UNIFICATION OF INTEGRABLE q-DIFFERENCE EQUATIONS

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Approval of the Graduate School of Natural and Applied Sciences.

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

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We have read the dissertation entitled **"Unification of integrable q-difference equations"** completed by **DUYGU SOYOĞLU** under supervision of **Assoc. Prof. Dr. Burcu Silindir Yantır** and co-supervision of **Assoc. Prof. Dr. Aslı Pekcan.** We certify that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

UNIFICATION OF INTEGRABLE q-DIFFERENCE EQUATIONS

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In this thesis our aim is to detect an equation which is a unification of integrable q-difference equations. This generalized equation, namely q-Hirota-Miwa equation, is in Hirota bilinear form. We search the existence of its integrability and find three-q-soliton solutions by Hirota's method. This generalized equation includes bilinear forms of several q-difference equations, such as q-analogues of Toda, KdV and sine-Gordon equations. Not only one of the most important point is to meet with suitable reductions for constructing bilinear forms from Hirota-Miwa equation, but also the key point is that Hirota bilinear forms must also recover their continuous bilinear forms. In this thesis, as a result of q-deformed Hirota bilinear forms reduced from q-Hirota-Miwa equation, we construct standard form of q-Toda, q-KdV and q-sine-Gordon equations as well as their three-q-soltions solutions.

Keywords: Integrability, q-exponential identity, q-soliton solutions, q-difference KdV equation, q-difference-q-difference Toda equation, q-difference sine-Gordon equation, Hirota direct method.

ÖZ

İNTEGRE EDİLEBİLEN q-FARK DENKLEMLERİNİN BİRLEŞTİRİLMESİ

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Bu tezdeki amacımız integre edilebilen q-fark denklemlerini birleştirici tek bir denklem elde etmektir. Bu genelleştirilmiş q-Hirota-Miwa adındaki denklem, Hirota bilineer formdadır. Bu denklemin integrallenebilirligini araştırdıktan sonra Hirota methodu ile üç-q-soliton çözümlerini bulduk. Bu denklem çeşitli q-fark denklemlerinin, Toda, KdV ve sine-Gordon gibi denklemlerin Hirota bilineer formlarını içermektedir. Ç alışmadaki en önemli nokta, bu bilineer formları oluşturmak için uygun kısıtların Hirota-Miwa denkleminden elde edilmesi ve bu lineer formların sürekli Hirota bilineer formlara indirgenmesidir. Bu tezde, q-Hirota-Miwa denkleminden elde edilen q-Hirota bilineer formlar sonucunda, q-Toda, q-KdV ve q-sine Gordon denklemlerinin standart formlarının yanı sıra üç-q-soliton çözümlerini de inşa ettik.

Anahtar Kelimeler: İntegre edilebilirlik, q-üstel özdeşlik, q-soliton çözümler, q-fark KdV denklemi, q-fark Toda denklemi, q-fark sine-Gordan denklemi, Hirota metodu.

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To my mother...

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Nomenclature

Symbol	Response
$h\mathbb{Z}$	$\{hn : n \in \mathbb{Z}, h > 0\}$
$q^{\mathbb{Z}}$	$\{q^n : n \in \mathbb{Z}, q > 1\}$
K_q	$q^{\mathbb{Z}} \cup \{0\}$
C^1	The set of continuously differentiable functions
C^n	The set of continuously differentiable functions of order n
D	Hirota D-operator
∂	Partial differential operator
D_q	q-Hirota D-operator

Symbol	Response
∂_q	q-derivative operator
σ	Forward jump operator
ρ	Backward jump operator
μ	Graininess function
E_q	q-forward jump operator
E_q^{-1}	q-backward jump operator
e_q^t	Jackson's q-exponential function
Δ	Central q-difference operator
Ω	q-difference operator
γ	q-sum operator

Chapter 1

Introduction

Up to present, by the virtue of the superposition principle a wide amount of studies have been made for linear differential equations. Critical progress for solving nonlinear equations has been widely approached especially for the last 40 years. Integrability of nonlinear partial differential equations has a critical role in physical applications since the integrable equations express real features in different areas of science, electronics, fluid mechanics, string theory, nonlinear optics, gravitation and field theory, etc. There does not exist a unique mathematical definition for integrability. Existence of infinite hierarchies of symmetries, conservation laws and regular behavior of solutions are some of the main definitions of this concept. Another integrability concept named C-integrability arises as existence of a transformation from nonlinear partial differential equations into linear equations [3]. Apart from the above mentioned integrability definitions, there is also a heuristic notion called S-integrability, the equations that are solved by the Inverse Scattering Method (IST). The method was found by Gardner, Greene, Kruskal, and Miura in 1967 [6] and used to solve initial value problems by assuming that the initial conditions tends to zero as $x \to \pm\infty$. This method is not a direct method but it is a nonlinear analogue of Fourier transform for nonlinear differential equations. In IST, the solutions of nonlinear partial differential equations are mapped into a potential function of another equation which determines its time evolution. Secondly the potential is determined by inverting the process and problem becomes to find the solution of linear ordinary differential equations with the time evolution of the scattering data. As it can be understood from its structure, method contains heavy machinery calculations but it is the first exact method to analyze the existence of soliton solutions and produces them in the form of exponential functions. The key point that we should emphasize on the word "soliton" is that it is a nonlinear wave with unchanging shape. Soliton phenomena was discovered in a shallow water channel while doing experiments to find the most efficient design for canal boats by Russel in 1834 [32]. In 1965, Zabusky and Kruskal [37] introduced the concept of solitons.

Definition. [5] A soliton is a solution of nonlinear differential or difference equation or a system, that

i. is a wave equation in permanent form.

ii. interacts with another one and after interaction they retain their identity.

iii. is localized i.e. (they are rapidly decaying functions) as $x \to \infty$ the wave vanishes.

Being inspired by Russel's research in [37] soliton solutions are studied by numerical experiments. The numerical results were brought a light for investigation the existence of many conservation laws which reveals another approach for integrability criteria. In [27], conserved densities of each order was found for KdV equation. While searching the conservation laws for KdV equation, Miura [28] introduced a transformation that reduces nonlinear partial differential equations to easy solvable ones. This transformation is a leading work for inverse scattering method and Lax pair [25]. Lax formalism is based on writing nonlinear equation as the compatibility condition of linear equations and finding the so-called Lax pair which allows us for a new aspect of integrability.

Noether [29], who was another scholar studying on the conservation laws, also discovered the connection between the conservation laws of a system and its symmetry properties. This relationship is a remarkable property for Hamiltonian structures [26] and recursion operator technique [31] which are also crucial indicators for integrability. Recursion operator constructs infinite number of symmetries with a mapping while Hamiltonian operator is a mapping between the symmetries and the co-symmetries. Also in [9], Gürses et al. developed a general procedure for finding the recursion operators for nonlinear integrable equations admitting Lax representation.

The necessity of existence of algebraic or analytic structures can also be taken as other definition for the integrability. For instance, Painlevé property, which states the singularities of the solutions in complex plane, is a sufficient condition for the integrability of partial differential equations [4].

Hirota developed his own technique to construct soliton solutions, called Hirota Direct Method in 1971 [14], by inspiring from exponential type of solutions. The most important differences between Hirota method and IST can be explained in two points. The first one is Hirota method is a direct method while IST is not. The second one is that Hirota method is algebraic while IST is analytic.

Another crucial turning point of Hirota's method is that it allows to check whether a nonlinear partial differential or difference equation is integrable or not [11, 12, 13]. Hirota's method has its own integrability condition in terms of its own derivative operator the so-called Hirota D-operator. In the literature the equations which can be written in terms of Hirota derivative operator and which have at least 3-soliton solutions are named as Hirota integrable equations. Moreover we want to emphasize that Hirota integrable equations are generally admitted to be integrable if they pass Painlevé property.

We mentioned about various different definitions of integrability so far. In the end, we want to gather them under a single roof.

Definition. The nonlinear differential or difference system or equation is said to be integrable if it satisfies one of the followings

i) the equation is linearized with a suitable variable transformation, e.g. a Miura transformation (i.e C-integrability),

ii) the nonlinear system can be solvable through inverse scattering transformation (i.e S-integrability),

iii) the system possesses the required number of independent integrals of motion (conserved quantities),

iv) the equation has bi-Hamiltonian structures,

v) the nonlinear equation has multi-soliton solutions,

vi) there exist Lax pair for the nonlinear equation,

vii) there exist infinite hierarchies of symmetries,

viii) there exist Recursion operator for the hierarchies of symmetries,

ix) the system passes the singularity confinement criterion (Painleve test).

Hirota Direct Method is the roof of this thesis. In the next chapter we present the method and emphasize the applicability of the method to discrete equations. For this purpose, we apply the method to a differential-difference type of an equation explicitly. The third chapter presents the findings of the pioneering article [33] which includes a vital type of equation, a q-difference equation analyzed by Hirota's method. In [33], Silindir stated that q-difference equations obtained by q-difference operators, are not isomorphic to lattice systems. Following this fact, in [33], it was shown that Hirota Direct Method is applicable to differentiable-q-difference Toda and q-difference-q-difference Toda equations to produce their q-soliton solutions. Based on the article [35], in the fourth chapter we present a unifying framework for q-discrete equations. We present a generalized q-difference equation namely q-Hirota-Miwa equation in Hirota bilinear form which arises as a roof for various q-discrete type of equations equipped with their q-soliton solutions. In continuation of this chapter, we construct not only appropriate q-deformed Hirota bilinear forms of sine-Gordan, KdV, Toda equations resulting from this generalized equation, but also their corresponding q-soliton solutions.

Significantly, these q-deformed Hirota bilinear forms provide q-analogues of corresponding equations as they fall into the associated continuous ones with limit procedure. In the final chapter, we explain the non-existence of multi-soliton solutions on q-differential equations by q-discretization via q-differential operators in the case of q-differential-q-difference Toda equation. In the final chapter, we conjecture the non-existence of other unifying approaches to derive multi-q-soliton solutions on arbitrary time scales with nonconstant step size by the use of classical Hirota perturbation.

Chapter 2

Hirota Direct Method

In this thesis, we focus on the soliton solutions of q-difference equations as well as the unification of such equations by the help of Hirota direct method. Before investigating q-difference equations, we mention about the behavior of the soliton solutions of Toda lattice and q-Toda lattice which was studied by Silindir in [33]. For this purpose, we briefly present Hirota direct method to construct soliton solutions.

Let

$$F[u] = F(x, t, u, u_x, u_t, u_{xx}, u_{tt}, \cdots) = 0, \tag{2.1}$$

be a nonlinear partial differential or difference equation. Due to the difficulty of finding solutions for nonlinear equations, it is useful to apply a transformation to simplify the nonlinear equation. This transformation is the first step of Hirota method. Before explaining the method, we explain Hirota's idea behind the transformation.

In [24], it was pointed out that the solutions of KdV equation are in the form of elliptic functions. Also, later on, Gardner et al. [6] showed that soliton type of solutions of KdV equation can only be given by the exponential functions. Hirota realized the connection between these two foundations and wrote the elliptic functions in terms of exponential functions which gives a rational function whose both denominator and numerator includes exponential functions. The transformation idea emerges when it is understood that it is necessary to convert the function by a rational dependent variable transformation and it brings out two coupled bilinear equations which are nonlinear equations of second degree.

In the light of this idea, the first step of Hirota's method is to rewrite the nonlinear equation $F[u]$ into a form by a transformation $u = T[f(x, t, \cdots)]$ where the new dependent variable comes out bilinearly. Here, it should be emphasized that all nonlinear equations cannot be transformed to single bilinear equations. For instance, Sine-Gordon (sG) [16], Modified Korteweg-de Vries (mKdV) [15], and nonlinear Schrödinger equations (nlS) [17] are some examples where transformations reduce to multiple bilinear forms [7].

When bilinear form of KdV equation was obtained, Hirota realized the distinction of Liebniz rule of differentiation of product of two functions. This variation implied the emergency to introduce Hirota D-operator which describes a new calculus with its special features.

Definition. [14] Assume V is a space of smooth functions f and g. Then Hirota D-operator $D : V \times V \rightarrow V$ is introduced as

$$[D_x^{n_1} D_y^{n_2} \ldots]\{f.g\} = [(\partial_x - \partial_{x^0})^{n_1} (\partial_y - \partial_{y^0})^{n_2} \ldots] f(x, y, \ldots) \times g(x^0, y^0, \ldots)|_{x^0=x, y^0=y, \ldots} \tag{2.2}$$

where $x, y, \ldots$ are independent variables and $i = 1, 2, \ldots, n_i \in Z^+$.

For instance, Hirota derivatives can be given as

$$D_y\{f.g\} = f_y g - f g_y, \tag{2.3}$$
$$D_y^2\{f.g\} = f_{yy} g - f_y g_y - f_y g_y + f g_{yy}, \tag{2.4}$$
$$D_y^3\{f.g\} = f_{yyy} g - 3 f_{yy} g_y + 3 f_y g_{yy} - f g_{yyy}. \tag{2.5}$$

Note that D-operator can also be described by the exponential identity especially for the study of difference type of equations.

Theorem 2.1 [21] Let $a(x)$, $b(x)$ be continuously differentiable functions and δ be a parameter. Then the following exponential identity holds

$$e^{\delta D_x}\{a(x).b(x)\} = e^{\delta\partial_y} a(x+y)b(x-y)\big|_{y=0} \tag{2.6}$$

$$= a(x+\delta)b(x-\delta). \tag{2.7}$$

Proof. Let us expand exponential function in a Taylor series for the left hand side of the equation (2.6).

$$\begin{aligned} e^{\delta D_x}\{a(x).b(x)\} &= \sum_{n=0}^{\infty} \frac{1}{n!}(\delta D_x)^n\{a(x).b(x)\} \\ &= \left(1 + \delta D_x + \frac{1}{2}\delta^2 D_x^2 + \frac{1}{3!}\delta^3 D_x^3 + \cdots\right)\{a(x).b(x)\} \\ &= a.b + \delta(a_x b - a b_x) + \frac{1}{2}\delta^2(a_{xx} - a_x b_x + b_{xx}) + \cdots \\ &= \left(a + \delta a_x + \frac{1}{2}\delta^2 a_{xx} + \cdots\right)\left(b - \delta b_x + \frac{\delta^2}{2} b_{xx} - \cdots\right), \end{aligned}$$

which gives Taylor series expansions of $a(x+\delta)b(x-\delta)$ around δ. □

The next step is to rewrite the bilinear form of $F[u]$ in terms of Hirota D-operator as

$$P(D)\{f.f\} = 0, \tag{2.8}$$

which is named as Hirota bilinear form. Here we must mention that, it is impossible to formulate how to write Hirota bilinear form of a given nonlinear partial differential or difference equation. Also, Hirota bilinear form of an equation is not always single, but it can be seen trilinear or multilinear forms [7] in some instances. It will be useful to touch upon another point regarding integrability. If an equation is completely integrable, then its Hirota bilinear form can be constructed. But the converse is not true i.e., some equations admit Hirota-bilinear forms although they are not integrable.

Proposition 2.2 [21] Let $P(D)$ be an arbitrary polynomial of D acting on two smooth functions $g(x, y, \ldots)$ and $f(x, y, \ldots)$ then we have

i. $P(D)\{g.f\} = P(-D)\{f.g\}$,

ii. $P(D)\{g.1\} = P(\partial)(g)$ and $P(D)\{1.g\} = P(-\partial)(g)$,

where ∂ is the usual partial differential operator.

Proof. For simplicity consider the case $P(D) = D_x^n$, $n \in Z^+$ since $P(D)$ is an arbitrary polynomial.

i. For the first part consider

$$
\begin{aligned}
P(D)\{g.f\} &= D_x^n\{g.f\} \\
&= \sum_{k=0} (-1)^k \binom{n}{k} g_{(n-k)x} f_{kx} \qquad (2.9) \\
&= g_{nx} f - n g_{(n-1)x} f_x + \ldots + (-1)^n g f_{nx}
\end{aligned}
$$

where g_{ix} and f_{ix} are the i^{th} order partial derivatives with respect to x. Rewriting the order in minus sign,

$$
\begin{aligned}
P(D)\{g.f\} =& g_{nx} f - n g_{(n-1)x} f_x + \cdots + (-1)^n g f_{nx} \\
=&(-1)^n [g f_{nx} - n g_x f_{(n-1)x} + \cdots + (-1)^{n-1} n g_{(n-1)x} f_x \\
&+ (-1)^n g_{nx} f] \\
=&P(-D)\{f.g\},
\end{aligned}
$$

which implies $P(D)\{g.f\} = P(-D)\{f.g\}$.

ii. For the second part of the proof, it is obvious that if we put 1 instead of f in (i), we get $P(D)\{g.1\} = P(\partial)(g)$ and from the first item we can conclude $P(D)\{1.g\} = P(-\partial)(g)$. □

The final step of Hirota's method is the ordinary perturbation technique, the so-called Hirota perturbation which produces soliton solutions. In this step we write an arbitrary function f in terms of a perturbation parameter ε in a way that

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4 + \cdots ,$$

where f_0 is constant and $\forall i = 1, 2, \cdots,\ f_i = f_i(x, t)$ are smooth functions. Consider the product $f.f$ and without loss of generality take $f_0 = 1$, then

$$f.f = 1.1 + \varepsilon(f_1.1 + 1.f_1) + \varepsilon^2(f_2.1 + f_1.f_1 + 1.f_2) + \varepsilon^3(f_3.1 + f_2.f_1 + f_1.f_2 + 1.f_3) + \cdots . \tag{2.10}$$

Substitution of (2.10) into (2.8), and the linearity property of $P(D)$ lead to

$$\begin{aligned} P(D)\{f.f\} =& P(D)\{1.1\} + \varepsilon P(D)\{1.f_1 + f_1.1\} + \varepsilon^2 P(D)\{f_2.1 + 1.f_2 + f_1.f_1\} \\ &+ \varepsilon^3 P(D)\{f_3.1 + 1.f_3 + f_1.f_2 + f_2.f_1\} + \cdots = 0. \end{aligned} \tag{2.11}$$

Theorem 2.3 [21] Let $P(D)\{f.f\} = 0$ where $P(D)$ is an arbitrary polynomial of D and $f(x, t, \cdots)$ is a smooth function. If the conditions

i. $P(D) = P(-D)$,

ii. $P(0) = 0$,

are satisfied, then $P(D)\{f.f\} = 0$ has at least two-soliton solutions.

By the virtue of the above theorem, we assume $P(0) = 0$ to guarantee the existence of 1-soliton solution . We equate the coefficients of ε^i to zero for each $i \geq 0$ to analyze the conditions for integrability of (2.8). Clearly $P(D)\{1.1\} = 0$. Secondly from the coefficient ε,

$$P(D)\{1.f_1 + f_1.1\} = 2P(\partial)\{f_1\} = 0. \tag{2.12}$$

The solution of the equation (2.12) turns out to be exponential function in general. Since other solutions are in terms of the function f_1, it is clear that $\forall i$, every f_i can be written in the form of f_1. However in this thesis we will show that the equation (2.12) does not always admit exponential type of solutions when D is in terms of q-forward jump operators. After finding one soliton solution by the virtue of (2.12), the method continues with two, and higher order soliton solutions by collecting the coefficients of ε^i for $i \geq 2$.

2.1 Toda Lattice Equation

In this section, as an illustration, Hirota method is used to find the soliton solution of Toda equation which was presented by Hirota in [19]. This equation is a mechanical model with a chain of particles as

$$\begin{aligned} &m\frac{d^2y_n}{dt^2} = a[e^{-br_n} - e^{-br_{n+1}}], \quad a, b \text{ are arbitrary parameters} \\ &r_n = y_n - y_{n-1}, \quad n \in Z^+ \end{aligned} \tag{2.13}$$

where m is mass of particle, a and b are repulsive and attractive forces respectively. In other words, Toda equation illustrates motion of an anharmonic one-dimensional lattice for n^{th} particle. If we suggest

$$V_n(t) = a[e^{-br_n} - 1], \tag{2.14}$$

which is the force of the n^{th} spring in lattice, equation (2.13) turns into

$$\frac{d^2}{dt^2}\log(1 + V_n(t)) = V_{n+1}(t) + V_{n-1}(t) - 2V_n(t), \quad V_n = V(n, t), \quad n \in Z, \quad t \in R. \tag{2.15}$$

In order to find soliton solutions of (2.15) let us begin with the first step i.e., the transformation.

