system with two quality types of returned materials and finished goods, contemplating the existence of five different types of refurbishment and two types of acquisition decisions. In addition to contributing to the literature on remanufacturing systems, this research advances the state of the art in the analysis of make-to-stock systems. To our knowledge, there is no existing study in the relevant literature that addresses such a make-to-stock system. Moreover, we demonstrate through numerical analysis that refurbishing and purchasing decisions depend on not only the per-unit profits for different refurbishment options but also the returned material, demand rates, and production times for different conversion options. Our numerical results also show that the optimal purchasing and conversion decisions are state-dependent threshold policies.

3. Problem framework

We consider a discrete-material flow remanufacturing system in which a single facility with a limited production capacity operates in a make-to-stock manner to fulfil the demand for a single product's first- and second-quality versions. The graphical illustration of the system we consider here is portrayed in Figure 1. As can be seen in the corresponding figure, the refurbisher keeps two different forms of inventory to perform her operations: (*i*) returned material and (*ii*) finished goods inventories. Specifically, in the system, two types of returned material stocks exist, and these stock types differ from each other only in terms of their qualities. Analogously, there are two kinds of finished goods stocks for high- and low-quality products. The returned materials can be the refurbisher's products returned at the end of their product life cycle or the returned products produced by other manufacturers. Due to the long product life cycle of the products, the return flow of the finished goods as the returned materials is not modelled explicitly.

At time t, the first- and second-quality returned material inventory levels, respectively, are $r_1(t)$ and $r_2(t)$ where $r_1(t), r_2(t) \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Similarly, at time t, the inventory levels of first- and second-quality finished goods respectively are $f_1(t)$ and $f_2(t)$ where $f_1(t), f_2(t) \in \mathbb{N} = \{0, 1, 2, \ldots\}$. As evident from their mathematical definitions, the inventories have no capacity constraints, meaning they can theoretically grow to infinity. Given

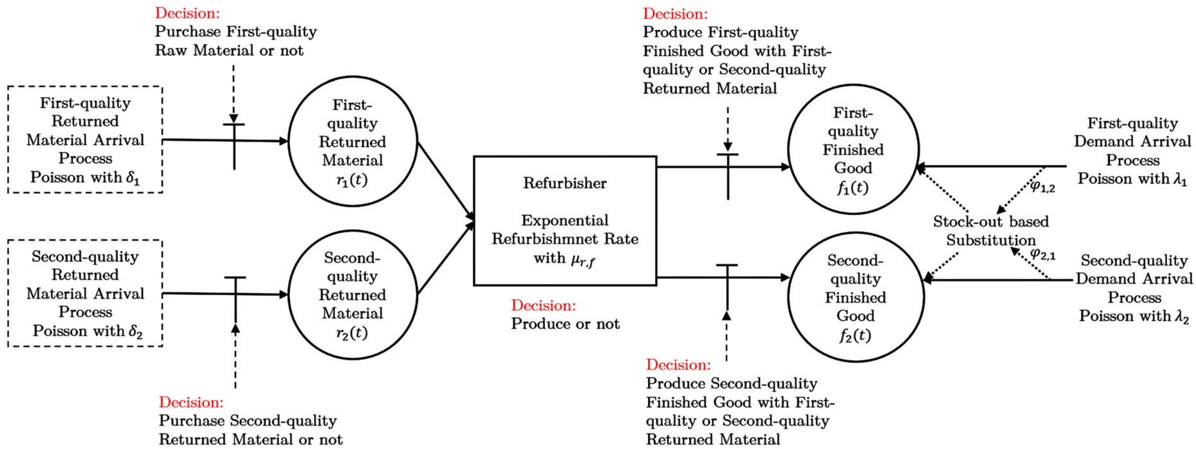


Figure 1. Purchasing, production and remanufacturing decisions for a system with two returned material and two finished good quality levels.

that inventory levels, market prices for the finished goods, and the returned materials are fully observed, the objective is to determine the optimal purchasing and refurbishing strategies that maximise the refurbisher's average profit over an infinite planning horizon. The system details are given in the subsequent sections.

3.1. Purchasing process

The arrival processes of the returned material types are modelled as two independent Poisson processes. Accordingly, the inter-arrival times of the first- and second-quality returned materials are independent of each other, and they are exponentially distributed with rates δ_1 and δ_2, respectively.

Each time a returned material unit at any quality level arrives at the system, the refurbisher decides whether to buy it. In case of no purchasing decision, the chance of purchasing the corresponding returned material unit is lost without any immediate penalty cost. In case of a purchasing decision, a unit purchasing cost is charged based on the returned material quality type. Specifically, the purchasing costs of first- and second-quality returned materials are p_1 and p_2, respectively. The quality difference between these two returned material types inherently leads to a difference in their purchasing prices (see, e.g. Galbreth and Blackburn 2010; Nadar et al. 2023).

The unit purchasing cost of first-quality returned materials is considered to be greater than the unit purchasing cost of second-quality returned materials, i.e. $p_1 > p_2$. Each purchased returned material unit is added to the corresponding inventory based on its quality. The costs of keeping one unit of stock in the first- and second-quality returned material buffers are h_1 and h_2, respectively.

3.2. Production process

At time t, the refurbisher can take five different types of decisions related to the production process. She may decide (*i*) not to produce, (*ii*) to produce a first-quality finished product from a first-quality returned material, (*iii*) to produce a first-quality finished product from a second-quality returned material, (*iv*) to produce a second-quality finished product from a first-quality returned material, and (*v*) to produce a second-quality finished product from a second-quality returned material. In case of no production decision, she does not incur any production cost. In this case, no change will originate from the production process in the returned and finished goods inventory levels. In case of a production decision, a single returned material unit is released into the system. The returned material's quality type being sent to the system is determined depending on what type of production decision she has taken. In this case, depending on the decision, the corresponding returned material inventory is decreased by one, and the related finished goods inventory is increased by one. In contrast, the other inventories stay the same.

The system operates with a limited production capacity, allowing it to process only one returned material unit at a time, with each unit requiring a random processing time. This is a standard model for production-inventory systems modelled as make-to-stock queues. The single processor assumption is a simplification of reality but enables us to capture the effect of endogenous lead times which are load dependent. More specifically, the system's returned material processing times follow an exponential distribution whose rate depends on the refurbisher's production decision. That is, which quality type of finished product is to be produced and which quality type of returned material is to be used in the production affect the processing times. The average time being spent to process a unit of returned material of quality r to a unit of finished good of quality f is denoted by $\mu_{r,f}$ where $r \in \{1,2\}$ and $f \in \{1,2\}$.

Once the production is completed, the finished good unit is placed in the corresponding inventory based on its quality type. The costs of keeping one unit of stock in the first- and second-quality finished good buffers are k_1 and k_2, respectively. Additionally, for transforming a unit of returned material of quality r to a unit of finished good of quality f, the system operator incurs a production (conversion) cost of $c_{r,f}$. While not restricting the model, these costs are possibly ordered as follows: $c_{2,1} > c_{1,1}$ and $c_{2,2} > c_{1,2}$.

3.3. Sales process

Two customer types arrive at the system; they essentially demand the same type of finished goods but at different quality levels. The customer arrival processes are modelled as two independent Poisson processes. The inter-arrival times of the customers demanding first- and second-quality finished goods are considered to be independent of each other. They are exponentially distributed with rates λ_1 and λ_2, respectively.

There exist two market prices, one for first-quality finished goods, s_1, and one for second-quality finished goods, s_2. The manufacturer earns s_1 and s_2 for each sales transaction of first- and second-quality finished goods, respectively. The quality difference between these two types of finished goods inherently leads to a difference in their sales prices. Correspondingly, the sales price for the first-quality finished goods is considered to be higher than the sales price for the second-quality finished goods, i.e. $s_1 > s_2$. These price parameters are considered to be exogenous (see, e.g. Karabağ and Tan 2019). In other words, the refurbisher is a price-taker. So, she cannot individually influence the market prices of finished goods and returned materials.

Regardless of its type, each customer arriving at the system requests only one unit of the finished goods; this request is met as long as the corresponding finished goods inventory is non-empty. Once an arriving demand is satisfied, the manufacturer receives the corresponding market sales price based on the customer's desired quality. The contribution of sales of a finished good of quality f that is produced by converting returned material of quality r is then $s_f - c_{r,f} - p_r$ where $r \in \{1,2\}$ and $f \in \{1,2\}$.

3.4. Demand substitution

We assume a stock-out-based substitution. Suppose a demand for a first-quality product arrives. In that case, if the first-quality stock is empty and a second-quality product is available, $\varphi_{1,2}$ percent of the customers substitute the first-quality product with a second-quality one. Similarly, suppose a demand for a second-quality product arrives. In that case, if the second-quality stock is empty and a first-quality product is available, $\varphi_{2,1}$ percent of the customers substitute the second-quality product with a first-quality one. If both finished goods inventories are empty, demand for either a first-quality or second-quality product is lost without immediate penalty.