Step 1: Transformation : We use the logarithmic transformation and get

$$V_n(t) := \frac{d^2}{dt^2}\log f(t, n) = \left(\frac{f(t, n)_t}{f(t, n)}\right)_t = \frac{f_{tt}(t, n).f(t, n) - f(t, n)_t f(t, n)_t}{f^2(t, n)}. \tag{2.16}$$

Step 2: Bilinearization: Our aim is to construct Hirota bilinear form. For this purpose we begin with using the anti-derivative of (2.15) with respect to t and use (2.16),

$$\log(1+V_n(t)) = [\log f(t,n+1) + \log f(t,n-1) - 2\log f(t,n)] \tag{2.17}$$

$$= \log \frac{f(t,n+1)f(t,n-1)}{f^2(n,t)}, \tag{2.18}$$

then we get

$$V_n(t) = \frac{f(t,n+1)f(t,n-1)}{f^2(t,n)} - 1. \tag{2.19}$$

Equating (2.19) to (2.16) we derive

$$\frac{f(t,n+1)f(t,n-1)}{f^2(t,n)} - 1 = \frac{f(t,n)_{tt}.f(t,n) - f(t,n)_t f(t,n)_t}{f^2(t,n)}, \tag{2.20}$$

which can be written as

$$f(t,n+1)f(t,n-1) - f^2(t,n) = f(t,n)_{tt}.f(t,n) - f(t,n)_t f(t,n)_t. \tag{2.21}$$

Right hand side of the equation is equivalent to $\frac{D_t^2\{f.f\}}{2}$. Note that the left hand side is equivalent to $(e^{D_n} + e^{-D_n} - 2)(f(t,n).f(t,n))$, thus we can write Hirota bilinear form as

$$[D_t^2 - (e^{D_n} + e^{-D_n} - 2)]\{f(t,n).f(t,n)\} = 0. \tag{2.22}$$

Equation (2.22) is called Hirota bilinear form of Toda lattice equation (2.15).

Step 3: We use finite perturbation expansion in the bilinear form. Inserting

$$f(t,n) = 1 + \varepsilon f^{(1)}(t,n) + \varepsilon^2 f^{(2)}(t,n) + \varepsilon^3 f^{(3)}(t,n) + \dots,$$

in (2.8) we get

$$
\begin{aligned}
P(D)\{f(t,n).f(t,n)\} &= \varepsilon^0 P(D)\{1.1\} + \varepsilon^1 P(D)\{f^{(1)}(t,n).1 + 1.f^{(1)}(t,n)\} \\
&+ \varepsilon^2 P(D)\{f^{(2)}(t,n).1 + 1.f^{(2)}(t,n) + f^{(1)}(t,n).f^{(1)}(t,n)\} \\
&+ \varepsilon^3 P(D)\{f^{(3)}(t,n).1 + 1.f^{(3)}(t,n) + f^{(2)}(t,n).f^{(1)}(t,n) \\
&+ f^{(1)}(t,n).f^{(2)}(t,n)\} + \cdots = 0,
\end{aligned}
$$

where ε is a constant called perturbation. We collect the coefficients of ε^i, □$i \geq 0$. The coefficient of ε^0 vanishes. The coefficient of ε^1 gives

$$P(D)\{f^{(1)}.1 + 1.f^{(1)}\} = P(D)\{f^{(1)}.1\} + P(D)\{1.f^{(1)}\} = 2P(\partial)f^{(1)}. \tag{2.23}$$

Our aim is to find a starting solution of the equation (2.23). One can show that one of the solution of this equation has exponential form.

$$f^{(1)}(t,n) = \alpha e^{\beta t + \gamma n}, \tag{2.24}$$

where α, β, γ are arbitrary constants. To understand the relation between the constants, the function (2.24) is inserted into (2.23).

$$
\begin{aligned}
[\partial_t^2 - (e^{\partial_n} + e^{-\partial_n} - 2)]f^{(1)}(t,n) &= \alpha\beta^2 e^{\beta t+\gamma n} - \alpha e^{\beta t + \gamma(n+1)} - \alpha e^{\beta t+\gamma(n-1)} + 2\alpha e^{\beta t+\gamma n} \\
&= \alpha e^{\beta t+\gamma n}[\beta^2 - e^{\gamma} - e^{-\gamma} + 2] = 0. \qquad (2.25)
\end{aligned}
$$

Thus we have

$$\beta^2 - e^{\gamma} - e^{-\gamma} + 2 = 0,$$

which implies

$$\beta^2 = e^{\gamma} + e^{-\gamma} - 2 = 4\sinh^2(\gamma/2),$$

and provides the relation between β and γ

$$\beta = \pm 2\sinh(\gamma/2). \tag{2.26}$$

The relation (2.26) is named as dispersion relation. For the coefficient of ε^2 we obtain

$$P(D)\{1.f^{(2)}+f^{(2)}.1+f^{(1)}.f^{(1)}\}=2P(\partial)f^{(2)}+P(D)\{f^{(1)}.f^{(1)}\}, \tag{2.27}$$

that can be written as

$$[D_t^2-(e^{D_n}+e^{-D_n}-2)]\{f^{(1)}.f^{(1)}\}=-2[\partial_t^2-(e^{\partial_n}+e^{-\partial_n}-2)]f^{(2)}. \tag{2.28}$$

Here we note that $f^{(1)}$, which is of the form (2.24), satisfies (2.28). Evaluating this equality, we concluded that $f^{(2)}=0$. Hence for $i\geq 2$, $f^{(i)}=0$. Hereupon for i-soliton solution, it can be accepted $f^{(k)}=0$, $k\geq i+1$. Resultantly, without loss of generality, taking $\varepsilon=1$, we obtain one-soliton solution of Toda lattice equation as

$$f=1+\alpha e^{\beta t+\gamma n}, \tag{2.29}$$

equipped with $\beta=\pm 2\sinh(\gamma/2)$. Now our next aim is to find two soliton solutions. For this purpose consider

$$f^{(1)}=\alpha_1 e^{\beta_1 t+\gamma_1 n}+\alpha_2 e^{\beta_2 t+\gamma_2 n}, \tag{2.30}$$

as a starting solution, where α_i, β_i and γ_i are arbitrary constants $\forall i=1,2$. Following the same steps, we collect the coefficients of ε^i, $\forall i\geq 0$. From ε^0, we get

$$P(D)\{1.1\}=D_t^2-(e^{D_n}+e^{-D_n}-2)\{1.1\}=0-(1+1-2)=0, \tag{2.31}$$

while from ε^1, we obtain

$$P(D)\{f^{(1)}.1+1.f^{(1)}\}=P(\partial)f^{(1)}=[\partial_t^2-(e^{\partial_n}+e^{-\partial_n}-2)]f^{(1)} \tag{2.32}$$

$$=[\partial_t^2-(e^{\partial_n}+e^{-\partial_n}-2)]\{\alpha_1 e^{\beta_1 t+\gamma_1 n}+\alpha_2 e^{\beta_2 t+\gamma_2 n}\}, \tag{2.33}$$

where

$$
\begin{aligned}
[\partial_t^2 - (e^{\partial_n} + e^{-\partial_n} - 2)]\{\alpha_1 e^{\beta_1 t+\gamma_1 n} + \alpha_2 e^{\beta_2 t+\gamma_2 n}\}
&= \alpha_1 e^{\beta_1 t+\gamma_1 n}[\alpha_1 \beta_1^2 - e^{\gamma_1} - e^{-\gamma_1} + 2] \\
&\quad + \alpha_2 e^{\beta_2 t+\gamma_2 n}[\alpha_2 \beta_2^2 - e^{\gamma_2} - e^{-\gamma_2} + 2] = 0,
\end{aligned}
$$

which reveals dispersion relation as

$$\beta_i^2 = e^{\gamma_i} + e^{-\gamma_i} - 2, \qquad i = 1, 2.$$

Similarly, ε^2 implies

$$P(D)\{1.f^{(2)} + f^{(2)}.1 + f^{(1)}.f^{(1)}\} = 2P(\partial)f^{(2)} + P(D)\{f^{(1)}.f^{(1)}\} = 0, \tag{2.34}$$

where

$$
\begin{aligned}
P(D)\{f^{(1)}.f^{(1)}\} &= -\alpha_1\alpha_2 e^{(\beta_1+\beta_2)t+(\gamma_1+\gamma_2)n}[(\beta_1 - \beta_2)^2 - e^{\gamma_1-\gamma_2} - e^{-\gamma_1+\gamma_2} + 2] \\
&= -2P(\partial)f^{(2)}.
\end{aligned} \tag{2.35}
$$

We conclude that $f^{(2)}$ is in the form of relations of two waves

$$f^{(2)} = A(1, 2)\alpha_1\, \alpha_2 e^{(\beta_1+\beta_2)t+(\gamma_1+\gamma_2)n}. \tag{2.36}$$

Plugging $f^{(2)}$ given in (2.36) into the equation (2.35), we get the interaction coefficient $A(1, 2)$,

$$A(1, 2) = -\frac{(\beta_1 - \beta_2)^2 - (e^{\gamma_1-\gamma_2} + e^{-\gamma_1+\gamma_2} - 2)}{(\beta_1 + \beta_2)^2 - (e^{\gamma_1+\gamma_2} + e^{-\gamma_1-\gamma_2} - 2)}. \tag{2.37}$$

Remark. We define the vector notations are defined as $p_i = (\alpha_i, \beta_i, \gamma_i)$ and

$$P(p_1) = \beta_1^2 - (e^{\gamma_1} + e^{-\gamma_1} - 2) = 0,$$

thus we have

$$P(p_1 \pm p_2) = (\beta_1 \pm \beta_2)^2 - (e^{\gamma_1\pm\gamma_2} + e^{-\gamma_1\pm\gamma_2} - 2).$$

Hence, we rewrite $A(1,2)$ as

$$A(1,2) = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)}.$$

Furthermore if we collect the coefficients of ε^3, we get $f^{(3)} = 0$. Thus the solution of two-soliton solutions is derived as

$$f = 1 + \alpha_1 e^{\beta_1 t + \gamma_1 n} + \alpha_2 e^{\beta_2 t + \gamma_2 n} + A(1,2)\alpha_1 \alpha_2 e^{(\beta_1+\beta_2)t + (\gamma_1+\gamma_2)n}. \tag{2.38}$$

To construct three-soliton solutions, we begin with

$$f^{(1)} = \alpha_1 e^{\beta_1 t + \gamma_1 n} + \alpha_2 e^{\beta_2 t + \gamma_2 n} + \alpha_3 e^{\beta_3 t + \gamma_3 n}, \tag{2.39}$$

where α_i and γ_i are constants for $i = 1, 2, 3$. Similarly we collect ε^i, $\forall i \geq 0$. The coefficients of ε^0 vanishes. From the coefficients of ε^1 we get

$$P(D)\{1.f^{(1)} + f^{(1)}.1\} = 0,$$

which gives the dispersion relation as

$$\beta_i^2 = e^{\gamma_i} + e^{-\gamma_i} - 2, \qquad i = 1, 2, 3. \tag{2.40}$$

Collecting the coefficient of ε^2, we get

$$P(D)\{1.f^{(2)} + f^{(1)}.f^{(1)} + f^{(2)}.1\} = P(D)\{f^{(1)}.f^{(1)}\} + 2P(\partial)f^{(2)} = 0. \tag{2.41}$$

One can find $f^{(2)}$ by evaluating $P(D)\{f^{(1)}.f^{(1)}\}$,

$$\begin{aligned} P(D)\{f^{(1)}.f^{(1)}\} = -2\ & \alpha_1\alpha_2 e^{(\beta_1+\beta_2)t+(\gamma_1+\gamma_2)n}[(\beta_1-\beta_2)^2 - e^{\gamma_1-\gamma_2} - e^{\gamma_2-\gamma_1} + 2] \\ & + \alpha_1\alpha_3 e^{(\beta_1+\beta_3)t+(\gamma_1+\gamma_3)n}[(\beta_1-\beta_3)^2 - e^{\gamma_1-\gamma_3} - e^{\gamma_3-\gamma_1} + 2] \\ & + \alpha_2\alpha_3 e^{(\beta_2+\beta_3)t+(\gamma_2+\gamma_3)n}[(\beta_2-\beta_3)^2 - e^{\gamma_2-\gamma_3} - e^{\gamma_3-\gamma_2} + 2]\ . \end{aligned} \tag{2.42}$$

If we insert (2.42) into (2.41), $\mathbf{f}^{(2)}$ arises as

$$\mathbf{f}^{(2)} = A(1,2)\alpha_1\alpha_2 e^{(\beta_1+\beta_2)t+(\gamma_1+\gamma_2)n} + A(1,3)\alpha_1\alpha_3 e^{(\beta_1+\beta_3)t+(\gamma_1+\gamma_3)n} + A(2,3)\alpha_2\alpha_3 e^{(\beta_2+\beta_3)t+(\gamma_2+\gamma_3)n}, \tag{2.43}$$

where

$$\begin{aligned} A(1,2) &= -\frac{P(p_1-p_2)}{P(p_1+p_2)} = -\frac{(\beta_1-\beta_2)^2 - e^{\gamma_1-\gamma_2} - e^{\gamma_2-\gamma_1}+2}{(\beta_1+\beta_2)^2 - e^{\gamma_1+\gamma_2} - e^{-\gamma_1-\gamma_2}+2}, \\ A(1,3) &= -\frac{P(p_1-p_3)}{P(p_1+p_3)} = -\frac{(\beta_1-\beta_3)^2 - e^{\gamma_1-\gamma_3} - e^{\gamma_3-\gamma_1}+2}{(\beta_1+\beta_3)^2 - e^{\gamma_1+\gamma_3} - e^{-\gamma_1-\gamma_3}+2}, \\ A(2,3) &= -\frac{P(p_2-p_3)}{P(p_2+p_3)} = -\frac{(\beta_2-\beta_3)^2 - e^{\gamma_2-\gamma_3} - e^{\gamma_3-\gamma_2}+2}{(\beta_2+\beta_3)^2 - e^{\gamma_2+\gamma_3} - e^{-\gamma_2-\gamma_3}+2}. \end{aligned} \tag{2.44}$$

Similarly, from the coefficient of ε^3 we find out

$$P(D)\{1.\mathbf{f}^{(3)}+\mathbf{f}^{(2)}.\mathbf{f}^{(1)}+\mathbf{f}^{(1)}.\mathbf{f}^{(2)}+\mathbf{f}^{(3)}.1\} = P(D)\{\mathbf{f}^{(2)}.\mathbf{f}^{(1)}+\mathbf{f}^{(1)}.\mathbf{f}^{(2)}\}+2P(\partial)\mathbf{f}^{(3)} = 0,$$

which implies

$$-2P(\partial)\mathbf{f}^{(3)} = P(D)\{\mathbf{f}^{(2)}.\mathbf{f}^{(1)} + \mathbf{f}^{(1)}.\mathbf{f}^{(2)}\}. \tag{2.45}$$

Our aim is to find $\mathbf{f}^{(3)}$. For this purpose we replace $\mathbf{f}^{(2)}$ given in (2.43) and $\mathbf{f}^{(1)}$ (2.39) into (2.45) which gives

$$\begin{aligned} & P(D)\{\mathbf{f}^{(1)}.\mathbf{f}^{(2)} + \mathbf{f}^{(2)}.\mathbf{f}^{(1)}\} \\ &= -2\Big[\alpha_3 A(1,2) e^{(\beta_1+\beta_2+\beta_3)t+(\gamma_1+\gamma_2+\gamma_3)n}[(\beta_3-\beta_1-\beta_2)^2 - e^{\gamma_3-\gamma_1-\gamma_2} - e^{-\gamma_3+\gamma_1+\gamma_2}+2] \\ &\quad + \alpha_1 A(2,3) e^{(\beta_1+\beta_2+\beta_3)t+(\gamma_1+\gamma_2+\gamma_3)n}[(\beta_1-\beta_2-\beta_3)^2 - e^{\gamma_1-\gamma_2-\gamma_2} - e^{-\gamma_1+\gamma_2+\gamma_3}+2] \\ &\quad + \alpha_2 A(1,3) e^{(\beta_1+\beta_2+\beta_3)t+(\gamma_1+\gamma_2+\gamma_3)n}[(\beta_2-\beta_1-\beta_3)^2 - e^{\gamma_2-\gamma_1-\gamma_2} - e^{-\gamma_2+\gamma_1+\gamma_3}+2]\Big], \end{aligned}$$

where $A(i,j)$ is given as in (2.44). We suggest $\mathbf{f}^{(3)}$ as

$$\mathbf{f}^{(3)} = A(1,2,3)\alpha_1\alpha_2\alpha_3 e^{(\beta_1+\beta_2+\beta_3)t+(\gamma_1+\gamma_2+\gamma_3)n}, \tag{2.46}$$

and substitute into the equality (2.45), $A(1,2,3)$ is found as

$$\begin{aligned} A(1,2,3) = &- \frac{\alpha_3 A(1,2)P(p_3-p_1-p_2)+\alpha_2 A(1,3)P(p_2-p_1-p_3)}{P(p_1+p_2+p_3)} \\ &+ \frac{\alpha_1 A(2,3)P(p_1-p_2-p_3)}{P(p_1+p_2+p_3)}, \end{aligned} \tag{2.47}$$

where

$$\begin{aligned} P(p_3-p_1-p_2) &= (\beta_3-\beta_1-\beta_2)^2 - e^{\gamma_3-\gamma_1-\gamma_2} - e^{-\gamma_3+\gamma_1+\gamma_2}+2, \\ P(p_2-p_1-p_3) &= (\beta_2-\beta_1-\beta_3)^2 - e^{\gamma_2-\gamma_1-\gamma_2} - e^{-\gamma_2+\gamma_1+\gamma_3}+2, \\ P(p_1-p_2-p_3) &= (\beta_1-\beta_2-\beta_3)^2 - e^{\gamma_1-\gamma_2-\gamma_2} - e^{-\gamma_1+\gamma_2+\gamma_3}+2, \\ P(p_1+p_2+p_3) &= (\beta_1+\beta_2+\beta_3)^2 - e^{\gamma_1+\gamma_2+\gamma_3} - e^{-\gamma_1-\gamma_2-\gamma_3}+2. \end{aligned}$$

On the other hand if we consider the coefficient of ε^4

$$P(D)\{1.f^{(4)}+f^{(1)}.f^{(3)}+f^{(2)}.f^{(2)}+f^{(3)}.f^{(1)}+f^{(4)}.1\}=0, \tag{2.48}$$

we get

$$A(1,2,3) = A(1,2)A(1,3)A(2,3). \tag{2.49}$$

Since both (2.47) and (2.49) represent $A(1,2,3)$, we equate them and reach to an expression called as three-soliton solution condition (3SSC) in the literature

$$\begin{aligned} &P(p_1+p_2+p_3)P(p_1+p_2)P(p_1-p_3)P(p_2-p_3)+ \\ &P(p_1-p_2-p_3)P(p_1+p_2)P(p_1+p_3)P(p_2-p_3)+ \\ &P(p_2-p_1-p_3)P(p_1+p_2)P(p_2+p_3)P(p_1-p_3)+ \\ &P(p_3-p_1-p_2)P(p_1+p_3)P(p_2+p_3)P(p_1-p_2)=0, \end{aligned} \tag{2.50}$$

equivalently

$$\sum_{\sigma=\pm1} P\left(\sum_{i=1}^{3} \sigma_i p_i\right) \prod_{i<j}^{(3)} P(\sigma_i p_i - \sigma_j p_j) = 0; \qquad i,j=1,2,3. \tag{2.51}$$

(3SSC) given with (2.51) is a restriction for $P(D)$. It determines that $P(D)$ cannot be selected arbitrarily but should satisfy equation (2.51). On the other hand, if an equation can be written in Hirota bilinear form and meets the 3-soliton solution condition, it is defined as Hirota integrable. Generally, Hirota integrability is extended to integrability with Painlevé test. In [11], it is concluded that Hirota integrability has a close relationship with traditional definitions of integrability.

Finally, the coefficients of ε^i for all $i \geq 5$ vanishes, and we obtain three-soliton solution of TLE equation as

$$\begin{aligned} f = 1 &+ (\alpha_1 e^{\beta_1 t+\gamma_1 n} + \alpha_2 e^{\beta_2 t+\gamma_2 n} + \alpha_3 e^{\beta_3 t+\gamma_3 n}) + (A(1,2)\alpha_1\alpha_2 e^{(\beta_1+\beta_2)t+(\gamma_1+\gamma_2)n} \\ &+ A(1,3)\alpha_1\alpha_3 e^{(\beta_1+\beta_3)t+(\gamma_1+\gamma_3)n} + A(2,3)\alpha_2\alpha_3 e^{(\beta_2+\beta_3)t+(\gamma_2+\gamma_3)n}) \\ &+ A(1,2,3)\alpha_1\alpha_2\alpha_3 e^{(\beta_1+\beta_2+\beta_3)t+(\gamma_1+\gamma_2+\gamma_3)n}. \end{aligned} \tag{2.52}$$

Chapter 3

q-discretization

A soliton does not change its amplitude, velocity, and shape after a collision with another soliton wave. Solitons, as is evident from its definition, are kinds of solutions of wave equations having nonlinearity and dispersion. In the previous chapter, the fastest algebraic method, Hirota Direct Method is explained to find soliton solutions. Hirota introduced the method for KdV equation in [14], since it is the simplest nonlinear partial differential equation. After a short time, Hirota showed that the method is not only valid for differential equations, but also for difference equations [18],[19]. Although for both for differential and difference equations, Hirota method can produce multi-soliton solutions, its application to q-analogues of equations remained unsolved until the pioneering article [33].

In [1], it was shown that an isomorphism does not exist between q-difference systems on R^{-} and the lattice systems on R. Motivated by this inequivalence in the pioneering article [33], q-discretization of equations are investigated. In [33], Silindir presented the q-analogue of Toda lattice equation and showed the existence of multi-soliton solutions by Hirota Direct Method. In [33] q-difference equations are constructed by means of q-difference operator while q-differential ones are built with the help of q-derivative operator. In the article, differential-q-difference, q-difference-q-difference and q-differential-q-difference Toda equations are presented. For differential-q-difference and q-difference-q-difference Toda equations, three-soliton solutions are found by Hirota's method. Contrary to the expectations the solutions are not exponential functions, however they are in the form of polynomial of power functions and they obey the usual soliton behaviors. This type of solutions are introduced as q-soliton solutions. For q-differential-q-difference Toda equation, Hirota's method is not valid. Silindir showed that even if q-differential-q-difference Toda equation can be written in Hirota bilinear form, the multisoliton solutions cannot be produced by Hirota direct method.

The theory presented in this chapter is based on the landmark article [33].

3.1 q-Exponential Identity

In this section, our aim is to present q-difference equations , so it is necessary to recall Hirota D-operator in terms of exponential function (2.6). To show the applicability of Hirota method to q-difference equations, we investigate the q-analogue of the exponential identity. For this purpose, we present the findings of the article [1]

Suppose $\sigma : \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R} \to \mathbb{R}$ are the forward and backward jump operators, respectively. If there exists inverse maps ρ^{-1} and σ^{-1} such that $\rho^{-1}(x) = \sigma(x)$ and $\sigma^{-1}(x) = \rho(x)$, $\forall x \in \mathbb{R}$, then σ and ρ are bijective and they determine discrete one parameter group of bijections on $\mathbb{R}$

$$n \mapsto \{\sigma_n : \mathbb{R} \to \mathbb{R}\}, \text{ where } \sigma_0 \equiv id_{\mathbb{R}}, \; n \in \mathbb{R}.$$

Instead of one parameter group of bijections, we introduce one parameter group of diffeomorphisms as we focus on systems of differential equations. Thus ,let $h \in \mathbb{R}$, and $h \mapsto \sigma_h$ be a continuous one-parameter group of diffeomorphisms.
We expand σ_h around $h = 0$,

$$\sigma_h(x) = x + h.\frac{d\sigma_h(x)}{dh}\bigg|_{h=0} + O(h^2), \tag{3.1}$$

which clearly implies that σ_h is generated by a vector field which we denote it by

$\chi(x)\partial x$, i.e. $\sigma_h(x) = x + h.\chi(x) + O(h^2)$, where $\chi(x)$ is smooth on R except at most finite number of points and it is said to be infinitesimal generator of the one-parameter group of diffeomorphisms. Referring [30], for such σ_h it is beneficial to compute one-parameter group in terms of exponentiation of the vector field

$$\sigma_h(x) = e^{h\chi(x)\partial_x}x, \tag{3.2}$$

if and only if

$$e^{h\chi(x)\partial_x}g(x) = g(e^{h\chi(x)\partial_x}x) = g(\sigma_h(x)), \tag{3.3}$$

where $g(x)$ is a smooth function. It is convenient to propose $\chi(x)\partial_x = x^{1-n}\partial_x$ on R. For $n = 1$ we have,

$$\sigma_h(x) = e^{h\partial_x}x = x + h \quad \text{iff} \quad e^{h\partial_x}g(x) = g(x + h), \tag{3.4}$$

which is nothing but forward jump operator in discrete variables. Furthermore, for $n = 0$, we have,

$$\sigma_h(x) = e^{hx\partial_x}x = e^h x = qx \quad \text{iff} \quad e^{hx\partial_x}g(x) = g(qx), \tag{3.5}$$

which turns out to be q-forward jump operator, equipped with $q = e^h$.

Definition. [33] The q-forward jump operator E_q is defined to be

$$E_q(g(x)) := e^{hx\partial_x}g(x) = g(qx), \tag{3.6}$$

where $x \in R$, h is a parameter and $g \in C^\infty(R)$. In the same manner, we define q-backward jump operator E_q^{-1} as

$$E_q^{-1}g(x) := e^{-hx\partial_x}g(x) = g\left(\frac{x}{q}\right). \tag{3.7}$$

Proposition 3.1 [35] q-forward jump operator E_q recovers its continuous case

$$\lim_{q\to 1} E_q(x) = \lim_{h\to 0} \sigma_h(x) = x. \tag{3.8}$$

Proof. Expanding Taylor series of $E_q(x)$ with respect to h near zero we have

$$\begin{aligned}\lim_{q\to 1} E_q(x) &= \lim_{h\to 0}[x + h\chi(x)\partial_x\{x\} + \frac{h^2}{2}(\chi(x)\partial_x)^2\{x\} + O(h^3)]\\ &= \lim_{h\to 0}[x + hx\partial_x\{x\} + \frac{h^2}{2}(x\partial_x)^2\{x\} + O(h^3)]\\ &= x,\end{aligned}$$

and also

$$\lim_{q\to 1} E_q(x) = \lim_{q\to 1} qx = x.$$

□

We stress that, discrete systems generated by $\chi(x)\partial x$ are not equivalent. In order to verify this fact, suggest $\chi(x) = x^{1-n}$, for odd n, and $n = 0$ to compare with $\chi^0(x^0) = 1$. Here we get $x^0 = \frac{1}{n}x^n$ which is a bijection on $R - \{0\}$. To sum up, all discrete systems determined by $\chi(x)\partial x = x^{1-n}\partial x$ (n is odd) are isomorphic to each other. On the other hand, if we compare $\chi(x) = x$ and $\chi^0(x^0) = 1$, $x = e^{x^0}$,the transformation is found which is not a bijection on R^-.

As it is evident, there is not an isomorphism between q-difference systems on R^- obtained by q-forward jump operators, $E_q = e^{hx\partial x}$, and lattice systems on R.

Hereafter, we q-discretize continuous equations by the use of q-forward jump operator E_q. Hence, we must present and prove the q-analogue of exponential identity, which was first introduced in [33]. This identity is based on E_q and is crucial to construct Hirota bilinear forms of q-differential equations.

Theorem 3.2 [33] Let $g(x)$, $f(x) \in C^\infty(R)$. Then we present the q-exponential identity as

$$e^{hxD_x}g(x)f(x) = E_q g(x)E_q^{-1}f(x), \quad x \in R, \tag{3.9}$$

where h and q do satisfy $q = e^h$ and D_x is given in (2.2).

Proof. Note that, the proof does not follow the proof of Theorem 2.1 since the operator $D_y := xD_x$ is not associative. Instead, integrate with respect to x, which implies $x = e^y$. Then

$$e^{hxD_x} f(x)g(x) = e^{hD_y} f(e^y)g(e^y). \tag{3.10}$$

Using the equation (3.4) and Theorem 2.1, we obtain

$$\begin{aligned} e^{hxD_x} g(x)f(x) &= e^{hD_y} g(e^y).f(e^y) = g(e^{y+h})f(e^{y-h}) \\ &= g(e^y e^h)f(e^y e^{-h}) = g(qx)f(\frac{x}{q}) = E_q g(x)E_q^{-1} f(x). \end{aligned}$$

□

3.2 The differential-q-difference Toda equation

In this section, our goal is to present the differential-q-difference Toda equation and its multi-soliton solutions. First, we present the difference operator, introduced in [33].

Definition. [33] The central q-difference operator Λ_x^2 acting to an arbitrary smooth function $g(x)$ is defined as

$$\Lambda_x^2 g(x) = g(qx) + g(\frac{x}{q}) - 2g(x), \qquad q = 1 \quad \text{and} \quad x \,\square\, R. \tag{3.11}$$

We introduce the differential q-difference Toda equation as

$$\frac{d^2}{dt^2} \log(1 + V(x, t)) = \Lambda_x^2 V(x, t) = V(qx, t) + V\left(\frac{x}{q}, t\right) - 2V(x, t), \tag{3.12}$$

where V is rapidly decaying function and $x, t \,\square\, R$. We mean by a rapidly decaying function, is a function whose all derivatives vanish as $| x | \to \square\infty$.

The next step is to write Hirota bilinear form of (3.12). For this purpose, we use the dependent variable transformation as

$$V(x, t) := \frac{d^2}{dt^2} \log(f(x, t)). \tag{3.13}$$

Similar to the previous discussions in Section (2.1), to reach Hirota bilinear form firstly we apply anti-derivative twice to (3.12) twice with respect to t

$$\log(1 + V(x,t)) = \partial_t^{-2} \quad \Lambda_x^2 V(x,t)$$

and then we use the transformation (3.13) to get

$$\log(1 + V(x,t)) = \Lambda_x^2 \log(f(x,t)).$$

Using Λ_x^2 defined in (3.11), we get

$$\log(1 + V(x,t)) = \log f(qx,t) + \log f(\frac{x}{q},t) - 2\log f(x,t) = \log \quad \frac{f(qx,t)f(\frac{x}{q},t)}{f^2(x,t)}$$

which implies

$$V(x,t) = \frac{f(qx,t)f(\frac{x}{q},t)}{f^2(x,t)} - 1.$$

If we put the described $V(x,t)$ in (3.13) on the left hand side of the above equation, we obtain

$$\frac{f_{tt}(x,t)f(x,t) - f_t^2(x,t)}{f^2(x,t)} = \frac{f(qx,t)f(\frac{x}{q},t)}{f^2(x,t)} - 1,$$

which gives us Hirota bilinear form of differential-q-difference Toda equation as follows

$$[D_t^2 - (e^{hxD_x} + e^{-hxD_x} - 2)]\{f(x,t).f(x,t)\} = 0. \qquad (3.14)$$

After obtaining Hirota bilinear form, the perturbation technique is used around a perturbation parameter ε to obtain soliton solutions where

$$P(D)\{f(x,t).f(x,t)\} = [D_t^2 - (e^{hxD_x} + e^{-hxD_x} - 2)]\{f(x,t).f(x,t)\}, \qquad (3.15)$$

and

$$f(x,t) = 1 + \varepsilon f^{(1)}(x,t) + \varepsilon^2 f^{(2)}(x,t) + \varepsilon^3 f^{(3)}(x,t) + \cdots .$$

We substitute this $f(x, t)$ into (3.15) then we get

$$P(D)\{f(x,t).f(x,t)\} = P(D)\{1.1\} + \varepsilon P(D)\{1.f^{(1)} + f^{(1)}.1\} + \varepsilon^2 P(D)\{1.f^{(2)} + f^{(2)}.1 + f^{(1)}.f^{(1)}\} + \varepsilon^3 P(D)\{1.f^{(3)} + f^{(3)}.1 + f^{(2)}.f^{(1)} + f^{(1)}.f^{(2)}\} + \varepsilon^4 P(D)\{1.f^{(4)} + f^{(4)}.1 + f^{(1)}.f^{(3)} + f^{(3)}.f^{(1)} + f^{(2)}.f^{(2)}\} + \cdots .$$

Our aim is to find one soliton solution of the equation of (3.12). Therefore, we begin to analyze the coefficients of ε. The coefficient of ε^0

$$P(D)\{1.1\} = [D_t^2 - (e^{hxD_x} + e^{-hxD_x} - 2)]\{1.1\},$$

vanishes clearly. From the coefficient of ε^1,

$$P(D)\{1.f^{(1)} + f^{(1)}.1\} = 2P(\partial)f^{(1)} = 2[\partial_t^2 - (e^{hx\partial_x} + e^{-hx\partial_x} - 2)]f^{(1)}. \tag{3.16}$$

From which the coefficient of ε^1, $f^{(1)}$ can be constructed.

Remark. The significant point of Hirota method is the form of the solution of (3.16). Here we cannot suggest an exponential form for $f^{(1)}$, since it does not satisfy the above equation (3.16). Our starting function should include a power function as

$$f^{(1)}(x,t) = x^{\alpha} e^{\beta t + \eta}, \tag{3.17}$$

where α, β, η are arbitrary constants, as a result of the q-discrete variable counterpart. We rename such distinctive solutions as q-solitons as defined explicitly in [33].

Definition. [33] We introduce a q-soliton solution as a solution which has classical soliton attitudes and additionally power counterparts for discrete variables.

If we substitute the starting solution (3.17) into the equation (3.16), we conclude

$$[D_t^2 - (e^{hxD_x} + e^{-hxD_x} - 2)]\{1.f^{(1)} + f^{(1)}.1\} = 2[\partial_t^2 - (e^{hx\partial_x} + e^{-hx\partial_x} - 2)]x^{\alpha}e^{\beta t+\eta}$$
$$= 2(\beta^2 - q^{\alpha} - q^{-\alpha} + 2)x^{\alpha}e^{\beta t+\eta} = 0. \tag{3.18}$$

Then the relation between the parameters results as

$$\beta^2 = q^{\alpha} + q^{-\alpha} - 2, \tag{3.19}$$

in order to satisfy (3.18). Here the relation (3.19) is said to be dispersion relation. The coefficient of ε^2 implies

$$P(D)\{1.f^{(2)} + f^{(2)}.1 + f^{(1)}.f^{(1)}\}$$
$$= [D_t^2 - (e^{hxD_x}e^{-hxD_x} - 2)]\{1.f^{(2)} + f^{(2)}.1 + x^{\alpha}e^{\beta t+\eta}.x^{\alpha}e^{\beta t+\eta}\} = 0,$$

where

$$P(D)\{f^{(1)}.f^{(1)}\} = [D_t^2 - (e^{hxD_x} + e^{-hxD_x} - 2)]\{x^{\alpha}e^{\beta t+\eta}.x^{\alpha}e^{\beta t+\eta}\}$$
$$= \beta^2 x^{2\alpha}e^{2(\beta t+\eta)} - \beta^2 x^{2\alpha}e^{2(\beta t+\eta)} - \frac{2(qx)^{\alpha}}{q}x^{\alpha}e^{2(\beta t+\eta)} - 2x^{2\alpha}e^{2(\beta t+\eta)} = 0.$$

Since $P(D)\{f^{(1)}.f^{(1)}\} = -2P(\partial)f^{(2)}$, with the above equivalence, $f^{(2)}$ will be determined. As a result, it is convenient to say □$n \geq 2$, $f^{(n)} = 0$ for one q-soliton solution. The generalization of this equation provides us for i^{th} solution $f^{(k)} = 0$ for all $k \geq i+1$. In the end, without loss of generality taking $\varepsilon = 1$, one q-soliton is

$$V(x,t) = \frac{\beta^2 x^{\alpha}e^{\beta t+\eta}}{(1 + x^{\alpha}e^{\beta t+\eta})^2}, \tag{3.20}$$

where $f(x,t) = 1 + x^{\alpha}e^{\beta t+\eta}$ and β, α are interrelated as (3.19). For two q-soliton solutions, we propose the solution as

$$f^{(1)} = x^{\alpha_1}e^{\beta_1 t+\eta_1} + x^{\alpha_2}e^{\beta_2 t+\eta_2}. \tag{3.21}$$

Similarly to the previous arguments, the coefficient of ε^1, gives the dispersion relation

$$\beta_i^2 = q^{\alpha_i} + q^{-\alpha_i} - 2 \qquad \forall i = 1, 2. \tag{3.22}$$

The coefficient of ε^2 gives the solution for $\mathbf{f}^{(2)}$ as

$$\mathbf{f}^{(2)} = A(1, 2)x^{\alpha_1+\alpha_2} e^{(\beta_1+\beta_2)t+\eta_1+\eta_2}, \tag{3.23}$$

where

$$A(1, 2) = -\frac{(\beta_1 - \beta_2)^2 - (q^{\alpha_1-\alpha_2} + q^{-\alpha_1+\alpha_2}) - 2}{(\beta_1 + \beta_2)^2 - (q^{\alpha_1+\alpha_2} + q^{-\alpha_1-\alpha_2}) - 2} = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)}.$$

Consequently, $\mathbf{f}^{(3)} = 0$ and the solution of two soliton is

$$\mathbf{f}(x, t) = 1 + x^{\alpha_1} e^{\beta_1 t+\eta_1} + x^{\alpha_2} e^{\beta_2 t+\eta_2} + A(1, 2)x^{\alpha_1+\alpha_2} e^{(\beta_1+\beta_2)t+\eta_1+\eta_2}. \tag{3.24}$$

Coming up to three soliton solution, we start with

$$\mathbf{f}^{(1)} = \sum_{i=1}^{3} x^{\alpha_i} e^{\beta_i t+\eta_i}. \tag{3.25}$$

Then the dispersion relation arises as

$$\beta_i^2 = q^{\alpha_i} + q^{-\alpha_i} - 2 \qquad i = 1, 2, 3, \tag{3.26}$$

and from the coefficient of ε^2, by the virtue of (3.25), $\mathbf{f}^{(2)}$ emerges as

$$\mathbf{f}^{(2)} = \sum_{i<j}^{(3)} A(i, j)x^{\alpha_i+\alpha_j} e^{(\beta_i+\beta_j)t+\eta_i+\eta_j}, \tag{3.27}$$

where

$$A(i, j) = -\frac{P(p_i - p_j)}{P(p_i + p_j)}, \quad i < j, \quad i, j = 1, 2, 3. \tag{3.28}$$

From the coefficient of ε^3, we obtain

$$\mathbf{f}^{(3)} = A(1, 2, 3)x^{\alpha_1+\alpha_2+\alpha_3} e^{(\beta_1+\beta_2+\beta_3)t+\eta_1+\eta_2+\eta_3}, \tag{3.29}$$

where

$$A(1,2,3) = -\frac{A(1,2)P(p_3 - p_1 - p_2) + A(1,3)P(p_2 - p_1 - p_3) + A(2,3)P(p_1 - p_2 - p_3)}{P(p_1 + p_2 + p_3)}. \tag{3.30}$$

The coefficient of ε^4 gives us a relation for $A(1,2,3)$,

$$A(1,2,3) = A(1,2)A(1,3)A(2,3). \tag{3.31}$$

If we combine the equations (3.30) and (3.31), the (3SSC) appears as

$$\sum_{\sigma_i=\pm 1} P\left(\sum_{i=1}^{3} \sigma_i p_i\right) \prod_{i<j}^{(3)} P(\sigma_i p_i - \sigma_j p_j) = 0, \qquad i, j = 1, 2, 3. \tag{3.32}$$

Finally, the three q-soliton solution is found as

$$\begin{aligned} f(x,t) = 1 &+ \sum_{i=1}^{3} x^{\alpha_i} e^{\beta_i t + \eta_i} + \sum_{i<j}^{(3)} A(i,j) x^{\alpha_i + \alpha_j} e^{(\beta_i + \beta_j)t + \eta_i + \eta_j} \\ &+ A(1,2,3) x^{\alpha_1 + \alpha_2 + \alpha_3} e^{(\beta_1 + \beta_2 + \beta_3)t + \eta_1 + \eta_2 + \eta_3}. \end{aligned} \tag{3.33}$$

Chapter 4

Unification of q-difference Equations

Hirota direct method, which converts nonlinear partial equations into bilinear equations, can be applied for a wide variety of differential and difference type of equations. In the previous chapter, following the work in [33] we presented that the method is applicable to q-difference type of equations to obtain q-soliton solutions and to seek for their integrability. In [33], differential-q-difference Toda and q-difference-q-difference Toda equation were investigated under Hirota's method and their q-soliton solutions were developed. Similarly, one can find q-soliton solutions of various q-difference type of equations. However, our aim is to find a main equation which unifies several q-difference type of equations. For this purpose, we introduce an equation in Hirota bilinear form, q-discrete analogue of Hirota-Miwa equation, which includes several q-difference equations. Following the approach in [33], in [35] we show the applicability of Hirota's method to such a unified q-difference soliton equation and develop its q-soliton solutions.

The important point is to analyze appropriate reductions on q-difference analogue of Hirota-Miwa equation, which provides proper q-deformed Hirota bilinear forms. The continuous limit of those Hirota bilinear forms meet the classical Hirota bilinear forms of various equations. Furthermore, such Hirota bilinear forms allows to derive standard form of q-difference equations under considerations. Moreover, in order to discuss the integrability, we utilize Hirota direct method and develop their 3-q-soliton solutions.

The theory presented in this chapter is based on the article [35].

4.1 q-difference analogue of Hirota-Miwa equation and its' q-soliton solutions

In this section, we analyze the unification of q-difference equations and the applicability of Hirota Direct Method on a generalized q-difference soliton equation. q-discrete analogue of Hirota-Miwa equation

$$P(D_1, D_2, D_3)\{f \cdot f\} := \sum_{i=1}^{3} \lambda_i \cosh(D_i)\{f \cdot f\} = 0, \tag{4.1}$$

where λ_i' s are parameters, D_i' s are given

$$D_i = a_i tD_t + b_i xD_x + c_i yD_y, \quad a_i, b_i, c_i \in \mathbb{R}, \quad i = 1, 2, 3 \tag{4.2}$$

as linear combinations of tD_t, xD_x, yD_y.

It is beneficial to express the q-analogue of exponential identity in three variables which is represented in q-shift operators.

Corollary 4.1 [35] We present the q-exponential identity in $t, x, y \in \mathbb{R}$ as follows

$$\exp(a_i tD_t + b_i xD_x + c_i yD_y)g(t, x, y)f(t, x, y) = g(p_i t, r_i x, q_i y)f\left(\frac{t}{p_i}, \frac{x}{r_i}, \frac{y}{q_i}\right), \tag{4.3}$$

where $e^{a_i} = p_i$, $e^{b_i} = r_i$, and $e^{c_i} = q_i$, for all $i = 1, 2, 3$, respectively and g, f are arbitrary smooth functions.