The main simplifying assumption in our approach is modelling the arrival and production times as exponential random variables. The historical data from the battery refurbisher that motivates our study is limited since the SME has been established recently. The Poisson arrival is a good approximation for the demand and returned material arrivals. Furthermore, since the company does not restrict its operations to refurbishing only its own products and it processes returned battery cells from a wide range of manufacturers, the return process of materials and the demand for refurbished products are independent. However, the production times are not expected to be exponential. Considering exponential arrival and production times allows modelling the problem as a Markovian Decision Process and derive the optimal policy as presented in the next section. When the historical data is available, modelling the arrival and production times as Markovian Arrival Processes also allows using the same MDP approach at the expense of more extensive computational effort.

4. Optimal control model and LP formulation

To effectively manage the system, the manufacturer employs a dynamic purchasing and production strategy based on the real-time system state. In the context of this study, the system state is represented as a four-dimensional vector, i.e. $\mathbf{s}(t) = (r_1(t), r_2(t), f_1(t), f_2(t))$ where $\mathbf{s}(t) \in \mathbb{N}^4$. The first two components of this vector denote the inventory levels of first- and second-quality returned materials at time t, whereas the last two components represent the inventory levels of first- and second-quality finished goods at time t.

The manufacturer's decisions regarding whether to purchase a unit of first-quality returned material and a unit of second-quality returned material, when the system state is $\mathbf{s}$ at time t, are respectively denoted by $u_{b_1}(\mathbf{s}(t))$ and $u_{b_2}(\mathbf{s}(t))$ where $u_{b_1}(\mathbf{s}(t)), u_{b_2}(\mathbf{s}(t)) \in \mathbb{B} = \{0,1\}$. In case of deciding to purchase a unit of first-quality returned material when the system state is $\mathbf{s}$ at time t, i.e. $u_{b_1}(\mathbf{s}(t)) = 1$, the manufacturer pays a unit procurement cost of p_1; otherwise, she pays nothing. Analogously, in case of deciding to purchase a unit of second-quality returned material when the system state is $\mathbf{s}$ at time t, i.e. $u_{b_2}(\mathbf{s}(t)) = 1$, she incurs a unit procurement cost of p_2; otherwise, no cost is incurred.

On the other hand, the manufacturer's decision regarding production, when the system state is $\mathbf{s}$ at time t, is denoted by $u_m(\mathbf{s}(t))$ where $u_m(\mathbf{s}(t)) \in \mathbb{M} = \{0, 1, 2, 3, 4\}$. If no production decision is made when the system state is $\mathbf{s}$ at time t (i.e. $u_m(\mathbf{s}(t)) = 0$), the manufacturer incurs no production costs. When the system state is $\mathbf{s}$ at time t, and the manufacturer decides to produce a unit of first-quality finished goods from a unit of first-quality (second-quality) returned material, denoted by $u_m(\mathbf{s}(t)) = 1$ ($u_m(\mathbf{s}(t)) = 3$), she incurs a unit production cost of $c_{1,1}$ ($c_{1,2}$). Similarly, if the decision is to produce a unit of second-quality finished goods from a unit of first-quality (second-quality) returned material at this instance, represented by $u_m(\mathbf{s}(t)) = 2$ ($u_m(\mathbf{s}(t)) = 4$), the manufacturer bears a unit production cost of $c_{2,1}$ ($c_{2,2}$).

As alluded to above, the refurbisher employs a dynamic purchasing and conversion policy $\ell \in \mathbb{U}$, which is determined based on the real-time system state $\mathbf{s}(t) = (r_1(t), r_2(t), f_1(t), f_2(t)) \in \mathbb{N}^4$. The policy ℓ is formally defined as a mapping $\ell : \mathbb{S} \to \mathbb{A}$, where $\mathbb{S} = \mathbb{N}^4$ represents the state space and $\mathbb{A} = \mathbb{B} \times \mathbb{B} \times \mathbb{M}$ denotes the action space. In another words, for any system state $\mathbf{s}(t)$ at time t, the policy ℓ prescribes an action in the form:

$$u^{\ell}(\mathbf{s}(t)) = \left(u_{b_1}(\mathbf{s}(t)), u_{b_2}(\mathbf{s}(t)), u_m(\mathbf{s}(t))\right), \tag{1}$$

where $u_{b_1}(\mathbf{s}(t))$, $u_{b_2}(\mathbf{s}(t))$, and $u_m(\mathbf{s}(t))$ correspond to purchasing and conversion decisions in the given state, and the set $\mathbb{U}$ thus encompasses all possible policies that map system states to purchasing and production actions, providing a structured framework for analyzing and determining optimal control strategies within the system.

Let $\mathbf{s}'(t')$ be the initial state of the system at time $t' = 0$. So, for given ℓ and $\mathbf{s}(t)$, the refurbisher's expected profit per unit of time over an infinite planning horizon is described as:

$$\phi^{\ell}(\mathbf{s}(t)) = \lim_{T\to\infty} \frac{1}{T} \mathrm{E}^{\ell}_{\mathbf{s}'(t')} \left[\sum_{i=1}^{2} \int_0^T s_i \, \mathrm{d}N_{Z_i}(t) - \sum_{i=1}^{2} \int_0^T p_i \, \mathrm{d}N_{G_i}(t) - \sum_{r=1}^{2} \sum_{f=1}^{2} \int_0^T c_{r,f} \, \mathrm{d}N_{M_i}(t) - \int_0^T H(S(t)) \, \mathrm{d}t \right]. \tag{2}$$

Note that in Equation (2), $H(S(t)) = h_1 R_1(t) + h_2 R_2(t) + k_1 F_1(t) + k_2 F_2(t)$ denotes the total holding cost being incurred for both returned materials and finished goods, $N_{Z_1}(t)$ and $N_{Z_2}(t)$ are the total numbers of customers requesting the first- and second-quality finished goods and being met until time t, $N_{G_1}(t)$ and $N_{G_2}(t)$ represent the total number of first- and second-quality returned materials procured until time t, and $N_{M_i}(t)$ represents the total number of finished good units of type i produced until time t.

Due to our problem structure, the time elapsed between each transition in the system is an exponentially distributed random variable. This problem characteristic makes the system memoryless, enabling us to restrict the decision epochs to when the system state changes. Such property makes it possible to represent our problem as a Markov decision process whose optimal solution belongs to the class of stationary policies and the system through time-independent state variables (Karabağ and Gökgür 2023). Correspondingly, given that the system state is time-independent, the optimisation problem that must be addressed to derive the optimal control policy ℓ^* can be formulated as:

$$\phi^*(\mathbf{s}) = \sup_{\ell \in \mathbb{U}} \phi^{\ell}(\mathbf{s}), \quad \forall\, \mathbf{s} \in \mathbb{N}^4. \tag{3}$$

The state trajectory of the system under a unichain policy eventually becomes confined to the recurrent class of states. This is attributable to the problem's finite action space and strictly positive, bounded costs and rewards. Given this unique characterisation, the average reward for all initial states, as well as the differential rewards of the recurrent states, are independent of the rewards obtained from the transient states (Bertsekas 2015; Puterman 2014). Considering this along with the standard argument from the theory of contraction mappings, it can be confirmed that Bellman's equation holds and that a deterministic control policy exists for this problem. Thus, the problem can be solved by using solution procedures such as value iteration, policy iteration, and linear programming.

4.1. LP formulation

In this study, we utilise a linear programming approach to numerically obtain the optimal policy for the system. This approach has been frequently applied to investigate and establish the optimal control policy within the domains of production control, as well as energy mode control and maintenance optimisation (see, e.g. Karabağ et al. 2024; Karabağ and Tan 2019; Loffredo et al. 2024; Tan, Karabağ, and Khayyati 2023a, 2023b). Using the LP approach empowers us to leverage state-of-the-art optimisation software packages, enabling a more expedient problem resolution. As a result, identifying pertinent optimal policies can be achieved in significantly less time compared to classical methods like value/policy iterations.

As alluded to in the problem definition, no capacity constraints are imposed on the quantities in the returned materials and finished goods inventories; theoretically, these quantities may increase indefinitely. Nevertheless, to meet the computational requirements of the linear programming approach, they must be truncated at specific levels, allowing the problem to be represented with a finite number of decision variables and constraints. In the literature, there are various methodologies for determining appropriate truncation levels. These levels are typically determined based on criteria such as ensuring that the steady-state probabilities at the truncated inventory levels remain below a specified tolerance or setting truncation levels sufficiently high such that any further increase in these levels results in a negligibly small change in the globally optimal reward (Karabağ et al. 2024; Loffredo et al. 2024; Tan, Karabağ, and Khayyati 2023a). In our approach, we specifically adopt the latter criterion, as it is widely used in the pertinent literature and easier to implement. Note that when inventory capacities are limited, the truncation level can be aligned with the corresponding capacity limits, enabling these constraints to be directly incorporated into the formulation.