In order to construct q-soliton solutions first we expand $f(t, x, y)$ around a formal perturbation parameter ε

$$f(t, x, y) = 1 + \varepsilon f^{(1)}(t, x, y) + \varepsilon^2 f^{(2)}(t, x, y) + \dots \quad . \tag{4.4}$$

Then substitution of (4.4) into our generalized Hirota bilinear form (4.1), we conclude

$$\begin{aligned} & P(D_1, D_2, D_3)\{f(t, x, y) \cdot f(t, x, y)\} \\ & = P(D_1, D_2, D_3)\{1.1\} + \varepsilon P(D_1, D_2, D_3)\{1 \cdot f^{(1)} + f^{(1)}.1\} \\ & + \varepsilon^2 P(D_1, D_2, D_3)\{1 \cdot f^{(2)} + f^{(2)}.1 + f^{(1)} \cdot f^{(1)}\} \\ & + \varepsilon^3 P(D_1, D_2, D_3)\{1 \cdot f^{(3)} + f^{(3)} \cdot 1 + f^{(1)} \cdot f^{(2)} + f^{(2)} \cdot f^{(1)}\} + \dots = 0 \end{aligned} \tag{4.5}$$

where we use the linearity of the polynomial $P(D_1, D_2, D_3)$. In the final step, we collect and vanish the coefficients of ε^i, for all $i \geq 0$. The coefficient of ε^0 is

$$P(D_1, D_2, D_3)\{1 \cdot 1\} = \lambda_1 + \lambda_2 + \lambda_3 .$$

It is helpful to remind the sufficient conditions to have at least two soliton solutions via Hirota method, stated in Theorem 2.3. Thus we have a restriction on parameters

$$P(0, 0, 0) = \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{4.6}$$

From the equation (4.5), the coefficient of ε^1 implies

$$\begin{aligned} & P(D_1, D_2, D_3)\{f^{(1)}.1 + 1 \cdot f^{(1)}\} = 2P(\partial_1, \partial_2, \partial_3)f^{(1)} = 2\sum_{i=1}^{3} \lambda_i \cosh(\partial_i) f^{(1)} \\ & = \sum_{i=1}^{3} \frac{\lambda_i}{2}(\exp(a_i t\partial_t + b_i x\partial_x + c_i y\partial_y) + \exp(-a_i t\partial_t - b_i x\partial_x - c_i y\partial_y))f^{(1)} = 0. \end{aligned} \tag{4.7}$$

From the previous experiences, as a consequence of change of variables, we must offer a solution that includes power form for q-discrete variables. Thus we start with the solution of the form

$$f^{(1)}(t, x, y) = \eta t^{\delta} x^{\zeta} y^{\theta}, \tag{4.8}$$

where $\eta, \delta, \zeta, \theta$ are arbitrary constants. Such kind of solutions provide q-soliton solutions, determined in Definition 3.2.

Inserting this solution into the equation (4.7), we derive the dispersion relation as

$$P(v) = \sum_{i=1}^{3} \lambda_i (p_i^{\delta} r_i^{\zeta} q_i^{\theta} + p_i^{-\delta} r_i^{-\zeta} q_i^{-\theta}) = 0, \tag{4.9}$$

where v is in vector notation $v = (\delta, \zeta, \theta)$.

Let us investigate the coefficient of ε^2 from (4.5)

$$P(D_1, D_2, D_3)\{1 \cdot f^{(2)} + f^{(1)} \cdot f^{(1)} + f^{(2)}.1\} = 0, \tag{4.10}$$

which implies

$$P(D_1, D_2, D_3)\{f^{(1)} \cdot f^{(1)}\} = -2P(\partial_1, \partial_2, \partial_3) f^{(2)}. \tag{4.11}$$

If we plug $f^{(1)}$ as defined in (4.8) into the left hand side of (4.11)

$$\begin{aligned} P(D_1, D_2, D_3)\{f^{(1)}.f^{(1)}\} &= \sum_{i=1}^{3} \lambda_i \cosh(D_i)\{\eta t^{\delta} x^{\zeta} y^{\theta}.\eta t^{\delta} x^{\zeta} y^{\theta}\} \\ &= \sum_{i=1}^{3} \lambda_i \eta^2 (p_i t)^{\delta} (r_i x)^{\zeta} (q_i y)^{\theta} (\frac{t}{p_i})^{\delta} (\frac{x}{r_i})^{\zeta} (\frac{y}{q_i})^{\theta} \\ &= (\lambda_1 + \lambda_2 + \lambda_3) \eta^2 t^{2\delta} x^{2\zeta} y^{2\theta} \end{aligned} \tag{4.12}$$

which vanishes by the assumption (4.6). Therefore we can say that $f^{(j)} = 0$, for all $j \geq 2$. If we generalize this for i-q-soliton solution, $f^{(k)} = 0$ for all $k \geq i + 1$. For simplicity, take $\varepsilon = 1$, then one q-soliton solution is

$$f(t, x, y) = 1 + \eta t^{\delta} x^{\zeta} y^{\theta}. \tag{4.13}$$

For two-q-soliton solutions, we begin with the solution

$$f^{(1)} = \sum_{i=1}^{2} \eta_i t^{\delta_i} x^{\zeta_i} y^{\theta_i},$$

where $\eta_i, \delta_i, \zeta_i$' s are constants for all $i = 1, 2$. The coefficient of ε^0 vanishes

$$P(D_1, D_2, D_3)\{1.1\} = \lambda_1 + \lambda_2 + \lambda_3 = 0 \qquad (4.14)$$

by the constraint (4.6). We continue to search for the coefficient of ε^1.

$$P(D_1, D_2, D_3)\{1.\mathbf{f}^{(1)} + \mathbf{f}^{(1)}.1\} = -2P(\partial_1, \partial_2, \partial_3)\mathbf{f}^{(2)}. \qquad (4.15)$$

We have to investigate $P(\partial_1, \partial_2, \partial_3)\mathbf{f}^{(1)}$ to find the dispersion relation, thus

$$P(\partial_1, \partial_2, \partial_3)\mathbf{f}^{(1)} = \sum_{i=1}^{3} \lambda_i \cosh(\partial_i)\{\eta_1 t^{\delta_1} x^{\zeta_1} y^{\theta_1} + \eta_2 t^{\delta_2} x^{\zeta_2} y^{\theta_2}\}$$

$$= \frac{\lambda_1}{2}\left[\eta_1 (p_1 t)^{\delta_1}(r_1 x)^{\zeta_1}(q_1 y)^{\theta_1} + \eta_2 (p_1 t)^{\delta_2}(r_1 x)^{\zeta_2}(q_1 y)^{\theta_2} + \eta_1 \left(\frac{t}{p_1}\right)^{\delta_1}\left(\frac{x}{r_1}\right)^{\zeta_1}\left(\frac{y}{q_1}\right)^{\theta_1}\right.$$
$$\left. + \eta_2 \left(\frac{t}{p_1}\right)^{\delta_2}\left(\frac{x}{r_1}\right)^{\zeta_2}\left(\frac{y}{q_1}\right)^{\theta_2}\right] + \frac{\lambda_2}{2}\left[\eta_1 (p_2 t)^{\delta_1}(r_2 x)^{\zeta_1}(q_2 y)^{\theta_1} + \eta_2 (p_2 t)^{\delta_2}(r_2 x)^{\zeta_2}(q_2 y)^{\theta_2}\right.$$
$$\left. + \eta_1 \left(\frac{t}{p_2}\right)^{\delta_1}\left(\frac{x}{r_2}\right)^{\zeta_1}\left(\frac{y}{q_2}\right)^{\theta_1} + \eta_2 \left(\frac{t}{p_2}\right)^{\delta_2}\left(\frac{x}{r_2}\right)^{\zeta_2}\left(\frac{y}{q_2}\right)^{\theta_2}\right] + \frac{\lambda_3}{2}\left[\eta_1 (p_3 t)^{\delta_1}(r_3 x)^{\zeta_1}(q_3 y)^{\theta_1}\right.$$
$$\left. + \eta_2 (p_3 t)^{\delta_2}(r_3 x)^{\zeta_2}(q_3 y)^{\theta_2} + \eta_1 \left(\frac{t}{p_3}\right)^{\delta_1}\left(\frac{x}{r_3}\right)^{\zeta_1}\left(\frac{y}{q_3}\right)^{\theta_1} + \eta_2 \left(\frac{t}{p_3}\right)^{\delta_2}\left(\frac{x}{r_3}\right)^{\zeta_2}\left(\frac{y}{q_3}\right)^{\theta_2}\right]$$
$$= \sum_{i=1}^{2} \eta_i t^{\delta_i} x^{\zeta_i} y^{\theta_i}\left[\frac{\lambda_1}{2}(p_1^{\delta_i} r_1^{\zeta_i} q_1^{\theta_i} + p_1^{-\delta_i} r_1^{-\zeta_i} q_1^{-\theta_i}) + \frac{\lambda_2}{2}(p_2^{\delta_i} r_2^{\zeta_i} q_2^{\theta_i} + p_2^{-\delta_i} r_2^{-\zeta_i} q_2^{-\theta_i})\right.$$
$$\left. + \frac{\lambda_3}{2}(p_3^{\delta_1} r_3^{\zeta_1} q_3^{\theta_1} + p_3^{-\delta_1} r_3^{-\zeta_1} q_3^{-\theta_1})\right]. \qquad (4.16)$$

Remember that (4.16) is equal to zero and since the terms $\eta_i t^{\delta_i} x^{\zeta_i} y^{\theta_i}$ cannot be identically zero, thus we get the dispersion relation among two-q-soliton solutions as

$$P(v_j) = P(\delta_j, \zeta_j, \theta_j) = \sum_{i=1}^{3} \frac{\lambda_i}{2}(p_i^{\delta_j} r_i^{\zeta_j} q_i^{\theta_j} + p_i^{-\delta_j} r_i^{-\zeta_j} q_i^{-\theta_j}) = 0, \quad \forall j = 1, 2. \quad (4.17)$$

The coefficient of ε^2 determines that

$$P(D_1, D_2, D_3)\{1 \cdot \mathbf{f}^{(2)} + \mathbf{f}^{(1)} \cdot \mathbf{f}^{(1)} + \mathbf{f}^{(2)} \cdot 1\} = 0, \qquad (4.18)$$

which implies

$$P(D_1, D_2, D_3)\{\mathbf{f}^{(1)} \cdot \mathbf{f}^{(1)}\} = -2P(\partial_1, \partial_2, \partial_3)\mathbf{f}^{(2)}. \tag{4.19}$$

Thus we have

$$\begin{aligned}
&P(D_1, D_2, D_3)\{\mathbf{f}^{(1)} \cdot \mathbf{f}^{(1)}\} \\
&= P(D_1, D_2, D_3)\{\eta_1 t^{\delta_1} x^{\zeta_1} y^{\theta_1}.\eta_1 t^{\delta_1} x^{\zeta_1} y^{\theta_1} + \eta_1 t^{\delta_1} x^{\zeta_1} y^{\theta_1}.\eta_2 t^{\delta_2} x^{\zeta_2} y^{\theta_2} \\
&\quad + \eta_2 t^{\delta_2} x^{\zeta_2} y^{\theta_2}.\eta_1 t^{\delta_1} x^{\zeta_1} y^{\theta_1} + \eta_2 t^{\delta_2} x^{\zeta_2} y^{\theta_2}.\eta_2 t^{\delta_2} x^{\zeta_2} y^{\theta_2}\}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\sum_{i=1}^{2} \frac{\lambda_i}{2} \; & \eta_1^2 (p_i t)^{\delta_1} (r_i x)^{\zeta_1} (q_i y)^{\theta_1} \left(\frac{t}{p_i}\right)^{\delta_1} \left(\frac{x}{r_i}\right)^{\zeta_1} \left(\frac{y}{q_i}\right)^{\theta_1} + \\
& \eta_1 (p_i t)^{\delta_1} (r_i x)^{\zeta_1} (q_i y)^{\theta_1} \eta_2 \left(\frac{t}{p_i}\right)^{\delta_2} \left(\frac{x}{r_i}\right)^{\zeta_2} \left(\frac{y}{q_i}\right)^{\theta_2} + \\
& \eta_2 (p_i t)^{\delta_2} (r_i x)^{\zeta_2} (q_i y)^{\theta_2} \eta_1 \left(\frac{t}{p_i}\right)^{\delta_1} \left(\frac{x}{r_i}\right)^{\zeta_1} \left(\frac{y}{q_i}\right)^{\theta_1} + \\
& \eta_2^2 (p_i t)^{\delta_2} (r_i x)^{\zeta_2} (q_i y)^{\theta_2} \left(\frac{t}{p_i}\right)^{\delta_2} \left(\frac{x}{r_i}\right)^{\zeta_2} \left(\frac{y}{q_i}\right)^{\theta_2} .
\end{aligned} \tag{4.20}$$

Utilizing the dispersion relation (4.17) in the equation (4.20), we get

$$\begin{aligned}
&P(D_1, D_2, D_3)\{\mathbf{f}^{(1)} \cdot \mathbf{f}^{(1)}\} \\
&\quad = \eta_1 \eta_2 t^{\delta_1+\delta_2} x^{\zeta_1+\zeta_2} y^{\theta_1+\theta_2} \sum_{i=1}^{2} \frac{\lambda_i}{2} [p_i^{\delta_1-\delta_2} r_i^{\zeta_1-\zeta_2} q_i^{\theta_1-\theta_2} + p_i^{\delta_2-\delta_1} r_i^{\zeta_2-\zeta_1} q_i^{\theta_2-\theta_1}],
\end{aligned} \tag{4.21}$$

and we can derive the form of $\mathbf{f}^{(2)}$ as

$$\mathbf{f}^{(2)} = A(1, 2)\eta_1 \eta_2 t^{\delta_1+\delta_2} x^{\zeta_1+\zeta_2} y^{\theta_1+\theta_2}. \tag{4.22}$$

One can explicitly discover the interaction coefficient $A(1, 2)$ by plugging (4.22) into (4.19)

$$-\eta_1\eta_2 A(1,2)\sum_{i=1}^{3}(p_i t)^{\delta_1+\delta_2}(r_i x)^{\zeta_1+\zeta_2}(q_i y)^{\theta_1+\theta_2}+\left(\frac{t}{p_i}\right)^{\delta_1+\delta_2}\left(\frac{x}{r_i}\right)^{\zeta_1+\zeta_2}\left(\frac{y}{q_i}\right)^{\theta_1+\theta_2}$$
$$=\eta_1\eta_2 t^{\delta_1+\delta_2}x^{\zeta_1+\zeta_2}y^{\theta_1+\theta_2}\sum_{i=1}^{3}\frac{\lambda_i}{2}[p_i^{\delta_1-\delta_2}r_i^{\zeta_1-\zeta_2}q_i^{\theta_1-\theta_2}+p_i^{-\delta_1+\delta_2}r_i^{-\zeta_1+\zeta_2}q_i^{-\theta_1+\theta_2}]. \tag{4.23}$$

Thus $A(1,2)$ is

$$\begin{aligned} A(1,2) &= -\frac{\sum_{i=1}^{3}\frac{\lambda_i}{2}[p_i^{\delta_1-\delta_2}r_i^{\zeta_1-\zeta_2}q_i^{\theta_1-\theta_2}+p_i^{-\delta_1+\delta_2}r_i^{-\zeta_1+\zeta_2}q_i^{-\theta_1+\theta_2}]}{\sum_{i=1}^{3}\frac{\lambda_i}{2}[p_i^{\delta_1+\delta_2}r_i^{\zeta_1+\zeta_2}q_i^{\theta_1+\theta_2}+p_i^{-\delta_1-\delta_2}r_i^{-\zeta_1-\zeta_2}q_i^{-\theta_1-\theta_2}]} \\ &= -\frac{P(v_1-v_2)}{P(v_1+v_2)}. \end{aligned} \tag{4.24}$$

Similar to the previous discussions for all $i \geq 3$, all terms $f^i = 0$ and taking $\varepsilon = 1$, two-q-solitons can be expressed as

$$f(t,x,y) = 1+\eta_1 t^{\delta_1}x^{\zeta_1}y^{\theta_1}+\eta_2 t^{\delta_2}x^{\zeta_2}y^{\theta_2}+A(1,2)\eta_1\eta_2 t^{\delta_1+\delta_2}x^{\zeta_1+\zeta_2}y^{\theta_1+\theta_2}. \tag{4.25}$$

For three-q-soliton solutions, we start with

$$f^{(1)}(t,x,y) = \sum_{i=1}^{3}\eta_i t^{\delta_i}x^{\zeta_i}y^{\theta_i}, \tag{4.26}$$

where $\eta, \delta_i, \eta_i, \zeta_i$ are constants for $i = 1, 2, 3$. We investigate the coefficient of ε^0

$$P(D_1, D_2, D_3)\{1\cdot 1\} = \sum_{i=1}^{3}\lambda_i\cosh(D_i)\{1.1\} = \lambda_1+\lambda_2+\lambda_3 = 0 \tag{4.27}$$

results from (4.6). The coefficient of ε^1 gives us

$$P(D_1, D_2, D_3)\{1\cdot f^{(1)} + f^{(1)}\cdot 1\} = -2P(\partial_1, \partial_2, \partial_3)f^{(1)}. \tag{4.28}$$

Writing (4.26) in the equation (4.28), we get

$$-2P(\partial_1,\partial_2,\partial_3)\{\eta_1 t^{\delta_1}x^{\zeta_1}y^{\theta_1}+\eta_2 t^{\delta_2}x^{\zeta_2}y^{\theta_2}+\eta_3 t^{\delta_3}x^{\zeta_3}y^{\theta_3}\}$$
$$=-\sum_{j=1}^{3}\sum_{i=1}^{3}\lambda_j(\eta_i t^{\delta_i+a_j}x^{\zeta_i+b_j}y^{\theta_i+c_j}+\eta_i t^{\delta_i-a_j}x^{\zeta_i-b_j}y^{\theta_i-c_j}), \tag{4.29}$$

which yileds the dispersion relation

$$P(\delta_j,\zeta_j,\theta_j)=\sum_{i=1}^{3}\frac{\lambda_i}{2}(p_i^{\delta_j}r_i^{\zeta_j}q_i^{\theta_j}+p_i^{-\delta_j}r_i^{-\zeta_j}q_i^{-\theta_j})=0, \quad \forall j=1,2,3. \tag{4.30}$$

From the coefficient of ε^2, we have

$$P(D_1,D_2,D_3)\{f^{(1)}\cdot f^{(1)}\}=-2P(\partial_1,\partial_2,\partial_3)f^{(2)}, \tag{4.31}$$

where

$$\begin{aligned}
&P(D_1,D_2,D_3)\{f^{(1)}\cdot f^{(1)}\}\\
&=\sum_{i=1}^{3}\frac{\lambda_i}{2}[\eta_1^2 t^{2\delta_1}x^{2\zeta_1}y^{2\theta_1}+\eta_1\eta_2 t^{\delta_1-\delta_2}x^{\zeta_1-\zeta_2}y^{\theta_1-\theta_2}(p_i^{\delta_1-\delta_2}r_i^{\zeta_1-\zeta_2}q_i^{\theta_1-\theta_2})\\
&+\eta_1\eta_3 t^{\delta_1-\delta_3}x^{\zeta_1-\zeta_3}y^{\theta_1-\theta_3}(p_i^{\delta_1-\delta_3}r_i^{\zeta_1-\zeta_3}q_i^{\theta_1-\theta_3})\\
&+\eta_1\eta_2 t^{-\delta_1+\delta_2}x^{-\zeta_1+\zeta_2}y^{-\theta_1+\theta_2}(p_i^{-\delta_1+\delta_2}r_i^{-\zeta_1+\zeta_2}q_i^{-\theta_1+\theta_2})+\eta_2^2 t^{2\delta_2}x^{2\zeta_2}y^{2\theta_2}\\
&+\eta_2\eta_3 t^{\delta_2-\delta_3}x^{\zeta_2-\zeta_3}y^{\theta_2-\theta_3}(p_i^{\delta_2-\delta_3}r_i^{\zeta_2-\zeta_3}q_i^{\theta_2-\theta_3})\\
&+\eta_1\eta_3 t^{-\delta_1+\delta_3}x^{-\zeta_1+\zeta_3}y^{-\theta_1+\theta_3}(p_i^{-\delta_1+\delta_3}r_i^{-\zeta_1+\zeta_3}q_i^{-\theta_1+\theta_3})\\
&+\eta_2\eta_3 t^{-\delta_2+\delta_3}x^{-\zeta_2+\zeta_3}y^{-\theta_2+\theta_3}(p_i^{-\delta_2+\delta_3}r_i^{-\zeta_2+\zeta_3}q_i^{-\theta_2+\theta_3})+\eta_3^2 t^{2\delta_3}x^{2\zeta_3}y^{2\theta_3}].
\end{aligned} \tag{4.32}$$

Using the dispersion relation (4.30) in (4.32), we obtain

$$-P(\partial)f^{(2)}=\sum_{i<j}^{(3)}\eta_i\eta_j t^{\delta_i+\delta_j}x^{\zeta_i+\zeta_j}y^{\theta_i+\theta_j}\left[\sum_{k=1}^{3}\frac{\lambda_k}{2}(q_k^{\delta_i-\delta_j}p_k^{\zeta_i-\zeta_j}r_k^{\theta_i-\theta_j}+q_k^{\delta_j-\delta_i}p_k^{\zeta_j-\zeta_i}r_k^{\theta_j-\theta_i})\right], \tag{4.33}$$

where $\sum_{i<j}^{(3)}$ is the summation of all elements for $i<j$ and $i,j=1,2,3$ and we

find the form of $\mathbf{f}^{(2)}$ as

$$\mathbf{f}^{(2)} = A(1,2)\eta_1\eta_2 \mathbf{t}^{\delta_1+\delta_2} x^{\zeta_1+\zeta_2} y^{\theta_1+\theta_2} + A(1,3)\eta_1\eta_3 \mathbf{t}^{\delta_1+\delta_3} x^{\zeta_1+\zeta_3} y^{\theta_1+\theta_3} + A(2,3)\eta_2\eta_3 \mathbf{t}^{\delta_2+\delta_3} x^{\zeta_2+\zeta_3} y^{\theta_2+\theta_3}. \quad (4.34)$$

More precisely,

$$\mathbf{f}^{(2)}(t,x,y) = \sum_{i<j}^{(3)} A(i,\mathbf{j})\eta_i\eta_j \mathbf{t}^{\delta_i+\delta_j} x^{\zeta_i+\zeta_j} y^{\theta_i+\theta_j}, \quad (4.35)$$

where $i < \mathbf{j}$, $i, \mathbf{j} = 1, 2, 3$ and

$$A(i,\mathbf{j}) = -\frac{\sum_{i=1}^{3} \frac{\lambda_i}{2}[q_i^{\delta_i-\delta_j} p_i^{\zeta_i-\zeta_j} \mathbf{r}_i^{\theta_i-\theta_j} + q_i^{-\delta_i+\delta_j} p_i^{-\zeta_i+\zeta_j} \mathbf{r}_i^{-\theta_i+\theta_j}]}{\sum_{i=1}^{3} \frac{\lambda_i}{2}[q_i^{\delta_i+\delta_j} p_i^{\zeta_i+\zeta_j} \mathbf{r}_i^{\theta_i+\theta_j} + q_i^{-\delta_i-\delta_j} p_i^{-\zeta_i-\zeta_j} \mathbf{r}_i^{-\theta_i-\theta_j}]} = -\frac{P(v_i - v_j)}{P(v_i + v_j)}. \quad (4.36)$$

Finally our aim is to obtain $\mathbf{f}^{(3)}$, so continue with the coefficient of ε^3, then

$$P(D_1, D_2, D_3)\{1 \cdot \mathbf{f}^{(3)} + \mathbf{f}^{(1)} \cdot \mathbf{f}^{(2)} + \mathbf{f}^{(2)} \cdot \mathbf{f}^{(1)} + \mathbf{f}^{(3)} \cdot 1\} = 0, \quad (4.37)$$

which can be written as

$$-P(D_1, D_2, D_3)\{\mathbf{f}^{(1)} \cdot \mathbf{f}^{(2)} + \mathbf{f}^{(2)} \cdot \mathbf{f}^{(1)}\} = P(D_1, D_2, D_3)\{1 \cdot \mathbf{f}^{(3)} + \mathbf{f}^{(3)}\}. \quad (4.38)$$

Consider the left hand side of (4.38)

$$\begin{aligned}
&P(D_1, D_2, D_3)\{\mathbf{f}^{(2)} \cdot \mathbf{f}^{(1)} + \mathbf{f}^{(1)} \cdot \mathbf{f}^{(2)}\} \\
&= 2P(D_1, D_2, D_3)\{(\eta_1 \mathbf{t}^{\delta_1} x^{\zeta_1} y^{\theta_1} + \eta_2 \mathbf{t}^{\delta_2} x^{\zeta_2} y^{\theta_2} + \eta_3 \mathbf{t}^{\delta_3} x^{\zeta_3} y^{\theta_3}) \cdot (A(1,2)\eta_1\eta_2 \mathbf{t}^{\delta_1+\delta_2} x^{\zeta_1+\zeta_2} y^{\theta_1+\theta_2} \\
&+ A(1,3)\eta_1\eta_3 \mathbf{t}^{\delta_1+\delta_3} x^{\zeta_1+\zeta_3} y^{\theta_1+\theta_3} + A(2,3)\eta_2\eta_3 \mathbf{t}^{\delta_2+\delta_3} x^{\zeta_2+\zeta_3} y^{\theta_2+\theta_3})\} \\
&= \sum_{i=1}^{3} \lambda_i \; A(1,2)\eta_1\eta_2 \mathbf{t}^{\delta_1+\delta_2} x^{\zeta_1+\zeta_2} y^{\theta_1+\theta_2}[\eta_1 \mathbf{t}^{\delta_1} x^{\zeta_1} y^{\theta_1}(p_i^{-\delta_1} \mathbf{r}_i^{-\zeta_1} q_i^{-\theta_1} + p_i^{\delta_1} \mathbf{r}_i^{\zeta_1} q_i^{\theta_1}) \\
&+ \eta_2 \mathbf{t}^{\delta_2} x^{\zeta_2} y^{\theta_2}(p_i^{-\delta_2} \mathbf{r}_i^{-\zeta_2} q_i^{-\theta_2} + p_i^{\delta_2} \mathbf{r}_i^{\zeta_2} q_i^{\theta_2}) + \eta_3 \mathbf{t}^{\delta_3} x^{\zeta_3} y^{\theta_3}(p_i^{-\delta_3} \mathbf{r}_i^{-\zeta_3} q_i^{-\theta_3} + p_i^{\delta_3} \mathbf{r}_i^{\zeta_3} q_i^{\theta_3})] \\
&+ A(1,3)\eta_1\eta_3 \mathbf{t}^{\delta_1+\delta_3} x^{\zeta_1+\zeta_3} y^{\theta_1+\theta_3}[\eta_1 \mathbf{t}^{\delta_1} x^{\zeta_1} y^{\theta_1}(p_i^{-\delta_1} \mathbf{r}_i^{-\zeta_1} q_i^{-\theta_1} + p_i^{\delta_1} \mathbf{r}_i^{\zeta_1} q_i^{\theta_1})
\end{aligned}$$

$$+\eta_2 t^{\delta_2} x^{\zeta_2} y^{\theta_2}(p_i^{-\delta_2} r_i^{-\zeta_2} q_i^{-\theta_2} + p_i^{\delta_2} r_i^{\zeta_2} q_i^{\theta_2}) + \eta_3 t^{\delta_3} x^{\zeta_3} y^{\theta_3}(p_i^{-\delta_3} r_i^{-\zeta_3} q_i^{-\theta_3} + p_i^{\delta_3} r_i^{\zeta_3} q_i^{\theta_3})]$$
$$+ A(2,3)\eta_2\eta_3 t^{\delta_2+\delta_3} x^{\zeta_2+\zeta_3} y^{\theta_2+\theta_3}[\eta_1 t^{\delta_1} x^{\zeta_1} y^{\theta_1}(p_i^{-\delta_1} r_i^{-\zeta_1} q_i^{-\theta_1} + p_i^{\delta_1} r_i^{\zeta_1} q_i^{\theta_1})$$
$$+\eta_2 t^{\delta_2} x^{\zeta_2} y^{\theta_2}(p_i^{-\delta_2} r_i^{-\zeta_2} q_i^{-\theta_2} + p_i^{\delta_2} r_i^{\zeta_2} q_i^{\theta_2}) + \eta_3 t^{\delta_3} x^{\zeta_3} y^{\theta_3}(p_i^{-\delta_3} r_i^{-\zeta_3} q_i^{-\theta_3} + p_i^{\delta_3} r_i^{\zeta_3} q_i^{\theta_3})] \ . \tag{4.39}$$

It can be seen that the summation (4.39) is simplified by the dispersion relation (4.30) and we get

$$\begin{aligned} &P(D_1, D_2, D_3)\{f^{(1)} \cdot f^{(2)} + f^{(2)} \cdot f^{(1)}\} \\ &= \sum_{i=1}^{3} \lambda_i \ \eta_1\eta_2\eta_3 t^{\delta_1+\delta_2+\delta_3} x^{\zeta_1+\zeta_2+\zeta_3} y^{\theta_1+\theta_2+\theta_3}[A(2,3)(p_i^{\delta_1-\delta_2-\delta_3} \\ &r_i^{\zeta_1-\zeta_2-\zeta_3} q_i^{\theta_1-\theta_2-\theta_3} + p_i^{-\delta_1+\delta_2+\delta_3} r_i^{-\zeta_1+\zeta_2+\zeta_3} q_i^{-\theta_1+\theta_2+\theta_3}) \\ &+ A(1,3)(p_i^{-\delta_1+\delta_2-\delta_3} r_i^{-\zeta_1+\zeta_2-\zeta_3} q_i^{-\theta_1+\theta_2-\theta_3} + p_i^{\delta_1-\delta_2+\delta_3} \\ &r_i^{\zeta_1-\zeta_2+\zeta_3} q_i^{\theta_1-\theta_2+\theta_3}) + A(1,2)(p_i^{\delta_1+\delta_2-\delta_3} r_i^{\zeta_1+\zeta_2-\zeta_3} q_i^{\theta_1+\theta_2-\theta_3} \\ &+ p_i^{-\delta_1-\delta_2+\delta_3} r_i^{-\zeta_1-\zeta_2+\zeta_3} q_i^{-\theta_1-\theta_2+\theta_3})] \ . \end{aligned} \tag{4.40}$$

By the equality (4.38), we can say that $f^{(3)}$ is of the form

$$f^{(3)} = A(1,2,3)\eta_1\eta_2\eta_3 t^{\delta_1+\delta_2+\delta_3} x^{\zeta_1+\zeta_2+\zeta_3} y^{\theta_1+\theta_2+\theta_3}. \tag{4.41}$$

To see the connection between $A(1,2)$, $A(1,3)$, $A(2,3)$ and $A(1,2,3)$ we expand the right hand side of (4.38) as

$$\begin{aligned} &P(D_1, D_2, D_3)\{1 \cdot f^{(3)} + f^{(3)} \cdot 1\} = 2P(\partial_1, \partial_2, \partial_3) f^{(3)} \\ &= P(\partial_1, \partial_2, \partial_3) A(1,2,3)\eta_1\eta_2\eta_3 t^{\delta_1+\delta_2+\delta_3} x^{\zeta_1+\zeta_2+\zeta_3} y^{\theta_1+\theta_2+\theta_3} \\ &= \sum_{i=1}^{3} \frac{\lambda_i}{2}[p_i^{\delta_1+\delta_2+\delta_3} r_i^{\zeta_1+\zeta_2+\zeta_3} q_i^{\theta_1+\theta_2+\theta_3} \\ &+ p_i^{-\delta_1-\delta_2-\delta_3} r_i^{-\zeta_1-\zeta_2-\zeta_3} q_i^{-\theta_1-\theta_2-\theta_3}] \\ &\times A(1,2,3)\eta_1\eta_2\eta_3 t^{\delta_1+\delta_2+\delta_3} x^{\zeta_1+\zeta_2+\zeta_3} y^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{4.42}$$

which implies the relationship between the coefficients $A(1,2)$, $A(1,3)$, and $A(2,3)$

with $A(1,2,3)$.

$$
\begin{aligned}
&A(1,2,3)p_i^{\delta_1+\delta_2+\delta_3}r_i^{\zeta_1+\zeta_2+\zeta_3}q_i^{\theta_1+\theta_2+\theta_3}+p_i^{-\delta_1-\delta_2-\delta_3}r_i^{-\zeta_1-\zeta_2-\zeta_3}q_i^{-\theta_1-\theta_2-\theta_3}\\
&=-\ A(1,2)(p_i^{\delta_1+\delta_2-\delta_3}r_i^{\zeta_1+\zeta_2-\zeta_3}q_i^{\theta_1+\theta_2-\theta_3}+p_i^{-\delta_1-\delta_2+\delta_3}r_i^{-\zeta_1-\zeta_2+\zeta_3}q_i^{-\theta_1-\theta_2+\theta_3})\\
&+A(1,3)(p_i^{\delta_1-\delta_2+\delta_3}r_i^{\zeta_1-\zeta_2+\zeta_3}q_i^{\theta_1-\theta_2+\theta_3}+p_i^{-\delta_1+\delta_2-\delta_3}r_i^{-\zeta_1+\zeta_2-\zeta_3}q_i^{-\theta_1+\theta_2-\theta_3})\\
&+A(2,3)(p_i^{\delta_1-\delta_2-\delta_3}r_i^{\zeta_1-\zeta_2-\zeta_3}q_i^{\theta_1-\theta_2-\theta_3}+p_i^{-\delta_1+\delta_2+\delta_3}r_i^{-\zeta_1+\zeta_2+\zeta_3}q_i^{-\theta_1+\theta_2+\theta_3})
\end{aligned}
\tag{4.43}
$$

leading to

$$
\begin{aligned}
A(1,2,3)=-&\frac{A(1,2)P(v_3-v_1-v_2)+A(1,3)P(v_2-v_1-v_3)}{P(v_1+v_2+v_3)}\\
&+\frac{A(2,3)P(v_1-v_2-v_3)}{P(v_1+v_2+v_3)}
\end{aligned}
\tag{4.44}
$$

If we write the coefficient of ε^4, then we know that $f^{(4)}=0$, but it reveals us another expression for $A(1,2,3)$

$$
A(1,2,3)=A(1,2)A(1,3)A(2,3). \tag{4.45}
$$

The equivalence of (4.44) and (4.45) for A(1, 2, 3) provides the three-soliton solution condition (see [35]) (3.32). In conclusion, the condition (3.32) on P, guarantees the existence of three-q-soliton solutions. Thus, we deduce the three-q-soliton solutions as

$$
\begin{aligned}
f(x,t)=1+&\sum_{i=1}^{3}\eta_i t^{\delta_i}x^{\zeta_i}y^{\theta_i}+\sum_{i<j}^{(3)}A(i,j)\eta_i\eta_j t^{\delta_i+\delta_j}x^{\zeta_i+\zeta_j}y^{\theta_i+\theta_j}\\
&+A(1,2)A(1,3)A(2,3)\eta_1\eta_2\eta_3 t^{\delta_1+\delta_2+\delta_3}x^{\zeta_1+\zeta_2+\zeta_3}y^{\theta_1+\theta_2+\theta_3}.
\end{aligned}
\tag{4.46}
$$

4.2 Reductions

In the previous section, we present q-discrete analogue of Hirota-Miwa equation (4.1) and find its three q-soliton solutions. Now, our aim is to obtain its special cases, i.e., q-analogues of Toda, KdV and sine-Gordon equations by determining suitable reductions on (4.1).

4.2.1 The q-difference-q-difference Toda equation

In [33], the q-difference-q-difference Toda equation was introduced and its three-q-soliton solutionsare developed.In this section, our goal is to reconstruct q-difference-q-difference Toda equation by a proper reduction on the generalized equation (4.1). For this purpose, we set

$$D_1 = h\tau D_\tau, \quad D_2 = \bar{h}yD_y, \quad D_3 = 0, \quad \lambda_1 = 2h^{-1}, \quad \lambda_2 = -2, \quad \lambda_3 = 2 - 2h^{-1}, \tag{4.47}$$

in (4.1) which results

$$\begin{aligned} P(D_1, D_2, D_3)\{f.f\} &= \sum_{i=1}^{3} \lambda_i \cosh(D_i)\{f.f\} \\ &= [2h^{-1}\cosh(h\tau D_\tau - 2\cosh(\bar{h}yD_y) + (2 - 2h^{-1})]\{f.f\} \\ &= [h^{-1}(e^{h\tau D_\tau} + e^{-h\tau D_\tau} - 2) - (e^{\bar{h}yD_y} + e^{-\bar{h}yD_y} - 2)]\{f.f\}, \end{aligned} \tag{4.48}$$

and leads us Hirota bilinear form of q-difference-q-difference Toda equation [33] as

$$\begin{aligned} &h^{-1}(e^{h\tau D_\tau} + e^{-h\tau D_\tau} - 2) \\ &\quad - (e^{\bar{h}yD_y} + e^{-\bar{h}yD_y} - 2)\ \{f(\tau, y) \cdot f(\tau, y)\} = 0. \end{aligned} \tag{4.49}$$

Next we prove that Hirota bilinear form (4.49) generalizes the continuous case as follows.

Proposition 4.2 [33] Hirota bilinear form of the q-difference-q-difference Toda equation

$$\begin{aligned} &h^{-1}(e^{h\tau D_\tau} + e^{-h\tau D_\tau} - 2) \\ &\quad - (e^{\bar{h}yD_y} + e^{-\bar{h}yD_y} - 2)\ \{f(\tau, y) \cdot f(\tau, y)\} = 0 \end{aligned} \tag{4.50}$$

reduces to Hirota bilinear form of the differential-q-difference Toda equation

$$[D_t{}^2 - (e^{(\bar{h}yD_y)} + e^{(-\bar{h}yD_y)} - 2)]\{f(t, y) \cdot f(t, y)\} = 0, \tag{4.51}$$

as h tends to zero.

Proof. To prove the proposition, we interchange h by h^2 in (4.50) and set $\tau = e^{ht}$, then $D_\tau = he^{ht}D_t$ implies $D_t = h\tau D_\tau$. Putting these transformations in the left hand side of the equation (4.50), we get,

$$\begin{aligned} h^{-1}(e^{h\tau D_\tau} + e^{-h\tau D_\tau} - 2) - (e^{\bar{h}yD_y} + e^{-\bar{h}yD_y} - 2) \\ = h^{-2}(e^{hD_t} + e^{-hD_t} - 2) - (e^{\bar{h}yD_y} + e^{-\bar{h}yD_y} - 2). \end{aligned} \tag{4.52}$$

Finally, if we consider the limit as $h \to 0$,

$$\lim_{h\to 0} \frac{e^{hD_t} + e^{-hD_t} - 2}{h^2} - (e^{\bar{h}yD_y} + e^{-\bar{h}yD_y} - 2) = D_t^2 - (e^{\bar{h}yD_y} + e^{-\bar{h}yD_y} - 2) \tag{4.53}$$

we end up with (4.51). □

Next we present the standard form of q-difference-q-difference Toda equation, stated in [33].

Proposition 4.3 [33] The standard form of the q-difference-q-difference Toda equation is

$$\Lambda_\tau^2 \log(1 + V(\tau, y)) = \Lambda_y^2 \log(1 + hV(\tau, y)). \tag{4.54}$$

Proof. We begin with (4.50) and expand the operators

$$\begin{aligned} h^{-1}[f(q\tau, y) \cdot f(\frac{\tau}{q}, y) + f(\frac{\tau}{q}, y) \cdot f(q\tau, y) - 2f(\tau, y)^2] \\ = f(\tau, \bar{q}y) \cdot f(\tau, \frac{y}{\bar{q}}) + f(\tau, \frac{y}{\bar{q}}) \cdot f(\tau, \bar{q}y) - 2f(\tau, y)^2, \end{aligned} \tag{4.55}$$

which implies

$$h^{-1} \frac{f(q\tau, y) \cdot f(\frac{\tau}{q}, y)}{f(\tau, y)^2} - 1 = \frac{f(\tau, \bar{q}y) \cdot f(\tau, \frac{y}{\bar{q}})}{f(\tau, y)^2} - 1. \tag{4.56}$$

Let us define the following transformation

$$V(\tau, y) := h^{-1} \frac{f(q\tau, y) \cdot f(\frac{\tau}{q}, y)}{f(\tau, y)^2} - 1 = \frac{f(\tau, \bar{q}y) \cdot f(\tau, \frac{y}{\bar{q}})}{f(\tau, y)^2} - 1. \tag{4.57}$$

To reach our aim, we use central-difference operator which is given in (3.11) and evaluate from the right hand side of (4.57)

$$\log(1+V(\tau,y)) = \log\left(\frac{f(\tau,\bar{q}y)\cdot f(\tau,\frac{y}{q})}{f(\tau,y)^2}\right) = \log f(\tau,\bar{q}y) + \log f\left(\tau,\frac{y}{\bar{q}}\right) - 2\log f(\tau,y)$$

leading to

$$\log(1+V(\tau,y)) = \Lambda_y^2 \log f\left(\tau, y\right), \tag{4.58}$$

and from the left hand side of (4.57) we get

$$\log(1+hV(\tau,y)) = \log\left(\frac{f(q\tau,y)\cdot f(\frac{\tau}{q},y)}{f(\tau,y)^2}\right) = \log f(q\tau,y) + \log f(\frac{\tau}{q},y) - 2\log f(\tau,y)$$

which is equivalent to

$$\log(1+hV(\tau,y)) = \Lambda_\tau^2 \log f(\tau,y). \tag{4.59}$$

Since $\log(1+V(\tau,y)) = \log(1+hV(\tau,y))$ from (4.57), the standard form is written as

$$\Lambda_\tau^2 \log(1+V(\tau,y)) = \Lambda_y^2 \log(1+hV(\tau,y)). \tag{4.60}$$

□

Up to now, we have found standard form and Hirota bilinear form of the equation. The next is to mention about its q-soliton solutions which were presented already in [33]. Here, we recompute them by using the reductions (4.47) in the findings of the Section 4.1.

Proposition 4.4 [35] We present one-q-soliton solution of q-difference-q-difference Toda equation (4.54) as

$$V(\tau,y) = \frac{\eta\tau^\delta y^\zeta[\bar{q}^\zeta + \bar{q}^{-\zeta} - 2]}{(1+\eta\tau^\delta y^\zeta)^2}, \tag{4.61}$$

with the dispersion relation

$$h^{-1}(q^\delta + q^{-\delta} - 2) = \bar{q}^\zeta + \bar{q}^{-\zeta} - 2 \tag{4.62}$$

is satisfied.