Furthermore, the LP-based solution approach cannot be used directly for CTMC problems. Therefore, it necessitates the transformation of the continuous-time problem into a discrete-time problem by applying the uniformization technique (Lippman 1975; Serfozo 1979). Following the completion of the transformation, the subsequent step entails the establishment of the dual linear programming formulation of the equivalent discrete-time problem. This formulation treats the long-run fraction of the time that the system spends in different states under particular decisions (Tan, Karabağ, and Khayyati 2023b). Using duality theory, the optimal policy for the problem can be derived from the associated dual problem's optimal solution. Specifically, the optimal solution to the dual problem would provide us with the steady-state distribution of the system under the optimal policy. The optimal policy is then characterised by mapping each decision variable having a positive value in the solution to its corresponding actions (see, e.g. Karabağ and Gökgür 2023; Loffredo et al. 2024).

Let $p(\mathbf{j} \mid \mathbf{s}, \mathbf{u})$ and $r_{\mathbf{s},\mathbf{u}}$ denote a transition probability from state $\mathbf{s}$ to state $\mathbf{j}$ under action $\mathbf{u}$ and a function that quantifies the revenue of taking action $\mathbf{u}$ in state $\mathbf{s}$, respectively. The decision variable of the formulation is represented by $\Pi_{\mathbf{s},\mathbf{u}}$ and it indicates the long-run fraction of the time that the system spends in state $\mathbf{s}$ when action $\mathbf{u}$ is taken. Considering the parameters and decision variables being introduced above, the linear programming formulation of our problem can be written, in a most generic way, as follows:

$$\max \sum_{\mathbf{s}\in\mathbb{N}^4} \sum_{\mathbf{u}\in\mathbb{U}^3} r_{\mathbf{s},\mathbf{u}} \times \Pi_{\mathbf{s},\mathbf{u}}, \tag{4}$$

subject to

$$\sum_{\mathbf{u}\in\mathbb{U}^3} \Pi_{\mathbf{j},\mathbf{u}} - \sum_{\mathbf{s}\in\mathbb{N}^4} \sum_{u\in\mathbb{U}^3} p(\mathbf{j} \mid \mathbf{s}, \mathbf{u}) \times \Pi_{\mathbf{s},\mathbf{u}} = 0, \quad \forall\, \mathbf{j} \in \mathbb{N}^4, \tag{5}$$

$$\sum_{\mathbf{s}\in\mathbb{N}^4} \sum_{\mathbf{u}\in\mathbb{U}^3} \Pi_{\mathbf{s},\mathbf{u}} = 1, \tag{6}$$

$$\Pi_{\mathbf{s},\mathbf{u}} \geq 0, \quad \forall\, \mathbf{s} \in \mathbb{N}^4, \forall\, u \in \mathbf{U}^3. \tag{7}$$

Equation (4) shows the objective function aiming to maximise the long-run average profit of the system. Equation (5) stands for the balance equation. Specifically, it ensures that each system state's flows into and out are equal in the long run. The decision variables of the formulation would form a probability mass function. Correspondingly, all decision variables must be greater than or equal to 0, and their sum should be 1. Equations (6) and (7) enable us to integrate these two conditions into the formulation. Note that for the sake of clarity, the explicit LP formulation for the problem we consider in this study is given in Appendix A.

5. Numerical analysis

In this section, we conduct an extensive numerical study to investigate the impact of system characteristics on various performance measures, including long-run average profit, average returned material inventory level, and average final product inventory level. In addition, we present the managerial insights derived from these numerical experiments. Before presenting the results, we provide a detailed overview of the long-run performance measures used in this study and introduce the parameter settings considered in the analysis.

5.1. *Performance measures & parameter sets*

In the numerical analysis, we consider several distinct performance measures to evaluate the system's long-term performance. Table 1 provides a comprehensive overview of these indicators, along with their mathematical definitions, which are crucial for analyzing the system's efficiency and guiding the decision-making process in managing returned materials and finished goods.

Table 1. Performance measures.

Notation	Definition
$\mathbb{N}_0^4$:	The set where both of the finished good buffers are 0: $\mathbb{N}_0^4 = \{(r_1, r_2, f_1, f_2) \mid f_1, f_2 \in \{0\}, r_1, r_2 \in \mathbb{N}\}$.
θ_0:	The long-run probabilities regarding the states in which both finished goods inventory levels are 0: $\theta_0 = \sum_{\mathbf{s}\in\mathbb{N}_0^4} \sum_{b_1\in\mathbb{B}} \sum_{b_2\in\mathbb{B}} \sum_{m\in\mathbb{M}} \Pi_{s,b_1,b_2,m}$.
$\mathbb{N}_1^4$:	The set where the first-quality finished goods buffer is 0: $\mathbb{N}_1^4 = \{(r_1, r_2, f_1, f_2) \mid f_1 \in \{0\}, f_2 \in \mathbb{N} \setminus \{0\}, r_1, r_2 \in \mathbb{N}\}$.
θ_1:	The long-run probabilities regarding the states in which the inventory level of the first-quality finished goods is 0 and the inventory level of the second-quality finished goods is non-zero: $\theta_1 = \sum_{\mathbf{s}\in\mathbb{N}_1^4} \sum_{b_1\in\mathbb{B}} \sum_{b_2\in\mathbb{B}} \sum_{m\in\mathbb{M}} \Pi_{s,b_1,b_2,m}$
$\mathbb{N}_2^4$:	The set where the second-quality finished goods buffer is 0: $\mathbb{N}_2^4 = \{(r_1, r_2, f_1, f_2) \mid f_1 \in \mathbb{N} \setminus \{0\}, f_2 \in \{0\}, r_1, r_2 \in \mathbb{N}\}$.
θ_2:	The long-run probabilities regarding the states in which the inventory level of the second-quality finished goods is 0 and the inventory level of the first-quality finished goods is non-zero: $\theta_2 = \sum_{\mathbf{s}\in\mathbb{N}_2^4} \sum_{b_1\in\mathbb{B}} \sum_{b_2\in\mathbb{B}} \sum_{m\in\mathbb{M}} \Pi_{s,b_1,b_2,m}$.
Π_m	The long-run probability distribution of the production decisions: $\Pi_m = \sum_{\mathbf{s}\in\mathbb{N}^4} \sum_{b_1\in\mathbb{B}} \sum_{b_2\in\mathbb{B}} \Pi_{s,b_1,b_2,m}$, where $\forall m \in \mathbb{M} = \{0, 1, 2, 3, 4\}$.
$\gamma_{1,1}$:	The percentage of the first-quality finished goods demand satisfied by converting the first-quality returned material: $\gamma_{1,1} = \mu_{1,1}\Pi_1/(\lambda_1(1-\theta_0-\theta_1) + \varphi_{2,1}\lambda_2\theta_2)$.
$\gamma_{2,1}$:	The percentage of the first-quality finished goods demand satisfied by converting the second-quality returned material: $\gamma_{2,1} = \mu_{2,1}\Pi_3/(\lambda_1(1-\theta_0-\theta_1) + \varphi_{2,1}\lambda_2\theta_2)$.
$\gamma_{1,2}$:	The percentage of the second-quality finished goods demand satisfied by converting the first-quality returned material: $\gamma_{1,2} = \mu_{1,2}\Pi_2/(\lambda_2(1-\theta_0-\theta_2) + \varphi_{1,2}\lambda_1\theta_1)$.
$\gamma_{2,2}$:	The percentage of the second-quality finished goods demand satisfied by converting the second-quality returned material: $\gamma_{2,2} = \mu_{2,2}\Pi_4/(\lambda_2(1-\theta_0-\theta_2) + \varphi_{1,2}\lambda_1\theta_1)$.
$E[R_1]$:	The long-run average inventory level of the first-quality returned materials: $E[R_1] = \sum_{\mathbf{s}\in\mathbb{N}^4} \sum_{b_1\in\mathbb{B}} \sum_{b_2\in\mathbb{B}} \sum_{m\in\mathbb{M}} r_1 \times \Pi_{s,b_1,b_2,m}$.
$E[R_2]$:	The long-run average inventory level of the second-quality returned materials: $E[R_2] = \sum_{\mathbf{s}\in\mathbb{N}^4} \sum_{b_1\in\mathbb{B}} \sum_{b_2\in\mathbb{B}} \sum_{m\in\mathbb{M}} r_2 \times \Pi_{s,b_1,b_2,m}$.
$E[F_1]$:	The long-run average inventory level of the first-quality finished goods: $E[F_1] = \sum_{\mathbf{s}\in\mathbb{N}^4} \sum_{b_1\in\mathbb{B}} \sum_{b_2\in\mathbb{B}} \sum_{m\in\mathbb{M}} f_1 \times \Pi_{s,b_1,b_2,m}$.
$E[F_2]$:	The long-run average inventory level of the second-quality finished goods: $E[F_2] = \sum_{\mathbf{s}\in\mathbb{N}^4} \sum_{b_1\in\mathbb{B}} \sum_{b_2\in\mathbb{B}} \sum_{m\in\mathbb{M}} f_2 \times \Pi_{s,b_1,b_2,m}$.
z:	The long-run average reward

The set of notations provided in the table, $\mathbb{N}_0^4$, $\mathbb{N}_1^4$, and $\mathbb{N}_2^4$, represent different inventory states for finished goods: $\mathbb{N}_0^4$ corresponds to both finished goods buffers being zero, $\mathbb{N}_1^4$ indicates that only the first-quality finished goods buffer is zero, and $\mathbb{N}_2^4$ signifies that only the second-quality buffer is zero. Understanding these states helps identify stock-out conditions and serves as a basis for calculating other important performance measures. The long-run probabilities, θ_0, θ_1, and θ_2, capture the likelihoods of these zero-inventory states, providing insights into the system's ability to meet demand and maintain service levels for both quality tiers of finished goods. Another critical performance indicator is Π_m, which describes the long-run probability distribution of the conversion decisions. This metric could be essential in evaluating the system's effectiveness in converting raw materials and returned goods into finished products.

The measures of γ, namely $\gamma_{1,1}$, $\gamma_{2,1}$, $\gamma_{1,2}$, and $\gamma_{2,2}$, represent the proportions of demand fulfilled for finished goods of different qualities by using returned materials of varying quality. More precisely, $\gamma_{r,f}$ denotes the percentage of demand for a given quality finished good f that is satisfied by converting a returned material of quality r, where $r,f \in \{1,2\}$. These metrics may provide insights into the environmental and resource recovery implications of conversion decisions. Specifically, converting lower-quality returned materials into higher-quality finished goods typically has a more negative environmental impact due to the extensive processing, energy, and resources required, which often results in increased waste and emissions. In contrast, producing lower-quality goods from higher-quality returns is generally less resource-intensive, as it requires minimal processing and makes better use of the material's existing quality. However, this approach may also raise concerns about the inefficient use of high-quality resources. Overall, these metrics underscore the trade-offs between meeting product quality demands and reducing environmental impacts.

On the other hand, the long-run average inventory levels, $E[R_1]$, $E[R_2]$, $E[F_1]$, and $E[F_2]$, provide a measure of the system's efficiency in managing the stocks of returned materials and finished goods. These averages are crucial for understanding the balance between holding costs and the system's capacity to meet demand. Moreover, z, the long-run average reward, serves as a comprehensive performance measure that encapsulates the economic sustainability of the system by considering revenues and costs associated with inventory levels and conversion policies.

In the numerical analysis, we address four alternative scenarios whose parameters are presented in Table 2. The parameter sets for these scenarios are constructed by considering the ranges employed in pertinent studies–specifically, studies Flapper, Gayon, and Lim (2014), Gayon, Vercraene, and Flapper (2017), and Nadar et al. (2023) that focus on the analysis of remanufacturing systems. It is also important to highlight that all financial parameters are normalised relative to the refurbishment cost and that for each scenario, all performance measures outlined in Table 1 are presented separately through tables and figures.

We utilise *Scenario*-1 to analyze how returned material arrival rates affect the system performance measures. Depending on whether the returned material arrivals are sufficient to meet the demand for a given quality finished good, the optimal conversion decision changes. In order to analyze these decisions under different cases depending on the sufficiency of the arrival rates with respect to the demand rates, we create 18 distinct problem instances

Table 2. Scenarios considered in the numerical analysis.

Scenarios	Parameter Sets
Scenario-1	$\mu_{1,1} = \mu_{1,2} = \mu_{2,1} = \mu_{2,2} = 4$, $\lambda_1 = 1.6$, $\lambda_2 = 1.5$, $\varphi_{1,2} = \varphi_{2,1} = 0.2$, $s_1 = 7$, $s_2 = 4$, $p_1 = 3$, $p_2 = 2$, $c_{1,1} = c_{1,2} = c_{2,1} = c_{2,2} = 1$, $h_1 = h_2 = 0.2$, $k_1 = k_2 = 0.25$, $\delta_2 \in \{0.5, 0.9, 1.3, 1.7, 2.1, 2.5\}$, $\delta_1 \in \{1.2, 1.6, 2\}$
Scenario-2	$\mu_{1,1} = \mu_{1,2} = \mu_{2,1} = \mu_{2,2} = 4$, $\lambda_1 = 1.6$, $\lambda_2 = 1.5$, $\varphi_{1,2} = \varphi_{2,1} = 0.2$, $\delta_1 = 1.6$, $k_1 = k_2 = 0.25$, $s_1 = 7$, $p_1 = 3$, $p_2 = 2$, $c_{1,1} = c_{1,2} = c_{2,1} = c_{2,2} = 1$, $h_1 = h_2 = 0.2$, $s_2 \in \{4, 5, 6, 7\}$, $\delta_2 \in \{1, 1.5, 2\}$
Scenario-3	$\mu_{1,1} = \mu_{1,2} = \mu_{2,1} = \mu_{2,2} = 4$, $\lambda_1 = 1.6$, $\lambda_2 = 1.5$, $\varphi_{1,2} = \varphi_{2,1} = 0.2$, $\delta_2 = 1.5$, $k_1 = k_2 = 0.25$, $s_1 = 7$, $s_2 = 5$, $p_1 = 3$, $p_2 = 2$, $c_{1,1} = c_{2,2} = 1$, $h_1 = h_2 = 0.2$, $\delta_1 \in \{1.2, 1.6, 2\}$, $c_{1,2} \in \{1, 0.9, 0.8, 0.7, 0.6\}$, $c_{2,1} \in \{1, 1.1, 1.2, 1.3, 1.4\}$,
Scenario-4	$\mu_{1,1} = \mu_{1,2} = \mu_{2,1} = \mu_{2,2} = 4$, $\lambda_1 = 1.6$, $\lambda_2 = 1.5$, $\varphi_{1,2} = \varphi_{2,1} = 0.2$, $s_1 = 7$, $s_2 = 4$, $p_1 = 3$, $p_2 = 2$, $c_{1,1} = c_{1,2} = c_{2,1} = c_{2,2} = 1$, $\delta_2 = 1.5$, $h_1 = h_2 = \{0.7, 0.6, 0.5, 0.4, 0.3\}$, $k_1 = k_2 \in \{0.7, 0.8, 0.9, 1.0, 1.1\}$ $\delta_1 \in \{1.2, 1.6, 2\}$

by varying the second-quality returned material arrival rate (δ_2) from 0.5 to 2.5 in increments of 0.4 and the first-quality returned material arrival rate (δ_1) from 1.2 to 2 in increments of 0.4.

Scenario-2 is designed to analyze the effects of finished goods sales prices on the system's performance measures. The difference between the sales price, the returned material purchasing cost, and the production cost yields the per-unit profit for each conversion option. The most profitable production option may not be followed in the optimal policy due to the effects of the arrival and demand rates, as analyzed in Scenario 1. In *Scenario*-2, we generate 12 different problem instances by varying the second-quality returned material arrival rate (δ_2) from 1 to 2 in increments of 0.5, and the sales price of the second-quality finished good (s_2) from 4 to 7 in increments of 1.

In order to analyze the effect of incorporating environmental considerations in the optimal conversion decisions, we vary the conversion costs in *Scenario*-3. As mentioned earlier, producing a higher-quality finished good using a lower-quality returned material might have a more negative environmental impact than producing a lower-quality finished good from a higher-quality returned material. This might be primarily due to the differences in the amount of material that can be effectively recovered in the finished product. We assume that these environmental effects are reflected in the production costs. In *Scenario*-3, we generate 15 different problem instances by ranging both costs of producing first-quality finished goods using second-quality returned materials and second-quality finished goods using first-quality returned materials in increments of 0.1 and the first-quality returned material arrival rate (δ_1) from 1.2 to 2 in increments of 0.4.

Lastly, *Scenario*-4 is designed to examine the impact of holding costs on the system's performance measures. The optimal conversion decision can be influenced by holding costs, as they, along with the sufficiency of returned material arrivals, determine the ability to meet the demand for a given quality of finished goods. To conduct such an analysis, we generate 15 distinct problem instances by varying the arrival rate of first-quality returned materials (δ_1) from 1.2 to 2 in increments of 0.4, along with adjusting the holding costs for raw materials ($h_1 = h_2$) from 0.7 to 0.3 and for finished goods ($k_1 = k_2$) from 0.7 to 1.1, both in increments of 0.1.