Proof. From the reductions (4.47), it is obviously seen that we have the below identifications

$$\begin{aligned} t=\tau, \quad x=y, \quad \theta=0, \quad a_1=h, \quad b_2=\bar{h}, \\ a_2=a_3=b_1=b_3=c_1=c_2=c_3=0. \end{aligned} \tag{4.63}$$

We will use these identifications to obtain one-q-soliton solution and the dispersion relation. We substitute above reductions into this $f^{(1)}$ to conclude

$$f^{(1)}=\eta t^{\delta} y^{\zeta}. \tag{4.64}$$

We verify the dispersion relation (4.62) by putting (4.64) into Hirota bilinear form (4.49), then we have

$$[h^{-1}(e^{(h\tau D_\tau)}+e^{(-h\tau D_\tau)}-2)-(e^{(\bar{h}\, yD_y)}+e^{(-\bar{h}yD_y)}-2)]\{\eta\tau^{\delta}y^{\zeta}\}=0,$$

which implies (4.62) immediately

$$h^{-1}(q^{\delta}+q^{-\delta}-2)-(\bar{q}^{\zeta}+\bar{q}^{-\zeta}-2)=0$$

where $e^h=q$ and $e^{\bar{h}}=\bar{q}$.

Secondly, if we put $f^{(1)}=1+\eta t^{\delta}y^{\zeta}$ into (4.57), we get

$$V(\tau,y)=\frac{f(\tau,\bar{q}y)\cdot f(\tau,\frac{y}{\bar{q}})}{f(\tau,y)^2}-1=\frac{\eta\tau^{\delta}y^{\zeta}(\bar{q}^{\zeta}+\bar{q}^{-\zeta}-2)}{(1+\eta t^{\delta}y^{\zeta})^2} \tag{4.65}$$

which finishes the proof. □

Remark. With the conditions τ, y □ q^{Z}, namely $\tau=q^n$ and $y=\bar{q}^m$, n, m □ Z, the q-difference-q-difference Toda equation (4.54) turns out to be

$$\begin{aligned} &\frac{(1+V(q^{n+1},(\bar{q})^m))(1+V(q^{n-1},(\bar{q})^m))}{(1+V(q^n,(\bar{q})^m))^2} \\ &=\frac{(1+hV(q^n,(\bar{q})^{m+1}))(1+hV(q^n,(\bar{q})^{m-1}))}{(1+hV(q^n,(\bar{q})^m))^2}, \end{aligned} \tag{4.66}$$

and it has one-q soliton solution explicitly

$$V(\tau, y) = \frac{\eta q^{n\delta}(\bar{q})^{m\zeta}\,[(\bar{q})^{\zeta} + (\bar{q})^{-\zeta} - 2]}{(1 + \eta q^{n\delta}(\bar{q})^{m\zeta})^2}. \tag{4.67}$$

Subsequently, two-q-soliton and three-q-soliton solutions can be written by using the reductions and identifications, respectively

$$f(\tau, y) = 1 + \eta_1 \tau^{\delta_1} y^{\zeta_1} + \eta_2 \tau^{\delta_2} y^{\zeta_2} + A(1,2)\eta_1\eta_2 \tau^{\delta_1+\delta_2} y^{\zeta_1+\zeta_2}, \tag{4.68}$$

where

$$A(1,2) = -\frac{h^{-1}(q^{\delta_1-\delta_2} + q^{-\delta_1+\delta_2} - 2) - (\bar{q}^{\zeta_1-\zeta_2} + \bar{q}^{-\zeta_1+\zeta_2} - 2)}{h^{-1}(q^{\delta_1+\delta_2} + q^{-\delta_1-\delta_2} - 2) - (\bar{q}^{\zeta_1+\zeta_2} + \bar{q}^{-\zeta_1-\zeta_2} - 2)},$$

and

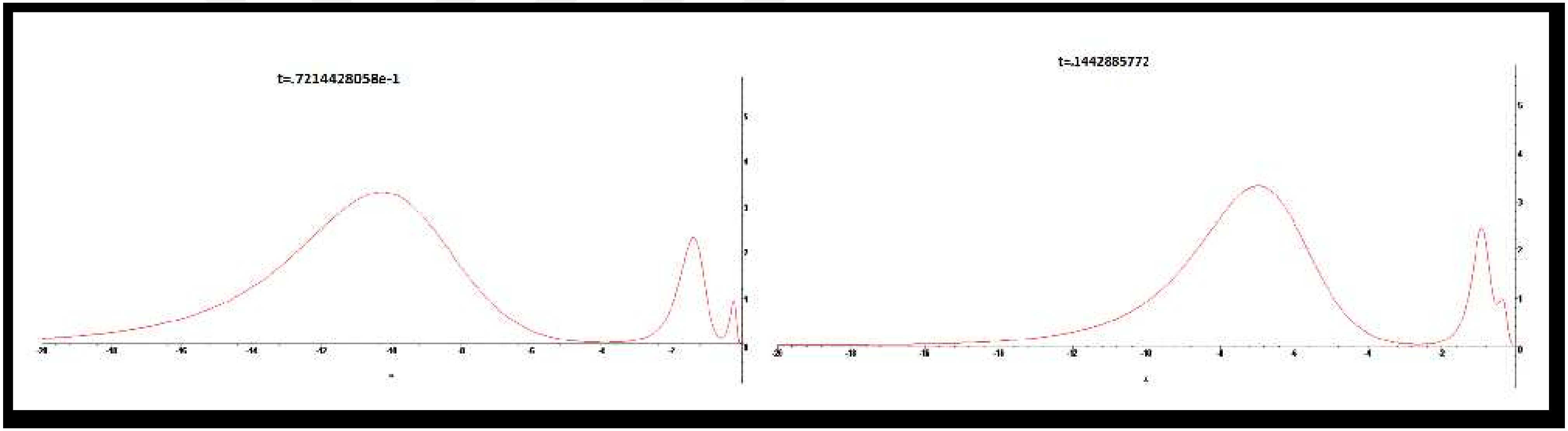
$$f(\tau, y) = 1 + \sum_{i=1}^{3} \eta_i \tau^{\delta_i} y^{\zeta_i} + \sum_{i<j}^{(3)} A(i,j)\eta_i\eta_j \tau^{\delta_i+\delta_j} y^{\zeta_i+\zeta_j}$$
$$+ A(1,2)A(1,3)A(2,3)\eta_1\eta_2\eta_3 \tau^{\delta_1+\delta_2+\delta_3} y^{\zeta_1+\zeta_2+\zeta_3},$$

where

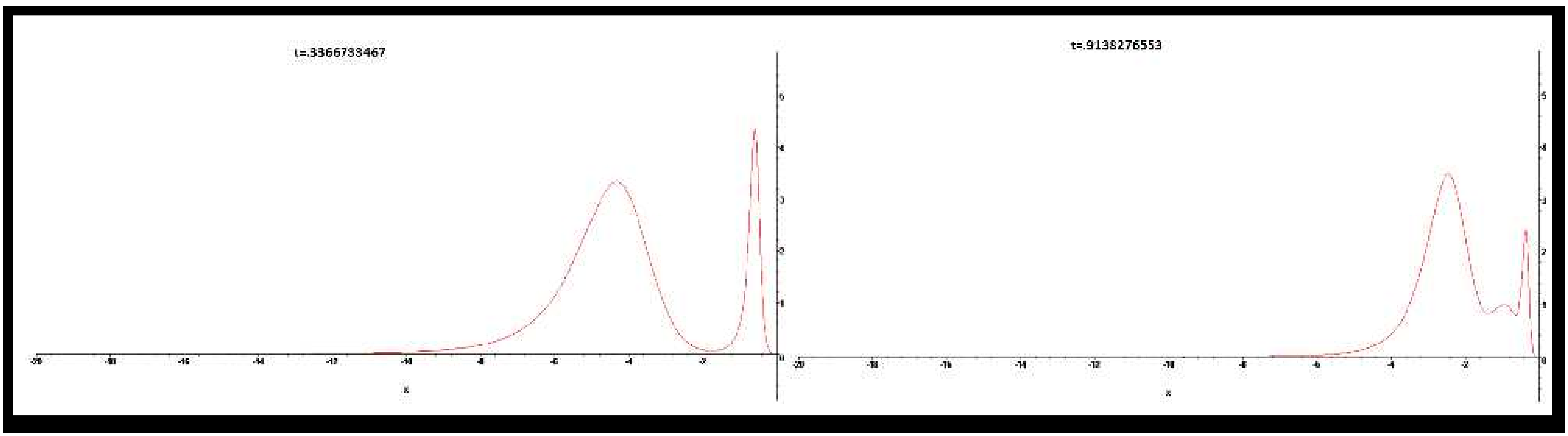
$$A(i,j) = -\frac{h^{-1}(q^{\delta_i-\delta_j} + q^{-\delta_i+\delta_j} - 2) - (\bar{q}^{\zeta_i-\zeta_j} + \bar{q}^{-\zeta_i+\zeta_j} - 2)}{h^{-1}(q^{\delta_i+\delta_j} + q^{-\delta_i-\delta_j} - 2) - (\bar{q}^{\zeta_i+\zeta_j} + \bar{q}^{-\zeta_i-\zeta_j} - 2)}, \quad i < j \quad i, j = 1, 2, 3.$$

Remark. We present the graph of three-q-soliton solutions of q-difference-q-difference Toda equation. The solitonic behavior of the waves can be observed from the graph. Since the form of the solutions is polynomials in power functions, we conclude that the length of the wave increases as x increases. In the graph we set $q = 1,25$, $\bar{q} = 2$, $h = \ln(q)$, $\delta_1 = 6$, $\delta_2 = 4$, and $\delta_3 = -7$ then from the dispersion relation we get $\zeta_1 = 3.487750814$, $\zeta_2 = -2.494747127$ and $\zeta_3 = -3.933630763$.

Figure 1: Three-q-soliton solutions of q-difference-q-difference Toda Equation

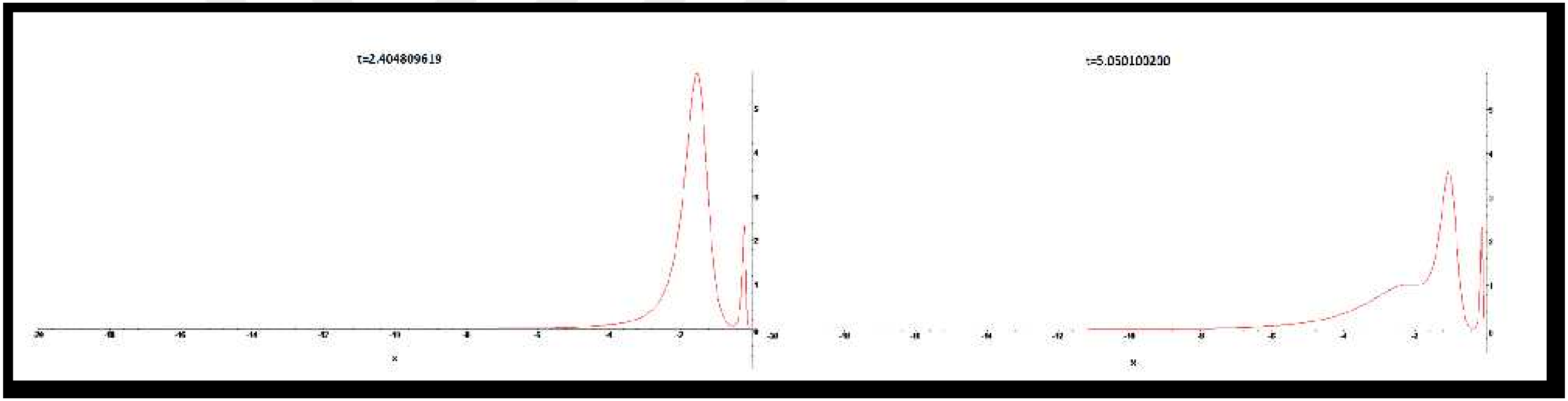


(a) (b)

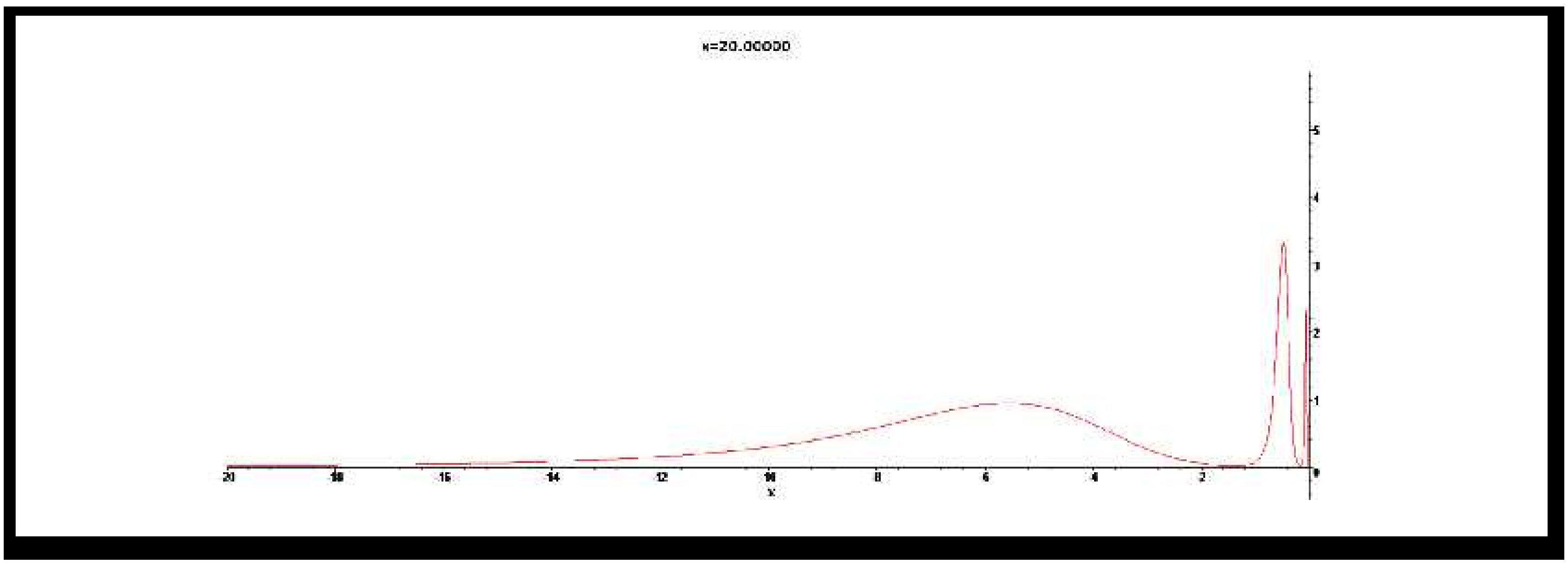


(c) (d)

Figure 1: Three-q-soliton solutions of q-difference-q-difference Toda Equation



(e) (f)



(g)

4.2.2 The q-difference-q-difference KdV equation

In this section, our first step is to find Hirota bilinear form of q-difference-q-difference KdV equation from the generalized equation (4.1). For this purpose we have to propose a Hirota bilinear form for the q-difference-q-difference KdV in a way that it reduces to the continuous one. Using the following reductions [35]

$$D_1 = \frac{1}{2}(3h^2\tau D_\tau + \bar{h}^2 y D_y),\ D_2 = \frac{1}{2}(h^2\tau D_\tau + 3\bar{h}^2 y D_y),\ D_3 = \frac{1}{2}(h^2\tau D_\tau - \bar{h}^2 y D_y),$$
$$\lambda_1 = 1, \quad \lambda_2 = 2h, \quad \lambda_3 = -1 - 2h, \tag{4.69}$$

in (4.1), we get

$$\sum_{i=1}^{3} \lambda_i \cosh(D_i)\{f \cdot f\} = [\lambda_1 \cosh(D_1) + \lambda_2 \cosh(D_2) + \lambda_3 \cosh(D_3)]\{f \cdot f\}$$
$$= \cosh(\frac{1}{2}(3h^2\tau D_\tau + \bar{h}^2 y D_y)) + 2h\cosh(\frac{1}{2}(h^2\tau D_\tau + 3\bar{h}^2 y D_y))$$
$$+ (-1-2h)\cosh(\frac{1}{2}(h^2\tau D_\tau - \bar{h}^2 y D_y))\{f.f\}$$
$$= \sinh\frac{h^2\tau D_\tau + \bar{h}^2 y D_y}{2}[h^{-1}\sinh(h^2\tau D_\tau) + 2\sinh(\bar{h}^2 y D_y)]\{f.f\}.$$

In conclusion

$$\sinh\frac{h^2\tau D_\tau + \bar{h}^2 y D_y}{2}[h^{-1}\sinh(h^2\tau D_\tau) + 2\sinh \bar{h}^2 y D_y]\{f(\tau, y)\cdot f(\tau, y)\} = 0, \tag{4.70}$$

is offered to be Hirota bilinear form of the q-difference-q-difference KdV equation.

Proposition 4.5 [35] Hirota bilinear form of q-difference-q-difference KdV equation

$$\sinh\frac{h^2\tau D_\tau + \bar{h}^2 y D_y}{2}[h^{-1}\sinh(h^2\tau D_\tau) + 2\sinh(\bar{h}^2 y D_y)]\{f(\tau, y)\cdot f(\tau, y)\} = 0, \tag{4.71}$$

recovers Hirota bilinear form of the continuous KdV equation [14]

$$[D_x(D_t + D_x^3)]\{f(t, x)\cdot f(t, x)\} = 0, \tag{4.72}$$

by taking small limit of h, $\bar{h}$.

Proof. Beginning with the equation (4.71), we have to reach the bilinear form (4.72). For this purpose, let $\tau = e^{ht}$, then (4.70) becomes

$$\sinh\left(\frac{hD_t + \bar{h}^2 yD_y}{2}\right)[h^{-1}\sinh(hD_t) + 2\sinh(\bar{h}^2 yD_y)]\{f \cdot f\} = 0. \tag{4.73}$$

Taking limit as h tends to 0 in (4.73), we obtain

$$\sinh\left(\frac{\bar{h}^2 yD_y}{2}\right)[D_t + 2\sinh(\bar{h}^2 yD_y)]\{f(t, y) \cdot f(t, y)\} = 0. \tag{4.74}$$

Secondly, setting $y = e^{\bar{h}x}$ implies $D_x = \bar{h}yD_y$, we get

$$\sinh\left(\frac{\bar{h}D_x}{2}\right)[D_t + 2\sinh(\bar{h}D_x)]\{f(t, x) \cdot f(t, x)\} = 0. \tag{4.75}$$

Last transition $D_t = \frac{\bar{h}^3 D_t}{3} - 2\bar{h}D_x$ in (4.75) gives us

$$\sinh\left(\frac{\bar{h}D_x}{2}\right)\left(\frac{\bar{h}^3 D_t}{3} - 2\bar{h}D_x + 2\sinh(\bar{h}D_x)\right)\{f \cdot f\} = 0. \tag{4.76}$$

To reach the continuous KdV equation in Hirota bilinear form (4.72), finally we divide (4.76) by $\bar{h}^4$ we have

$$\lim_{h\to 0}\left(\frac{e^{\bar{h}D_x/2} - e^{-\bar{h}D_x/2}}{2\bar{h}^4}\right)\left(\frac{\bar{h}^3 D_t}{3} - 2\bar{h}D_x + e^{\bar{h}}D_x - e^{-\bar{h}D_x}\right) = \frac{1}{6}(D_t D_x + D_x^4) \tag{4.77}$$

by using L'Hospitals Rule four times. □

Hence (4.71) is the general form of Hirota bilinear form of (4.72).

After showing the bilinear form of q-difference-q-difference KdV equation, our aim is to find its standard form but first we give basic properties and definitions to describe the details.

Lemma 4.6 [21] Let $f, g \in C^\infty(R)$. Then the followings hold

(i) $\sinh\frac{1}{2}(D_2-D_3)\{\sinh\frac{1}{2}(D_2+D_3)\cdot 2\sinh D_1 f\cdot f\}.\{\cosh(\frac{1}{2}(D_2+D_3)-D_1)f\cdot f\}=\sinh D_1\{\cosh D_2 f\cdot f\}\cdot\{\cosh D_3 f\cdot f\}$,

(ii) $\cosh D_1\{\cosh D_2 f\cdot f\}\cdot\{\cosh D_2 f\cdot f\}=\cosh D_2\{\cosh D_1 f\cdot f\}.\{\cosh D_1 f\cdot f\}$,

(iii) $e^{\beta\partial_x}(\frac{f}{g})=e^{\beta D_x}\{f\}.\{\frac{g}{\cosh(\beta D_x)g.g}\}$,

where D_i's satisfy (4.2) and β is a constant.

Proof. We prove the second and third items, since the first one is similar.

(ii) Beginning with $\cosh D_2\{f(t,x)\cdot f(t,x)\}=f(t+p_2,x+r_2)f(t-p_2,x-r_2)$ gives us

$$\begin{aligned}&\cosh D_1\{\cosh D_2 f\cdot f\}\cdot\{\cosh D_2 f\cdot f\}\\&=f(t+p_1+p_2,x+r_1+r_2)f(t+p_2-p_1,x+r_2-r_1)\\&f(t-p_2+p_1,x-r_2+r_1)f(t-p_2-p_1,x-r_2-r_1).\end{aligned}\tag{4.78}$$

On the other hand

$$\begin{aligned}&\cosh D_2\{\cosh D_1 f\cdot f\}\cdot\{\cosh D_1 f\cdot f\}\\&=f(t+p_2+p_1,x+r_2+r_1)f(t+p_2-p_1,x+r_2-r_1)\\&f(t-p_2+p_1,x-r_2+r_1)f(t-p_2-p_1,x-r_2-r_1).\end{aligned}\tag{4.79}$$

The equivalence of the equations (4.78) and (4.79) ends the proof of (ii).