5.2. *Effect of returned material arrival rates on the system performance measures*

Table 3 demonstrates that as the second-quality returned material arrival rate, δ_2, gets larger, the percentage of the first-quality finished goods demand satisfied by converting the second-quality returned material $\gamma_{2,1}$ increases and the percentage of the first-quality finished goods demand satisfied by converting the first-quality returned material $\gamma_{1,1}$ decreases.

Given the cost and revenue structures introduced in *Scenario*-1 are considered, one can say that the contribution of the first-quality finished goods to the profit when they are produced from the second-quality returned materials is higher than the others. Furthermore, with a high second-quality returned material arrival rate, the refurbisher has more opportunities to acquire lower-cost returned materials. Namely, the second-quality returned materials yield an opportunity to enhance the profit by converting them into first-quality finished goods (see Table 5). This will make the refurbisher less inclined to convert returned materials into second-quality finished goods and lead to a significant loss in sales for this product type. As a result of this purchasing and conversion strategy, the average inventory levels of the second-quality returned materials ($E[R_2]$) and the first-quality finished goods ($E[F_1]$) increase. In contrast, the lost sales rates for second-quality finished goods are notably higher than those for first-quality finished goods. All these observations can be confirmed by checking the relevant parts in Figure 2 and Table 4.

In Table 3, there is an upward trend in the percentage of first-quality finished goods demand satisfied by converting the first-quality returned materials $\gamma_{1,1}$ as the first-quality returned material rate increases. On the other hand, a slightly decreasing trend is noticeable in the percentage of first-quality finished goods demand satisfied by converting the second-quality returned materials $\gamma_{2,1}$. As the arrival rate of first-quality returned materials rises, the refurbisher will capitalise on the increasing

Table 3. Performance measures – I for *Scenario*-1.

	$\delta_1/\lambda_1 < 1$				$\delta_1/\lambda_1 = 1$				$\delta_1/\lambda_1 > 1$			
δ_2	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$
0.5	67.7%	32.3%	3.9%	87.1%	70.1%	29.9%	0%	100%	70.8%	29.2%	0%	100%
0.9	40.9%	59.1%	0%	100%	43.5%	56.5%	1.2%	98.8%	43.8%	56.9%	0%	100%
1.3	32.5%	67.5%	0%	100%	33.2%	66.8%	0%	100%	34.4%	65.6%	0%	100%
1.7	18.1%	81.9%	0%	100%	26.8%	73.2%	0%	100%	27.3%	72.7%	0%	100%
2.1	10.1%	89.9%	0%	100%	10.6%	89.4%	0%	100%	11.6%	88.4%	0%	100%
2.5	4.2%	95.8%	0.1%	99.9%	2.9%	97.1%	0%	100%	4.8%	95.2%	0%	100%

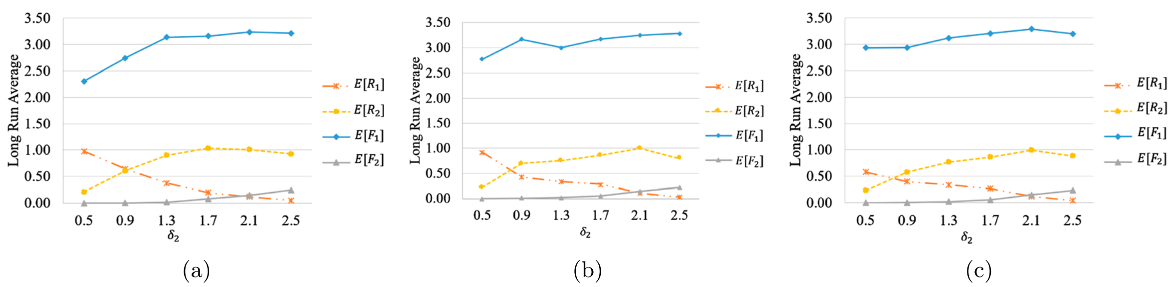


Figure 2. Average inventory levels associated with the returned materials and the finished goods when (a) $\delta_1/\lambda_1 < 1$, (b) $\delta_1/\lambda_1 = 1$, and (c) $\delta_1/\lambda_1 > 1$.

opportunity for purchasing returned materials. She opts to use first-quality returned materials to produce first-quality finished goods rather than utilising her capacity and available returned materials to produce second-quality finished goods. This conversion strategy is driven by the higher marginal profitability of first-quality finished goods compared to the other. The long-run production rates (Π_m's) given in Table 4 also confirm all these observations. As a result of this conversion strategy, the lost sales rate for second-quality finished goods also starts to increase (see Table 4).

5.3. *Effect of sales prices on the system performance measures*

In case of a sufficient number of second-quality returned material arrivals, the refurbisher opts to produce more first- and second-quality finished goods by converting second-quality returned materials because converting second-quality returned materials to fulfil customer demands yields a higher marginal profit than alternative conversion options. Due to this reason, as the second-quality returned material arrival rate increases, the refurbisher will tend to purchase the second-quality returned

Table 4. Performance measures – II for *Scenario*-1.

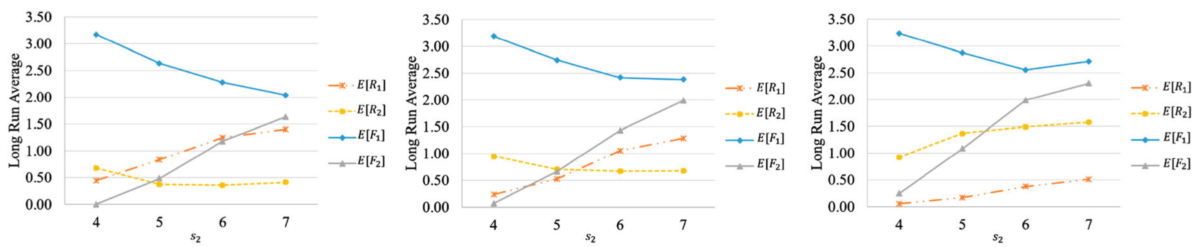
	$\delta_1/\lambda_1 < 1$				$\delta_1/\lambda_1 = 1$				$\delta_1/\lambda_1 > 1$			
δ_2	z	θ_0	θ_1	θ_2	z	θ_0	θ_1	θ_2	z	θ_0	θ_1	θ_2
0.5	4.32	18.6%	0%	81.4%	4.59	12.0%	0%	88.0%	4.74	9.9%	0%	90.1%
0.9	5.10	11.5%	0%	88.1%	5.21	8.1%	0%	91.1%	5.29	8.3%	0%	91.2%
1.3	5.32	8.9%	0%	89.6%	5.41	8.5%	0%	89.3%	5.46	7.0%	0%	90.8%
1.7	5.67	7.7%	0%	84.3%	5.57	7.1%	0%	87.2%	5.61	6.4%	0%	88.0%
2.1	5.90	6.6%	0%	78.7%	5.92	6.2%	0%	79.3%	5.93	5.6%	0%	79.7%
2.5	6.07	6.1%	0%	69.2%	6.19	5.1%	0%	73.0%	6.09	5.9%	0%	71.0%

Table 5. Performance measures – III for *Scenario*-1.

	$\delta_1/\lambda_1 < 1$					$\delta_1/\lambda_1 = 1$					$\delta_1/\lambda_1 > 1$				
δ_2	Π_0	Π_1	Π_2	Π_3	Π_4	Π_0	Π_1	Π_2	Π_3	Π_4	Π_0	Π_1	Π_2	Π_3	Π_4
0.5	61.3%	26.2%	0%	12.5%	0%	58.2%	29.3%	0%	12.5%	0%	57.2%	30.3%	0%	12.5%	0%
0.9	57.9%	17.2%	0%	24.8%	0.1%	56.1%	19.0%	0%	24.6%	0.3%	56.3%	18.8%	0%	24.7%	0.2%
1.3	56.3%	14.0%	0%	29.1%	0.6%	55.9%	14.4%	0%	28.9%	0.8%	55.1%	15.1%	0%	28.9%	0.8%
1.7	53.8%	7.8%	0%	35.4%	3.0%	54.1%	11.7%	0%	32.0%	2.1%	53.8%	12.0%	0%	32.0%	2.1%
2.1	51.2%	4.4%	0%	38.9%	5.5%	51.1%	4.6%	0%	38.9%	5.4%	50.8%	5.1%	0%	38.6%	5.5%
2.5	48.0%	1.8%	0%	40.9%	9.3%	48.3%	1.3%	0%	42.2%	8.2%	48.4%	2.1%	0%	40.9%	8.6%

Table 6. Performance measures – I for *Scenario*-2.

s_2	$\delta_2/\lambda_2<1$				$\delta_2/\lambda_2=1$				$\delta_2/\lambda_2>1$			
	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$
4	43.5%	56.5%	0%	100%	22.8%	77.2%	0.01%	99.9%	4.9%	95.1%	0.01%	99.9%
5	56.7%	43.3%	47.1%	52.9%	38.1%	61.9%	28.3%	71.7%	10.9%	89.1%	5.2%	94.8%
6	60.1%	39.9%	57.4%	42.6%	45.3%	54.7%	38.6%	61.4%	16.3%	83.7%	11.2%	88.8%
7	59.4%	40.6%	60.9%	39.1%	43.1%	56.9%	45.9%	54.1%	15.7%	84.3%	15.6%	84.4%



(a) (b) (c)

Figure 3. Average inventory levels associated with returned materials and finished goods when (a) $\delta_2/\lambda_2<1$, (b) $\delta_2/\lambda_2=1$, and (c) $\delta_2/\lambda_2>1$.

material as much as she can, increasing the average inventory level of second-quality returned material $E[R_2]$. Correspondingly, there will be a decrease in the average inventory level of first-quality returned material $E[R_1]$.

Note that both Table 6 and Figure 3 corroborate all these observations. If one focuses on a specific row in Table 6, and examines the values of δ_2/λ_2 from left to right, it becomes clear that an increase in the arrival rate of second-quality returned materials δ_2 leads to an increase in the percentage of the first-quality finished goods demand satisfied by converting the second-quality returned material $\gamma_{2,1}$ and the percentage of the second-quality finished goods demand satisfied by converting the second-quality returned material $\gamma_{2,2}$. Meanwhile, this leads to a reduction in the percentage of the first-quality finished goods demand satisfied by converting the first-quality returned material $\gamma_{1,1}$ and the percentage of the second-quality finished goods demand satisfied by converting the first-quality returned material $\gamma_{1,2}$.

As the levels of both sales price of second-quality finished goods s_2 and second-quality returned material arrival rate δ_2 increase, the marginal profit of second-quality finished goods demonstrates a corresponding increase, thereby amplifying its significance in the refurbisher's revenue. So, the refurbisher will be inclined to increase the production of second-quality finished goods, resulting in a notable decrease in the lost sales rates for second-quality finished goods θ_2 (see, Tables 7 and 8). Furthermore, due to the limited production capacity, this leads to a slight increase in the lost sales rates for first-quality finished goods θ_1. The results presented in Figure 3 confirm these observations. In the corresponding figure, one can see that with an increase in the sales

Table 7. Performance measures – II for *Scenario*-2.

s_2	$\delta_2/\lambda_2<1$				$\delta_2/\lambda_2=1$				$\delta_2/\lambda_2>1$			
	z	θ_0	θ_1	θ_2	z	θ_0	θ_1	θ_2	z	θ_0	θ_1	θ_2
4	5.21	8.1%	0.0%	91.1%	5.64	6.9%	0.0%	86.0%	6.08	5.8%	0.1%	69.4%
5	5.38	8.9%	0.4%	52.8%	5.96	6.8%	0.7%	47.4%	6.78	4.9%	1.0%	32.5%
6	6.17	10.1%	2.6%	26.1%	6.91	7.0%	2.9%	20.0%	7.89	4.6%	3.5%	12.7%
7	7.22	11.0%	5.6%	15.3%	8.11	6.6%	4.6%	1.5%	9.20	3.9%	3.7%	8.7%

Table 8. Performance measures – III for *Scenario*-2.

s_2	$\delta_2/\lambda_2<1$					$\delta_2/\lambda_2=1$					$\delta_2/\lambda_2>1$				
	Π_0	Π_1	Π_2	Π_3	Π_4	Π_0	Π_1	Π_2	Π_3	Π_4	Π_0	Π_1	Π_2	Π_3	Π_4
4	56.1%	19.0%	0%	24.6%	0.3%	53.7%	10.0%	0%	33.7%	2.7%	47.8%	2.1%	0%	40.8%	9.3%
5	45.4%	22.8%	6.8%	17.4%	7.6%	42.2%	15.5%	4.9%	25.1%	12.3%	36.3%	4.4%	1.2%	35.7%	22.4%
6	39.0%	22.2%	13.8%	14.7%	10.3%	34.8%	17.0%	10.7%	20.5%	16.9%	31.0%	6.2%	3.5%	31.6%	27.8%
7	37.4%	20.5%	17.1%	14.0%	11.0%	32.6%	15.7%	14.3%	20.7%	16.8%	29.3%	5.9%	5.1%	31.7%	27.9%

price of the second-quality finished goods, the average inventory level of first-quality finished goods $E[F_1]$ slightly decreases. In contrast, the average inventory level of second-quality finished goods $E[F_2]$ sharply increases.

5.4. Effect of production costs on the system performance measures

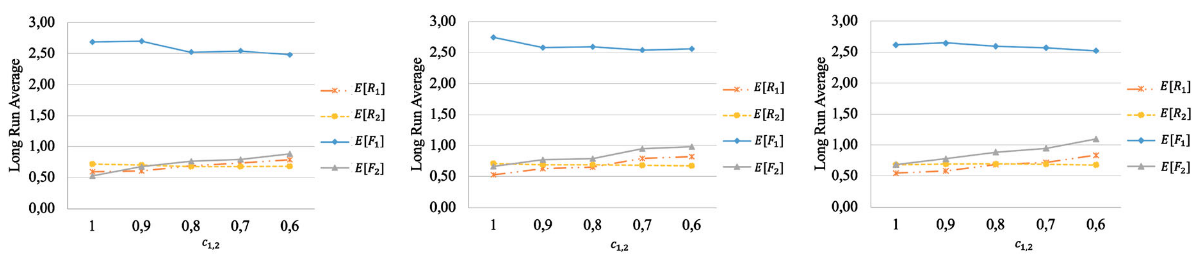
In Figure 4, it is evident that a slight decrease occurs in the average inventory levels of first-quality finished goods with a rise in the cost associated with the conversion from second-quality returned materials to first-quality finished goods and a decrease in the cost associated with the conversion from first-quality returned materials to second-quality finished goods. The elevated cost acts as a deterrent for the refurbisher, influencing a reduction in the production of first-quality finished goods using second-quality returned materials. As a result, the lost sales rate of first-quality finished goods increases (see Table 10), and the rate of conversion from second-quality returned materials to first-quality finished goods decreases (see Table 9).

On the other hand, Figure 4 illustrates a slight increase in the average inventory levels of first-quality finished goods as the cost of converting first-quality returned materials to second-quality finished goods decreases and the cost of converting second-quality returned materials to first-quality finished goods increases. In contrast to the case we discussed above, the diminished cost motivates the refurbisher to engage in the production of second-quality finished goods using first-quality returned materials. Consequently, an observed decrease in the lost sales rate of second-quality finished goods is accompanied by an increase in the rate of conversion from first-quality returned materials to second-quality finished goods (see Tables 9 and 10).

Last but not least, for both cases, a noticeable reduction in the level of average profit –on average, it is around 5%-6%– can be observed. The principal reason behind this fact is a reduction in the marginal profit of the most profitable conversion option due to an increase in the associated conversion cost (Table 11).

5.5. Effect of holding costs on the system performance measures

In this section, the effects of holding costs related to both returned materials and finished products on the performance metrics of the system are examined through numerical analyses. Table 12 indicates that as the holding



(a) (b) (c)

Figure 4. Average inventory levels associated with returned materials and finished goods when (a) $\delta_1/\lambda_1 < 1$, (b) $\delta_1/\lambda_1 = 1$, and (c) $\delta_1/\lambda_1 > 1$.

Table 9. Performance measures – I for *Scenario*-3.

	$\delta_1/\lambda_1 < 1$				$\delta_1/\lambda_1 = 1$				$\delta_1/\lambda_1 > 1$			
$c_{1,2}$&$c_{2,1}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$
1.0 & 1.0	36.3%	63.7%	21.8%	78.2%	38.1%	61.9%	28.3%	71.7%	38.6%	61.4%	32.4%	67.6%
0.9 & 1.1	36.2%	63.8%	30.2%	69.8%	39.1%	60.9%	33.6%	66.4%	39.8%	60.2%	34.6%	65.4%
0.8 & 1.2	37.2%	62.8%	33.0%	67.0%	39.4%	60.5%	33.9%	66.1%	44.2%	55.8%	31.7%	68.3%
0.7 & 1.3	37.7%	62.3%	33.6%	66.4%	43.8%	56.1%	31.4%	68.7%	44.9%	55.1%	31.8%	68.2%
0.6 & 1.4	40.0%	60.0%	31.0%	69.0%	44.1%	55.9%	32.1%	67.9%	43.5%	56.5%	38.3%	61.7%

Table 10. Performance measures – II for *Scenario*-3.