(iii) We begin with the right hand side of $e^{\beta\partial_x}(\frac{f}{g})=\frac{e^{\beta D_x}\{f.g\}}{\cosh(\beta D_x)g.g}$,

$$e^{\beta D_x}\{f.g\}=f(x+\beta)g(x-\beta),\tag{4.80}$$

and

$$\cosh(\beta D_x)\{g.g\}=g(x+\beta)g(x-\beta).\tag{4.81}$$

Dividing equation (4.80) to (4.81), the left hand side is obtained clearly as

$$\frac{f(x+\beta)}{g(x+\beta)} = e^{\beta\partial_x}(\frac{f}{g}).$$

□

Definition. [35] The q-difference operator δ_x, acting on a function $h(x)$, is given as

$$\delta_x h(x) := h(qx) - h(\frac{x}{q}), \quad x \in R, \quad q = 1. \tag{4.82}$$

Proposition 4.7 [35] We present the standard form of the q-difference-q-difference KdV equation

$$\delta_\tau(\frac{1}{V(\tau, y)}) = -2h^{1/2}\delta_y V(\tau, y), \tag{4.83}$$

with the dependent variable transformation

$$V(\tau, y) := -\frac{f(\tau, \bar{q}y)f(\tau, \frac{y}{q})}{f(q\tau, y)f(\frac{\tau}{q}, y)}, \tag{4.84}$$

where $e^h = q$ and $e^{\bar{h}} = \bar{q}$.

Proof. In order to facilitate our work, we begin with an equivalent form of q-difference-q-difference KdV equation (4.71) and simplify it by changing h^2 and $\bar{h}^2$ by h and $\bar{h}$, respectively. Then we obtain

$$\begin{aligned} &\sinh\frac{h\tau D_\tau - \bar{h}yD_y}{2}\{\sinh\frac{h\tau D_\tau + \bar{h}yD_y}{2}[h^{-1/2}\sinh(h\tau D_\tau) \\ &+ 2\sinh(\bar{h}yD_y)]f\cdot f\}\cdot\{\cosh\frac{h\tau D_\tau - \bar{h}yD_y}{2}f\cdot f\} = 0. \end{aligned} \tag{4.85}$$

We separate the equation (4.85)

$$\sinh\left(\frac{h\tau D_\tau - \bar{h}yD_y}{2}\right)\{\sinh\left(\frac{h\tau D_\tau + \bar{h}yD_y}{2}\right)\cdot h^{-1/2}\sinh(h\tau D_\tau)f\cdot f\} \cdot\{\cosh\left(\frac{h\tau D_\tau - \bar{h}yD_y}{2}\right)f\cdot f\}$$
$$= -\sinh\left(\frac{h\tau D_\tau - \bar{h}yD_y}{2}\right)\{\sinh\left(\frac{h\tau D_\tau + \bar{h}yD_y}{2}\right)\cdot 2\sinh(\bar{h}yD_y)]f\cdot f\} \cdot\{\cosh\left(\frac{h\tau D_\tau - \bar{h}yD_y}{2}\right)f\cdot f\} \tag{4.86}$$

In equation (4.86), using the property of property (i) of Lemma with $D_1 = h\tau D_\tau$, $D_2 = h\tau D_\tau$ and $D_3 = \bar{h}yD_y$, left hand side of equality becomes

$$\sinh\left(\frac{h\tau D_\tau - \bar{h}yD_y}{2}\right)\{\sinh\left(\frac{h\tau D_\tau + \bar{h}yD_y}{2}\right)\cdot h^{-1/2}\sinh(h\tau D_\tau)f\cdot f\} \cdot\{\cosh\left(\frac{h\tau D_\tau - \bar{h}yD_y}{2}\right)f\cdot f\}$$
$$= h^{-1/2}\sinh(h\tau D_\tau)\{\cosh(h\tau D_\tau)f\cdot f\}.\{\cosh(\bar{h}yD_y)f\cdot f\} \tag{4.87}$$

and the right hand side arises

$$-\sinh\left(\frac{h\tau D_\tau - \bar{h}yD_y}{2}\right)\{\sinh\left(\frac{h\tau D_\tau + \bar{h}yD_y}{2}\right)\cdot 2\sinh(\bar{h}yD_y)]f\cdot f\} \cdot\{\cosh\left(\frac{h\tau D_\tau - \bar{h}yD_y}{2}\right)f\cdot f\}$$
$$= -2\sinh(\bar{h}yD_y)\{\cosh(h\tau D_\tau)f\cdot f\}.\{\cosh(\bar{h}yD_y)f\cdot f\}. \tag{4.88}$$

Combining the equations (4.87) and (4.88), we get

$$h^{-1/2}\sinh(h\tau D_\tau)\{\cosh(h\tau D_\tau)f\cdot f\}.\{\cosh(\bar{h}yD_y)f\cdot f\}$$
$$= -2\sinh(\bar{h}yD_y)\{\cosh(h\tau D_\tau)f\cdot f\}.\{\cosh(\bar{h}yD_y)f\cdot f\}. \tag{4.89}$$

If we divide (4.89) with the results in (ii) of Lemma, equipped with $D_1 = h\tau D_\tau$,

$D_2 = \bar{h}yD_y$, we reach

$$\frac{h^{-1/2}\sinh(h\tau D_\tau)(\cosh(h\tau D_\tau)f\cdot f).(\cosh(\bar{h}yD_y)f\cdot f)}{\cosh(h\tau D_\tau)(\cosh(\bar{h}yD_y)f\cdot f)\cdot(\cosh(\bar{h}yD_y)f\cdot f)}$$
$$= \frac{-2\sinh(\bar{h}yD_y)(\cosh(h\tau D_\tau)f\cdot f).(\cosh(\bar{h}yD_y)f\cdot f)}{\cosh(\bar{h}yD_y)(\cosh(h\tau D_\tau f\cdot f))\cdot(\cosh(h\tau D_\tau)f\cdot f)},$$

and using (iii) of Lemma, equation yields as

$$h^{-1/2}\sinh(h\tau\partial_\tau)(\frac{\cosh(h\tau D_\tau)f\cdot f}{\cosh(\bar{h}yD_y)f\cdot f}) = -2\sinh(\bar{h}y\partial_y)(\frac{\cosh(\bar{h}yD_y)f\cdot f}{\cosh(h\tau D_\tau)f\cdot f}). \quad (4.90)$$

If we make use of the q-exponential identity on . (4.90) we derive

$$\frac{f(q^2\tau,y)f(\tau,y)}{f(q\tau,\bar{q}y)\cdot f(q\tau,\frac{y}{\bar{q}})} - \frac{f(\tau,y)f(\frac{\tau}{q^2},y)}{f(\frac{\tau}{q},\bar{q}y)\cdot f(\frac{\tau}{q},\frac{y}{\bar{q}})} = 2h^{1/2}\left[\frac{f(\tau,y)f(\tau,\frac{y}{\bar{q}^2})}{f(q\tau,\frac{y}{\bar{q}})\cdot f(\frac{\tau}{q},\frac{y}{\bar{q}})} - \frac{f(\tau,\bar{q}^2y)f(\tau,y)}{f(q\tau,\bar{q}y)\cdot f(\frac{\tau}{q},\bar{q}y)}\right]. \quad (4.91)$$

where $e^h = q$ and $e^{\bar{h}} = \bar{q}$. Using the dependent variable transformation defined in (4.84), in the equation (4.91), we obtain

$$\delta_\tau(\frac{1}{V(\tau,y)}) = -2h^{1/2}\delta_y V(\tau,y), \quad (4.92)$$

by the help of q-difference operator δ defined in (4.82). □

Proposition 4.8 [35] One-q-soliton solution of the q-difference-q-difference KdV equation (4.83) is

$$V(\tau,y) = -\frac{[1+\eta\tau^\delta(\bar{q}y)^\zeta + \eta\tau^\delta(\bar{q})^{-\zeta}y^\zeta + \eta^2\tau^{2\delta}y^{2\zeta}]}{(1+\eta(q\tau)^\delta y^\zeta)(1+\eta\tau^\delta(q)^{-\delta}y^\zeta)}, \quad (4.93)$$

with the dispersion relation

$$(\bar{q})^{\frac{\zeta}{2}}[q^{\frac{3\delta}{2}} - q^{-\frac{\delta}{2}}] + (\bar{q})^{\frac{-\zeta}{2}}[q^{\frac{-3\delta}{2}} - q^{\frac{\delta}{2}}] + 2h\{q^{\frac{\delta}{2}}[(\bar{q})^{\frac{3\zeta}{2}} - (\bar{q})^{-\frac{\zeta}{2}}] + q^{\frac{-\delta}{2}}[(\bar{q})^{\frac{-3\zeta}{2}} - (\bar{q})^{\frac{\zeta}{2}}]\} = 0 \quad (4.94)$$

satisfied.

Proof. Recall the reductions (4.69) which we have used to obtain bilinear form of q-difference KdV equation. If we match these reductions with general bilinear form we can conclude the identifications

$$t=\tau, \quad x=y, \quad \theta=0, \quad a_1=\frac{3h}{2}, \quad b_1=\frac{\bar{h}}{2}, \tag{4.95}$$
$$a_2=a_3=\frac{h}{2}, \quad b_2=\frac{3\bar{h}}{2}, \quad b_3=-\frac{\bar{h}}{2}, \quad c_1=c_2=c_3=0.$$

Leading us to the function $f^{(1)}=1+\eta\tau^{\delta}y^{\zeta}$ with identifications (4.95). If we substitute $f^{(1)}$ into $V(\tau,y)$ defined in the equation (4.84), we obtain

$$V(\tau,y)=\frac{(1+\eta\tau^{\delta}(\bar{q}y)^{\zeta})(1+\eta\tau^{\delta}(\frac{y}{\bar{q}})^{\zeta})}{(1+\eta(q\tau)^{\delta}y^{\zeta})(1+\eta(\frac{\tau}{q})^{\delta}y^{\zeta})},$$

where $e^h=q$ and $e^{\bar{h}}=\bar{q}$. Hence one-q-soliton solution arises as (4.93). Moreover from the coefficient of ε^1

$$P(D_1,D_2,D_3)\{f^{(1)}\cdot 1+1\cdot f^{(1)}\}=2P(\partial_1,\partial_2,\partial_3)\eta\tau^{\delta}y^{\zeta}. \tag{4.96}$$

We obtain the dispersion relation

$$(\bar{q})^{\frac{\zeta}{2}}[q^{\frac{3\delta}{2}}-q^{-\frac{\delta}{2}}]+(\bar{q})^{\frac{-\zeta}{2}}[q^{\frac{-3\delta}{2}}-q^{\frac{\delta}{2}}]+2h\{q^{\frac{\delta}{2}}[(\bar{q})^{\frac{3\zeta}{2}}-(\bar{q})^{-\frac{\zeta}{2}}]+q^{\frac{-\delta}{2}}[(\bar{q})^{\frac{-3\zeta}{2}}-(\bar{q})^{\frac{\zeta}{2}}]\}=0,$$

where we plug Hirota bilinear form (4.71) and $f^{(1)}=\eta t^{\delta}y^{\zeta}$. □

Remark. Furthermore, if $\tau, y \in q^{Z}$, i.e., $\tau=q^n$ and $y=\bar{q}^m$, $n,m \in Z$, then one-q-soliton solution (4.93) can be rewritten as

$$V(\tau,y)=-\frac{[1+\eta q^{n\delta}(\bar{q})^{\zeta(m+1)}+\eta q^{n\delta}(\bar{q})^{\zeta(m-1)}+\eta^2\tau^{2n\delta}y^{2m\zeta}]}{(1+\eta(q)^{\delta(n+1)}(\bar{q})^{m\zeta})(1+\eta q^{\delta(n-1)}(\bar{q})^{m\zeta})}. \tag{4.97}$$

Besides, two-q-soliton and three-q-soliton solutions can be written by using the reductions and identifications as

$$f(\tau,y)=1+\eta_1\tau^{\delta_1}y^{\zeta_1}+\eta_2\tau^{\delta_2}y^{\zeta_2}+A(1,2)\eta_1\eta_2\tau^{\delta_1+\delta_2}y^{\zeta_1+\zeta_2} \tag{4.98}$$

where

$$\begin{aligned}A(1,2) = & -h^{-1}(\bar{q})^{\frac{\zeta_1-\zeta_2}{2}}[q^{\frac{3\delta_1-3\delta_2}{2}} - q^{\frac{-\delta_1+\delta_2}{2}}] + (\bar{q})^{\frac{-\zeta_1+\zeta_2}{2}}[q^{\frac{-3\delta_1+3\delta_2}{2}} - q^{\frac{\delta_1-\delta_2}{2}}] \\ & + \{q^{\frac{\delta_1-\delta_2}{2}}[(\bar{q})^{\frac{3\zeta_1-3\zeta_2}{2}} - (\bar{q})^{\frac{-\zeta_1+\zeta_2}{2}}] + q^{\frac{-\delta_1+\delta_2}{2}}[(\bar{q})^{\frac{-3\zeta_1+3\zeta_2}{2}} - (\bar{q})^{\frac{\zeta_1-\zeta_2}{2}}]\} \\ & h^{-1}(\bar{q})^{\frac{\zeta_1+\zeta_2}{2}}[q^{\frac{3\delta_1+3\delta_2}{2}} - q^{\frac{-\delta_1-\delta_2}{2}}] + (\bar{q})^{\frac{-\zeta_1-\zeta_2}{2}}[q^{\frac{-3\delta_1-3\delta_2}{2}} - q^{\frac{\delta_1+\delta_2}{2}}] \\ & - \{q^{\frac{\delta_1+\delta_2}{2}}[(\bar{q})^{\frac{3\zeta_1+3\zeta_2}{2}} - (\bar{q})^{\frac{-\zeta_1-\zeta_2}{2}}] + q^{\frac{-\delta_1-\delta_2}{2}}[(\bar{q})^{\frac{-3\zeta_1-3\zeta_2}{2}} - (\bar{q})^{\frac{\zeta_1+\zeta_2}{2}}]\}\end{aligned}$$

and

$$\begin{aligned}f(\tau, y) = 1 + \sum_{i=1}^{3} \eta_i \tau^{\delta_i} y^{\zeta_i} + \sum_{i<j}^{(3)} A(i,j)\eta_i\eta_j\tau^{\delta_i+\delta_j}y^{\zeta_i+\zeta_j} \\ + A(1,2)A(1,3)A(2,3)\eta_1\eta_2\eta_3\tau^{\delta_1+\delta_2+\delta_3}y^{\zeta_1+\zeta_2+\zeta_3}\end{aligned}$$

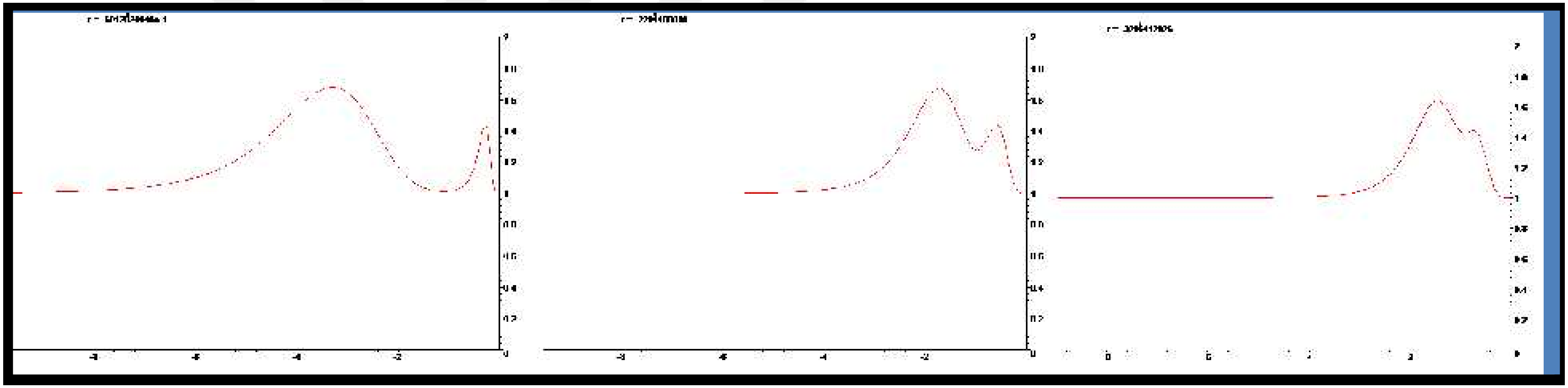
where

$$\begin{aligned}A(i,j) = & -h^{-1}(\bar{q})^{\frac{\zeta_i-\zeta_j}{2}}[q^{\frac{3\delta_i-3\delta_j}{2}} - q^{\frac{-\delta_i+\delta_j}{2}}] + (\bar{q})^{\frac{-\zeta_i+\zeta_j}{2}}[q^{\frac{-3\delta_i+3\delta_j}{2}} - q^{\frac{\delta_i-\delta_j}{2}}] \\ & + \{q^{\frac{\delta_i-\delta_j}{2}}[(\bar{q})^{\frac{3\zeta_i-3\zeta_j}{2}} - (\bar{q})^{\frac{-\zeta_i+\zeta_j}{2}}] + q^{\frac{-\delta_i+\delta_j}{2}}[(\bar{q})^{\frac{-3\zeta_i+3\zeta_j}{2}} - (\bar{q})^{\frac{\zeta_i-\zeta_j}{2}}]\} \\ & h^{-1}(\bar{q})^{\frac{\zeta_i+\zeta_j}{2}}[q^{\frac{3\delta_i+3\delta_j}{2}} - q^{\frac{-\delta_i-\delta_j}{2}}] + (\bar{q})^{\frac{-\zeta_i-\zeta_j}{2}}[q^{\frac{-3\delta_i-3\delta_j}{2}} - q^{\frac{\delta_i+\delta_j}{2}}] \\ & - \{q^{\frac{\delta_i+\delta_j}{2}}[(\bar{q})^{\frac{3\zeta_i+3\zeta_j}{2}} - (\bar{q})^{\frac{-\zeta_i-\zeta_j}{2}}] + q^{\frac{-\delta_i-\delta_j}{2}}[(\bar{q})^{\frac{-3\zeta_i-3\zeta_j}{2}} - (\bar{q})^{\frac{\zeta_i+\zeta_j}{2}}]\}\end{aligned}$$

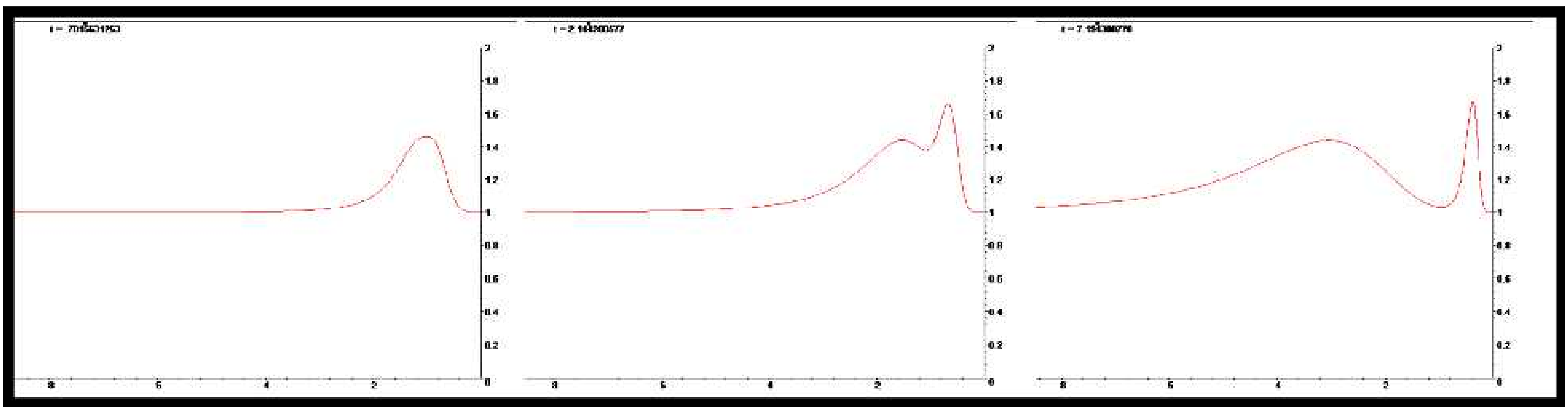
for all $i < j$ and $i, j = 1, 2, 3$.

Remark. We present the graph of two-q-soliton solutions of q-difference-q-difference KdV equation. The solitonic behavior of the waves can be observed from the graph. Since the form of the solutions is polynomials in power functions, we conclude that the length of the wave increases as x increases.

Figure 2: Two-q-soliton solutions of q-difference-q-difference KdV Equation



(a) (b) (c)



(d) (e) (e)

We set $q = 1,25$, $\bar{q} = 2$, $h = \ln(q)$, $\delta_1 = -6$ and $\delta_2 = 4$, then from the dispersion relation we get $\zeta_1 = -3.015059298$ and $\zeta_2 = -2.251766226$.

4.2.3 The q-difference sine-Gordon equation

The last equation that we determined from our generalized equation (4.1) is q-analogue sine-Gordon equation. Our aim is to write Hirota bilinear form for q-difference sine-Gordon equation. Thus we propose the reductions

$$\begin{aligned} D_1 &= h^2\tau D_\tau + \bar{h}^2 y D_y, \\ D_2 &= h^2\tau D_\tau + \bar{h}^2 y D_y + kzDz, \\ D_3 &= h^2\tau D_\tau - \bar{h}^2 y D_y, \end{aligned} \tag{4.99}$$

$$\lambda_1 = 1, \quad \lambda_2 = h\bar{h}, \quad \lambda_3 = -1 - h\bar{h}, \tag{4.100}$$

on (4.1) which reveals the bilinear form as

$$\begin{aligned} &[2\sinh(\bar{h}^2 y D_y)\sinh(h^2\tau D_\tau) + h\bar{h}\cosh(h^2\tau D_\tau + \bar{h}^2 y D_y + kzDz) \\ &- h\bar{h}\cosh(h^2\tau D_\tau - \bar{h}^2 y D_y)]\{f(\tau, y, z)\cdot f(\tau, y, z)\} = 0. \end{aligned} \tag{4.101}$$

As we have proved for the previous equations, we will show how the bilinear form of q-difference sine-Gordon (4.101) falls into Hirota bilinear form of classical one [16]. Because of the decomposed Hirota bilinear form of the continuous sine-Gordon equation, we require a proper decomposition form from (4.101). To revise the equation, first we have to mention about periodicity on q-numbers.

Definition. [35] A function $h(x)$ is called as q^m-periodic function if

$$h(q^m t) = h(t), \quad q > 1, \ m \in Z, \ t \in K_q. \tag{4.102}$$

Proposition 4.9 [35] Hirota bilinear form of q-difference sine-Gordon equation, (4.101) recovers its continuous form [16]

$$D_x D_t\{\bar{g}.\bar{f}\} = \bar{g}.\bar{f}, \tag{4.103}$$

$$D_x D_t\{\bar{f}.\bar{f} - \bar{g}.\bar{g}\} = 0, \tag{4.104}$$

as h and $\bar{h}$ tend to zero.

Proof. Suppose that f in (4.101) is $\bar{q}^2$-periodic function i.e.,

$$f(\bar{q}^2 z) = f(z),$$

or

$$f(\bar{q}z) = f(\frac{z}{q})$$

Then if we consider a function f and its q-shifted version, say $\tilde{f}$, i.e., we have

$$e^{kz\partial_z} f(z) = f(\bar{q}z) := \tilde{f}(z),$$

provided that $e^k = \bar{q}$. We use this periodicity in Hirota bilinear equation (4.101) i.e., $\cosh(kzD_z)\{f.f\} = 2f(\bar{q}z).f(\frac{z}{q}) = 2\tilde{f}(z).\tilde{f}(z)$ and divide (4.101) by $2(h\bar{h})^{-1}$,

$$\begin{aligned}(h\bar{h})^{-1}\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau) - \frac{1}{2}\cosh(h^2\tau D_\tau - \bar{h}^2 yD_y)\ \{f(\tau,y,z)\cdot f(\tau,y,z)\} \\ = -\frac{1}{2}\cosh(h^2\tau D_\tau + \bar{h}^2 yD_y)\{\tilde{f}(\tau,y,z).\tilde{f}(\tau,y,z)\}.\end{aligned} \tag{4.105}$$

If we utilize from the property

$$\begin{aligned}\cosh(h^2\tau D_\tau - \bar{h}^2 yD_y)\{f.f\} - \cosh(h^2\tau D_\tau + \bar{h}^2 yD_y)\{\tilde{f}.\tilde{f}\} \\ = \cosh(h^2 yD_y)[\cosh(\bar{h}^2\tau D_\tau)\{f.f - \tilde{f}.\tilde{f}\}]\end{aligned}$$

then the equation (4.105) turns into

$$\begin{aligned}&[(h\bar{h})^{-1}\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau)]\{f\cdot f\} \\ &= \frac{1}{2}\cosh(\bar{h}^2 yD_y)\cosh(h^2\tau D_\tau)]\{f\cdot f - \tilde{f}.\tilde{f}\}.\end{aligned} \tag{4.106}$$

If we set

$$f = \bar{f} + i\bar{g} \quad \tilde{f} = \bar{f} - i\bar{g}, \tag{4.107}$$

on (4.106), we have

$$\begin{aligned}&[(h\bar{h})^{-1}\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau)]\{\bar{g}\cdot\bar{f}\}\\&=\cosh(\bar{h}^2 yD_y)\cosh(h^2\tau D_\tau)]\{\bar{g}\cdot\bar{f}\}\end{aligned} \tag{4.108}$$

and

$$\sinh(\bar{h}^2 yD_y)\sinh(h^2\tau D_\tau)\{\bar{f}.\bar{f}-\bar{g}.\bar{g}\}=0. \tag{4.109}$$

We set $y = e^{\bar{h}x}$ and $\tau = e^{ht}$ in (4.108) and (4.109) and take limit as $h, \bar{h} \to 0$

$$\begin{aligned}&\lim_{h\to 0}\lim_{h\to 0}(h\bar{h})^{-1}\sinh(\bar{h}yD_y)\sinh(h\tau D_\tau)]\{\bar{g}\cdot\bar{f}\}\\&\qquad=\lim_{h\to 0}\lim_{h\to 0}\cosh(\bar{h}yD_y)\cosh(h\tau D_\tau)]\{\bar{g}\cdot\bar{f}\}\end{aligned} \tag{4.110}$$

which gives us

$$D_x D_t\{\bar{g}.\bar{f}\} = \bar{g}.\bar{f}.$$

Similarly we conclude the second counterpart of bilinear form

$$D_x D_t\{\bar{f}.\bar{f} - \bar{g}.\bar{g}\} = 0.$$

□

In the above proposition we have showed the decomposed bilinear forms (4.108), (4.109) bring us Hirota bilinear forms of continuous sine-Gordon equation. Our second aim is to construct the standard form of q-difference sine-Gordon equation.

Definition. [35] The q-sum operator Γ_x, operating on any function $f(x)$ is defined as

$$\Gamma_x f(x) := f(qx) + f(\frac{x}{q}), \quad x \,\square\, R,\ q = 1. \tag{4.111}$$

Proposition 4.10 [35] The standard form of the q-difference sine-Gordon equation is

$$\sin[\delta_y \delta_\tau \varphi(\tau, y)] = (h\bar{h})^{1/2} \sin[\Gamma_y \Gamma_\tau \varphi(\tau, y)], \tag{4.112}$$

where q-sum operator is defined in (4.111) and q-difference operator is defined in(4.82).

Proof. We interchange h^2 and $\bar{h}^2$ by h and $\bar{h}$, respectively and we get

$$[(h\bar{h})^{-1/2} \sinh(\bar{h}\, y D_y) \sinh(h\tau\, D_\tau)]\{\bar{g}.\bar{f}\} = \cosh(\bar{h} y D_y) \cosh(h\tau\, D_\tau)]\{\bar{g}.\bar{f}\}, \tag{4.113}$$

$$\sinh(\bar{h}\, y D_y) \sinh(h\tau\, D_\tau)\{\bar{f}.\bar{f} - \bar{g}.\bar{g}\} = 0. \tag{4.114}$$

Since $\bar{f}$ is complex conjugate of f given in (4.107) we can rewrite $\bar{f} := \exp^\rho . \cos(\varphi)$, $\bar{g} := \exp^\rho . \sin(\varphi)$ and convert the equations (4.113) as

$$\begin{aligned}&(1 - (h\bar{h})^{1/2}) \exp^{\rho(q\tau, py)+\rho(\frac{\tau}{q}, \frac{y}{p})} . \sin(\varphi(q\tau, py) + \varphi(\frac{\tau}{q}, \frac{y}{p}))\\ &= (1 + (h\bar{h})^{1/2}) \exp^{\rho(q\tau, \frac{y}{p})+\rho(\frac{\tau}{q}, py)} . \sin(\varphi(q\tau, \frac{y}{p}) + \varphi(\frac{\tau}{q}, py)),\end{aligned} \tag{4.115}$$

where $e^h = q$ and $e^{\bar{h}} = p$. Secondly, rewriting (4.114) we get

$$\begin{aligned}&\exp^{\rho(q\tau, py)+\rho(\frac{\tau}{q}, \frac{y}{p})} . \cos(\varphi(q\tau, py) + \varphi(\frac{\tau}{q}, \frac{y}{p}))\\ &\qquad = \exp^{\rho(q\tau, \frac{y}{p})+\rho(\frac{\tau}{q}, py)} . \cos(\varphi(q\tau, \frac{y}{p}) + \varphi(\frac{\tau}{q}, py)).\end{aligned} \tag{4.116}$$

From the equations (4.115) and (4.116) we find the below relation for the function φ

$$\begin{aligned}&\sin(\varphi(q\tau, py) + \varphi(\frac{\tau}{q}, \frac{y}{p}) - \varphi(q\tau, \frac{y}{p}) - \varphi(\frac{\tau}{q}, py))\\ &= (h\bar{h})^{1/2} \sin(\varphi(q\tau, py) + \varphi(\frac{\tau}{q}, \frac{y}{p}) + \varphi(q\tau, \frac{y}{p}) + \varphi(\frac{\tau}{q}, py)).\end{aligned} \tag{4.117}$$

which implies (4.112) □

In the end we give one-q-soliton solution.

Proposition 4.11 [35] One-q-soliton solution of the q-difference sine-Gordon equation (4.112) is

$$\varphi = 4\tan^{-1}(\eta\tau^{\alpha}y^{\beta}z^{\gamma}), \tag{4.118}$$

with the dispersion relation

$$(q^{\alpha}-q^{-\alpha})(p^{\beta}-p^{-\beta})+h\bar{h}[q^{\alpha}(p^{\beta}(\bar{q})^{\gamma}-p^{-\beta})+q^{-\alpha}(p^{-\beta}(\bar{q})^{-\gamma}-p^{\beta})]=0. \tag{4.119}$$

Proof. The reductions (4.99),(4.100) lead the identifications as

$$t=\tau, x=y, y=z \quad a_1=a_2=a_3=h, \quad b_1=b_2=\bar{h}, \tag{4.120}$$

$$c_2=k \quad b_3=-\bar{h}, \quad c_1=c_3=0, \tag{4.121}$$

and we can conclude the starting solution as $f^{(1)}=\eta\tau^{\alpha}y^{\beta}z^{\gamma}$. We have $\bar{f}$ and $\bar{g}$. From $\bar{f}=\exp(\rho).\cos(\varphi)$, $\bar{g}=\exp(\rho).\sin(\varphi)$ it can be found out that

$$\frac{\bar{g}}{\bar{f}}=\tan\varphi.$$

If we write expansion $\bar{f}$ and $\bar{g}$ around ε,

$$\bar{f}=1+\varepsilon^2 f^{(2)}+\varepsilon^4 f^{(4)}+\ldots$$
$$\bar{g}=\varepsilon f^{(1)}+\varepsilon^3 f^{(3)}+\ldots$$

and suggest $\bar{g}/\bar{f}=f^{(1)}$, with taking $\varepsilon=1$. Then one-q-soliton solution can be obtained as (4.118). Further, the dispersion relation (4.119) can be obtained by using the reductions (4.120), (4.121) on (4.9). □

Chapter 5

q-differential-q-difference Toda equation

Up to this chapter, we have analyzed q-discretization of equations by the use of q-difference operator. In this chapter our aim is to q-discretize the equations by q-derivative operator ∂_q [22]

$$\partial_{q,t}h(t) = \frac{h(qt) - h(t)}{qt - t}, \tag{5.1}$$

where h is q-differentiable function. The fundamental feature of this chapter is to analyze whether Hirota method is applicable to q-differential equations determined by the q-derivative operator ∂_q. For this purpose we present the q-analogue of Hirota D-operator. This chapter is based on the article [33].

Definition. [33] q-Hirota D-operator acting on q-differentiable functions f, g defined as

$$D^m_{q,t}\{f.g\} := (\partial_{q,t} - \partial_{q,t^0})^m f(t)g(t^0)|_{t^0=t}, \quad m \in Z^+. \tag{5.2}$$

We emphasize here that if we take limit as $q \to 1$, $\partial_{q,t} \to \partial_t$ and $D_{q,t} \to D_t$.

Note that q-Hirota D-operator satisfies the properties similar to continuous case.

Proposition 5.1 [33] Let $P(D_q)$ be a polynomial in $D_{q,t}$, then

(i) $P(D_q)\{g \cdot f\} = P(-D_q)\{f \cdot g\}$,

(ii) $P(D_q)\{g \cdot 1\} = P(\partial_{q,t})g; \quad P(D_q)\{1 \cdot g\} = P(-\partial_{q,t})g$,

where g, f are q-differentiable functions.

Proof. Since $P(D_q)$ refers to a polynomial let us consider simply $P(D_q) = D^m_{q,t}$. Using the definition (5.2)

$$
\begin{aligned}
P(D_q)\{g \cdot f\} &= (D_{q,t})^m g(t)f(t) \\
&= (-1)^m g(t)\partial^m_{q,t} f(t) + (-1)^{m-1} m\partial_{q,t} g(t)\partial^{m-1}_{q,t} f(t) + \cdots + f(t)\partial^m_{q,t} g(t) \\
&= (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \partial^{m-k}_{q,t} f(t)\partial^k_{q,t} g(t) \\
&= (D_{q,t})^m f(t)g(t) = P(-D_q)\{f \cdot g\},
\end{aligned}
$$

where $\binom{m}{k}$ is binomial coefficient and $\partial^j_{q,t}$ is the j^{th} q-derivative with respect to t. Second property is a consequence of (i) when we plug 1 instead of one of the functions. □

We present the following equation, introduced in [33],

$$f(x,t)\partial^2_{q,t} f(x,t) - (\partial_{q,t} f(x,t))^2 - [f(qx,t)f(\frac{x}{q},t) - f^2(x,t)] = 0 \qquad (5.3)$$

for q-differential-q-difference Toda equation. It is clear that the left hand side of (5.3) is $D^2_{q,t}$ in terms of q-Hirota-D-operator (5.2). Hence, we can conclude Hirota bilinear form of q-differential-q-difference Toda equation as

$$P(D)\{f(x,t) \cdot f(x,t)\} = [D^2_{q,t} - (e^{hxD_x} + e^{-hxD_x} - 2)]\{f \cdot f\} = 0. \qquad (5.4)$$

If we take limit as $q \to 1$ and $h \to 0$ in (5.4), Hirota bilinear form of q-differential-q-difference Toda equation falls into Hirota bilinear form of differential-difference Toda equation (2.22). This result brings us the preciseness of Hirota bilinear form of q-differential-q-difference Toda equation. Similar to the previous equations, we follow the same steps for perturbation technique. Beginning with the coefficient of ε^0, we get

$$P(D)\{1.1\} = [D^2_{q,t} - (e^{hxD_x} + e^{-hxD_x} - 2)]\{1.1\} = 0 \tag{5.5}$$

obviously. Secondly, we collect the coefficient of ε^1 and get

$$P(D)\{1 \cdot f^{(1)} + f^{(1)} \cdot 1\} = P(\partial) f^{(1)} = \partial^2_{q,t} - (e^{hx\partial_x} + e^{-hx\partial_x} - 2)] f^{(1)} = 0. \tag{5.6}$$

The solution of (5.6) should not be only in polynomial of a power form but also in q-exponential form. Hence taking $f^{(1)}$ as

$$f^{(1)} = \zeta x^{\beta} e_q^{\eta t}, \tag{5.7}$$

and plugging it in (5.6), we obtain the dispersion relation

$$\eta^2 = q^{\beta} + q^{-\beta} - 2, \tag{5.8}$$

where ζ, β are arbitrary constants and $e_q^{\eta t}$ is Jackson's q-exponential function [23],

$$e_q^t = \sum_{n=0} \frac{t^n}{[n]!}, \tag{5.9}$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$, $[n]! = [n][n-1]\cdots[1]$, for all $n \geq 1$, $[0]! = 1$. We continue to vanish the coefficients. From the coefficient of ε^2, we get

$$P(D)\{1 \cdot f^{(2)} + f^{(1)}.f^{(1)} + f^{(2)} \cdot 1\} = 0, \tag{5.10}$$

which yields to

$$-2[\partial^2_{q,t} - (e^{hx\partial_x} + e^{-hx\partial_x} - 2)] f^{(2)} = \tag{5.11}$$

$$[D^2_{q,t} - (e^{hxD_x} + e^{-hxD_x} - 2)]\{f^{(1)}.f^{(1)}\}. \tag{5.12}$$

When we put $f^{(1)}$ defined in (5.7) into (5.12), then $f^{(2)} = 0$ and one q-soliton solution is $f^{(1)} = 1 + \zeta x^{\beta} e_q^{\eta t}$. If we continue in this way to construct two-q-soliton solutions, we may choose $\mathbf{f}^{(1)}$ as

$$\mathbf{f}^{(1)} = \zeta_1 x^{\beta_1} e_q^{\eta_1 t} + \zeta_2 x^{\beta_2} e_q^{\eta_2 t}, \tag{5.13}$$

where ζ_i, β_i are arbitrary constants $\forall i = 1, 2$. Then dispersion relation emerges similarly as

$$\eta_i^2 = q^{\beta_i} + q^{-\beta_i} - 2, \quad \forall i = 1, 2. \tag{5.14}$$

Let us investigate the coefficient of ε^2 from

$$-2[\partial_{q,t}^2 - (e^{hx\partial_x} + e^{-hx\partial_x} - 2)]\mathbf{f}^{(2)} = \tag{5.15}$$

$$[D_{q,t}^2 - (e^{hxD_x} + e^{-hxD_x} - 2)]\{\mathbf{f}^{(1)}.\mathbf{f}^{(1)}\} \tag{5.16}$$

which gives us

$$P(\partial)\mathbf{f}^{(2)} = -\zeta_1\zeta_2[(\eta_1 - \eta_2)^2 - (q^{\beta_1-\beta_2} + q^{-\beta_1+\beta_2} - 2)]x^{\beta_1+\beta_2} e_q^{\eta_1 t} e_q^{\eta_2 t}. \tag{5.17}$$

The form of $\mathbf{f}^{(2)}$ has two chances. First one is

$$\mathbf{f}^{(2)} = A(1, 2)x^{\beta_1+\beta_2} e_q^{(\eta_1+\eta_2)t}. \tag{5.18}$$

If we put (5.18) into (5.17), we must have additive property of q-exponentials [?] as

$$e_q^m e_q^n = e_q^{m+n}, \tag{5.19}$$

which holds only if m, n are q-commuting variables as $mn = qnm$. In this case our condition yields as $\eta_1\eta_2 t^2 = q\eta_1\eta_2 t^2$ and it is satisfied if and only if either $\eta_1 = 0$ $(\beta_1 = 0)$ or $\eta_2 = 0$ $(\beta_2 = 0)$. As a second choice, if we select $\mathbf{f}^{(2)}$ as

$$\mathbf{f}^{(2)} = A(1, 2)x^{\beta_1+\beta_2} e_q^{\eta_1 t} e_q^{\eta_2 t}. \tag{5.20}$$

But in this case, $A(1, 2)$ depends on t, since we have the product rule for q-derivate as $\partial_q(a(t)b(t)) = a(t)\partial_q(b(t)) + b(qt)\partial_q a(t)$. Both in two cases carry out the nonexistence of $\mathbf{f}^{(2)}$. Some other approaches for $\mathbf{f}^{(2)}$ even very general ones fall into these two cases.

In conclusion, although q-differential type of equations have Hirota bilinear forms, they cannot produce multi soliton solutions by Hirota perturbation.

Hence, we conjecture that it is not possible to construct a different unifying approach than the one constructed in the article [35] and in this thesis, for integrable equations on arbitrary time scales with nonconstant step size; by classical Hirota perturbation.

On regular time scales with arbitrary graininess, integrability of equations is analyzed and developed in [2, 8, 34, 36] by Lie algebraic setting and construction of bi-Hamiltonian structures.

To sum up, in spite of previous discussions, on such time scales, we conjecture that classical Hirota perturbation does not produce another unifying framework for integrable δ-differential equations.

Chapter 6

Conclusion

Hirota Direct Method is one of the most preferred method not to only find soliton solutions, but also investigate the integrability nature of a partial differential or difference equation. However this method has not been applied to q discrete type of equations up to the article [33]. Silindir [33] proved that the method can be applicable to q-difference type of equations such as differential-q-difference Toda equation and q-difference-q-difference Toda equation in order to construct the desired q-soliton solutions. Inspired by this foundation, we presented a generic equation, q-analogue of Hirota-Miwa equation and found its three-q-soliton solutions by Hirota direct method. Besides, based on Hirota-Miwa equation, we presented Hirota bilinear forms of q-difference-q-difference Toda, q-difference-q-difference KdV, and q-difference sine-Gordon equations. This is a vital development as Hirota bilinear forms of such equations consisted of not only q-soliton solutions but also their standard forms.

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VITA

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