	$\delta_1/\lambda_1 < 1$				$\delta_1/\lambda_1 = 1$				$\delta_1/\lambda_1 > 1$			
$c_{1,2}$&$c_{2,1}$	z	θ_0	θ_1	θ_2	z	θ_0	θ_1	θ_2	z	θ_0	θ_1	θ_2
1.0 & 1.0	5.89	8.2%	0.4%	51.8%	5.96	6.8%	0.7%	47.4%	6.02	6.6%	0.8%	43.3%
0.9 & 1.1	5.80	8.0%	0.6%	46.6%	5.89	7.3%	1.0%	39.3%	5.94	6.3%	0.9%	39.3%
0.8 & 1.2	5.72	8.9%	1.0%	39.8%	5.82	7.2%	1.0%	38.7%	5.88	6.3%	1.2%	33.9%
0.7 & 1.3	5.65	8.6%	1.0%	38.9%	5.75	7.0%	1.3%	33.5%	5.82	6.1%	1.3%	32.9%
0.6 & 1.4	5.58	8.6%	1.1%	37.6%	5.70	6.8%	1.3%	32.7%	5.77	5.9%	1.6%	28.6%

Table 11. Performance measures – III for *Scenario*-3.

	δ_1/λ_1 <1					δ_1/λ_1 =1					δ_1/λ_1 >1				
$c_{1,2}\&c_{2,1}$	Π_0	Π_1	Π_2	Π_3	Π_4	Π_0	Π_1	Π_2	Π_3	Π_4	Π_0	Π_1	Π_2	Π_3	Π_4
1.0 & 1.0	44.6%	14.7%	3.3%	25.7%	11.7%	42.2%	15.5%	4.9%	25.1%	12.3%	40.9%	15.5%	6.1%	24.7%	12.7%
0.9 & 1.1	42.9%	14.5%	5.2%	25.6%	11.9%	40.3%	15.5%	6.8%	24.1%	13.3%	39.5%	16.0%	7.1%	24.1%	13.4%
0.8 & 1.2	41.6%	14.5%	6.4%	24.5%	12.9%	40.0%	15.6%	6.9%	24.0%	13.5%	37.9%	17.5%	7.1%	22.1%	15.4%
0.7 & 1.3	41.2%	14.7%	6.6%	24.3%	13.1%	38.3%	17.2%	7.0%	22.0%	15.4%	37.5%	17.7%	7.3%	21.8%	15.7%
0.6 & 1.4	40.7%	15.5%	6.4%	23.4%	14.0%	38.0%	17.3%	7.3%	21.9%	15.5%	36.2%	17.0%	9.5%	22.1%	15.2%

Table 12. Performance measures – II for *Scenario*-4.

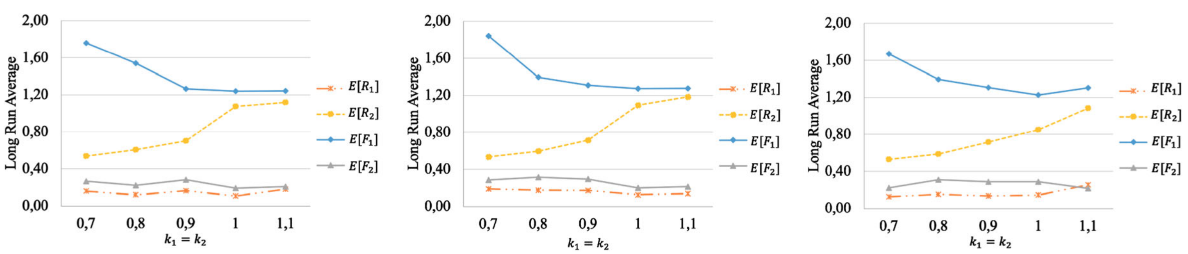
	δ_1/λ_1 <1				δ_1/λ_1 =1				δ_1/λ_1 >1			
$h_1 = h_2\&k_1 = k_2$	z	θ_0	θ_1	θ_2	z	θ_0	θ_1	θ_2	z	θ_0	θ_1	θ_2
0.7 & 0.7	4.33	17.3%	0.9%	59.6%	4.38	15.0%	0.9%	60.1%	4.41	17.7%	0.7%	62.7%
0.6 & 0.8	4.21	20.0%	0.9%	60.3%	4.26	19.6%	2.0%	51.1%	4.31	19.4%	1.9%	51.9%
0.5 & 0.9	4.13	22.0%	2.2%	49.8%	4.18	19.9%	2.2%	50.6%	4.22	19.8%	2.1%	51.4%
0.4 & 1.0	4.08	21.0%	1.2%	59.9%	4.12	19.3%	1.3%	60.9%	4.16	19.9%	2.3%	51.2%
0.3 & 1.1	4.06	20.6%	1.4%	58.7%	4.10	19.0%	1.4%	59.8%	4.13	17.8%	1.4%	60.6%

costs of finished products increase, the system's profitability, z decreases. The main reason for this outcome is the tendency to maintain lower levels of finished product inventory due to the rising holding costs, which in turn results in an upward trend in the average lost sales. The average inventory levels of first and second quality finished products presented in Figure 5, $E[F_1]$ and $E[F_2]$, also support this observation. This situation also causes the refurbisher to exhibit a tendency to increase the average idle time over the long term, namely Π_0, as shown by Table 13.

On the other hand, Figure 5 indicates that when the holding cost of returned materials starts to decrease, there is a significant increase in the average inventory level, particularly for second-quality returned materials. This shows that the manufacturer prefers to keep inventory on the upstream side rather than the downstream side and attempts to meet the demand using this strategy. Such an inventory strategy leads to an upward trend in the average lost sales such as θ_0, θ_1 and θ_2, as reflected in the average lost sales figures presented in Table 12. Additionally, the conversion ratios (γ's) provided in Table 14 demonstrate that as the holding cost for returned materials decreases, the manufacturer increasingly uses second-quality products to meet the demand for first-quality products.

5.6. Insights into the structure of optimal purchasing and conversion policies

In all numerical instances examined in this study, it has been observed that the optimal purchasing and conversion decisions exhibit characteristics consistent with state-dependent threshold policies. Building upon



(a) (b) (c)

Figure 5. Average inventory levels associated with returned materials and finished goods when (a) δ_1/λ_1 <1, (b) δ_1/λ_1 =1, and (c) δ_1/λ_1 >1.

Table 13. Performance measures – III for *Scenario*-4.

	δ_1/λ_1 <1					δ_1/λ_1 =1					δ_1/λ_1 >1				
$h_1 = h_2\&k_1 = k_2$	Π_0	Π_1	Π_2	Π_3	Π_4	Π_0	Π_1	Π_2	Π_3	Π_4	Π_0	Π_1	Π_2	Π_3	Π_4
0.7 & 0.7	54.0%	10.3%	0.0%	26.9%	8.7%	52.5%	12.0%	0.1%	26.1%	9.3%	55.3%	9.0%	0.0%	28.3%	7.4%
0.6 & 0.8	56.4%	8.0%	0.0%	28.2%	7.4%	53.6%	10.5%	0.2%	24.8%	10.9%	53.7%	9.4%	1.2%	26.0%	9.8%
0.5 & 0.9	55.2%	8.1%	0.9%	26.0%	9.8%	53.8%	9.5%	1.1%	25.5%	10.1%	53.9%	10.2%	0.3%	24.9%	10.7%
0.4 & 1.0	57.1%	7.1%	0.2%	28.6%	7.0%	56.1%	8.0%	0.4%	28.4%	7.2%	54.0%	8.9%	1.2%	26.1%	9.8%
0.3 & 1.1	56.5%	6.9%	0.2%	28.7%	7.7%	55.6%	8.1%	0.1%	28.3%	8.0%	54.9%	8.8%	0.2%	28.1%	8.0%

Table 14. Performance measures – I for *Scenario*-4.

$h_1 = h_2 \& k_1 = k_2$	$\delta_1/\lambda_1 < 1$				$\delta_1/\lambda_1 = 1$				$\delta_1/\lambda_1 > 1$			
	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{1,2}$	$\gamma_{2,2}$
0.7 & 0.7	27.7%	72.3%	0.4%	99.6%	31.4%	68.6%	0.6%	99.4%	24.2%	75.8%	0.2%	99.8%
0.6 & 0.8	22.0%	78.0%	0.1%	99.9%	29.7%	70.3%	2.2%	97.8%	26.6%	73.4%	10.5%	89.5%
0.5 & 0.9	23.7%	76.3%	8.5%	91.5%	27.0%	73.0%	10.2%	89.8%	29.0%	71.0%	2.3%	97.7%
0.4 & 1.0	19.8%	80.2%	3.3%	96.7%	22.0%	78.0%	5.3%	94.7%	25.4%	74.6%	11.2%	88.8%
0.3 & 1.1	19.3%	80.7%	1.9%	98.1%	22.2%	77.8%	0.8%	99.2%	23.9%	76.1%	2.4%	97.6%

these observations, one may conjecture that these optimal policies adhere to a structured framework resembling the following:

(i) For any given state $\hat{\mathbf{s}} = (\hat{r_1}, \hat{r_2}, \hat{f_1}, \hat{f_2})$, there exists a threshold level $Y^*_{b_1}(\hat{r_2}, \hat{f_1}, \hat{f_2})$ such that

$$Y^*_{b_1}(\hat{r_2}, \hat{f_1}, \hat{f_2}) = \min_{r_1} \left\{ \left(r_1, \hat{r_2}, \hat{f_1}, \hat{f_2}\right) \mid u^*_{b_1}\left(r_1, \hat{r_2}, \hat{f_1}, \hat{f_2}\right) = 0 \right\}, \quad (8)$$

where $u^*_{b_1}(.)$ denotes the optimal purchasing decision for first-quality returned materials in any given state. Correspondingly, for any given state $\hat{\mathbf{s}} = (\hat{r_1}, \hat{r_2}, \hat{f_1}, \hat{f_2})$, it is optimal to purchase an arriving unit of first-quality returned material if $\hat{r_1} < Y^*_{b_1}(\hat{r_2}, \hat{f_1}, \hat{f_2})$; otherwise, it is optimal not to purchase.

(ii) For any given state $\hat{\mathbf{s}} = (\hat{r_1}, \hat{r_2}, \hat{f_1}, \hat{f_2})$, there exists a threshold level $Y^*_{b_2}(\hat{r_1}, \hat{f_1}, \hat{f_2})$ such that

$$Y^*_{b_2}(\hat{r_1}, \hat{f_1}, \hat{f_2}) = \min_{r_2} \left\{ \left(\hat{r_1}, r_2, \hat{f_1}, \hat{f_2}\right) \mid u^*_{b_2}\left(r_1, \hat{r_2}, \hat{f_1}, \hat{f_2}\right) = 0 \right\}, \quad (9)$$

where $u^*_{b_2}(.)$ represents the optimal purchasing decision for second-quality returned materials in any given state. Correspondingly, for any given state $\hat{\mathbf{s}} = (\hat{r_1}, \hat{r_2}, \hat{f_1}, \hat{f_2})$, it is optimal to purchase an arriving unit of second-quality returned material if $\hat{r_2} < Y^*_{b_2}(\hat{r_1}, \hat{f_1}, \hat{f_2})$; otherwise, it is optimal not to purchase.

(iii) For any given state $\hat{\mathbf{s}} = (\hat{r_1}, \hat{r_2}, \hat{f_1}, \hat{f_2})$ where $\hat{r_1} + \hat{r_2} = b$, there exists a threshold level $Y^*_{m_1}(\hat{r_2}, \hat{f_1}, \hat{f_2})$ such that

$$Y^*_{m_1}(\hat{r_2}, \hat{f_1}, \hat{f_2}) = \min_{r_1} \left\{ \left(r_1, b - r_1, \hat{f_1}, \hat{f_2}\right) \mid u^*_{m_1}\left(r_1, b - r_1, \hat{f_1}, \hat{f_2}\right) = 1 \right\}, \quad (10)$$

where $u_{m_1}(.)$ represents the optimal choice between producing first-quality finished goods utilising either first-quality returned materials or second-quality returned materials. That is, for any given state $\hat{\mathbf{s}} = (\hat{r_1}, \hat{r_2}, \hat{f_1}, \hat{f_2})$ where $\hat{r_1} + \hat{r_2} = b$, it is optimal to produce the first-quality finished goods using the first-quality returned materials if $\hat{r_1} > Y^*_{m_1}(\hat{r_2}, \hat{f_1}, \hat{f_2})$; otherwise, it is optimal to use the second-quality returned materials to produce the first-quality finished goods.

(iv) For any given state $\hat{\mathbf{s}} = (\hat{r_1}, \hat{r_2}, \hat{f_1}, \hat{f_2})$ where $\hat{r_1} + \hat{r_2} = a$, there exists a threshold level $Y^*_{m_2}(\hat{r_2}, \hat{f_1}, \hat{f_2})$ such that

$$Y^*_{m_2}(\hat{r_2}, \hat{f_1}, \hat{f_2}) = \min_{r_1} \left\{ \left(r_1, a - r_1, \hat{f_1}, \hat{f_2}\right) \mid u^*_{m_2}\left(r_1, a - r_1, \hat{f_1}, \hat{f_2}\right) = 2 \right\}, \quad (11)$$

where $u_{m_2}(.)$ represents the optimal choice between producing second-quality finished goods utilising either first-quality returned materials or second-quality returned materials. That is, for any given state $\hat{\mathbf{s}} = (\hat{r_1}, \hat{r_2}, \hat{f_1}, \hat{f_2})$ where $\hat{r_1} + \hat{r_2} = a$, it is optimal to produce the second-quality finished goods using the first-quality returned materials if $\hat{r_1} > Y^*_{m_2}(\hat{r_2}, \hat{f_1}, \hat{f_2})$; otherwise, it is optimal to use the second-quality returned materials to produce the second-quality finished goods.

While the numerical experiments confirm the above threshold structure, a complete proof requires checking a large number of inequalities that must be simultaneously satisfied. For instance, we can observe that if a strong general property such as multi-modularity holds, the threshold structure is guaranteed. However, establishing multi-modularity in a four dimensional state space requires checking that a large number of inequalities propagate. In addition, some of these inequalities are far from trivial. This complexity renders such a mathematical analysis intractable. Should it become feasible to analytically characterise the structures of the optimal purchasing, production, and remanufacturing decisions, a suite of policies amenable to tuning via sequential decision-making methodologies such as reinforcement learning or some tailored-based heuristics could also be developed. Therefore, the comprehensive analysis of these aspects is deferred to future research endeavours.

6. Conclusions

In this study, we derive the purchasing, production, and remanufacturing decisions for a refurbisher that refurbishes first- and second-quality returned materials to produce and sell first- and second-quality products in a make-to-stock fashion. In the model we developed, there are five refurbishing decisions (refurbishing first-quality returned material to first-quality finished good, first to second, second to first, second to second, and do not produce) and two purchasing decisions (for the first- and second-quality returned materials). When the production and arrival times are exponential, the optimal policy is determined by solving the Markovian Decision Process formulation of the optimal control problem using a linear programming approach.

The difference between the sales price, returned material purchasing cost, and production cost yields the per-unit profit for each refurbishing option. When the returned material rate for the desired returned material quality choice is sufficient to satisfy the demand for the desired finished good quality choice, the most profitable production option can be followed with a threshold-type purchasing policy that guarantees returned material availability according to the given returned material holding cost and a threshold-type production policy that satisfies the demand according to the given finished good holding cost. However, when the arrival rates and demand rates are different, it is not possible to follow the myopic per-unit profit-driven policy. Through extensive numerical experiments, we show that the optimal refurbishment policy is driven by the per-unit profits for different refurbishment options and also by the returned material, demand rates, and production times for different conversion options. For all numerical instances examined within this study, it is observed that the optimal purchasing and conversion decisions adhere to state-dependent threshold policies.

It might be an interesting research direction to prove the optimal policy structure for some special cases which allow for a state-space dimension reduction of the model considered here.

Lastly, this study can be extended in several ways. In this study, we have numerically derived the optimal control policies for procuring returned materials and converting them into reusable finished goods by employing the Linear Programming approach commonly used in solving Markov Decision Processes. Due to the high dimensionality of the state space, analytically characterising the structures of the optimal control policies remains challenging and is an open question for further investigation. Another extension of this work could involve modelling inter-event times using phase-type distributions rather than using exponential distributions. This modification would enable the model to represent any distribution observed in practical case studies, thereby enhancing its flexibility and sophistication. Additionally, such a modelling approach would facilitate the implementation of data-driven policies and solution methods, including artificial intelligence and reinforcement learning techniques. However, incorporating phase-type distributions would increase the complexity of the problem by adding new dimensions to the state space to track the phases of the distributions. Moreover, the number of quality classes for returned materials can be extended beyond two, allowing for a more granular representation of varying quality levels. Introducing multiple quality classes would enable the model to better capture the complexities of real-world returned material streams. Additionally, the option of incorporating virgin raw materials into the production process could be considered, offering a more comprehensive model of material sourcing options. Furthermore, incorporating sales price decisions as an extension could allow for an integrated analysis of pricing, inventory, and production strategies, enabling a more comprehensive and better understanding, particularly for firms transitioning from price-takers to market influencers. However, including these additional factors–multiple quality classes, virgin raw materials, and sales price decisions–would significantly increase the model's complexity. Such extensions would require careful consideration of the additional computational challenges introduced by the expanded state space and decision variables.

In conclusion, we show that considering the per-unit profits of different refurbishing options based on the quality levels of returned materials and finished goods, along with the returned material, demand, and production rates, enables the formulation of optimal purchasing, production, and remanufacturing decisions that not only maximise profitability but also potentially contribute to environmental benefits by optimising the use of returned materials.

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Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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