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Limit theorems for the spacings of weak records

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Abstract Let $W(1), W(2), \ldots$ be weak record values obtained from a sample of independent variables with common discrete distribution. In the present paper, we derive weak and strong limit theorems for the spacings $W(n+m) - W(n), m \geq 1, n \to \infty$.

Keywords Weak records · Spacings of weak records · Convergence in distribution · Almost sure convergence

1 Introduction

Let $X_1, X_2, \ldots$ be independent random variables with common distribution function F which has support on non-negative integers. The sequences of weak record times $L(n)$ $(n \geq 1)$ and weak record values $W(n)$ $(n \geq 1)$ were introduced in Vervaat (1973) as

$$
\begin{aligned}
L(1) &:= 1, \qquad L(n+1) := \min\left\{j : j > L(n),\ X_j \geq X_{L(n)}\right\}, \\
W(n) &:= X_{L(n)}.
\end{aligned}
$$

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Weak records have been studied later in Stepanov (1992, 1993), Aliev (1998, 1999), López-Blázquez and Wesołowski (2001), Wesołowski and Ahsanullah (2001), Stepanov et al. (2003), Wesołowski and López-Blázquez (2004), Dembińska and López-Blázquez (2005), Dembińska and Stepanov (2006), Danielak and Dembińska (2007). The material related to weak records is presented in the books of Arnold et al. (1998) and Nevzorov (2001).

If F is continuous, then weak record values and times agree with record values and times, which are intensively discussed in the literature; see e.g. Resnick (1987), Arnold et al. (1998) and Nevzorov (2001).

The joint density function of weak record values is given by

$$\mathbf{P}\{W(1) = k_1, \ldots, W(n) = k_n\} = \mathbf{P}\{X_1 = k_n\} \prod_{i=1}^{n-1} \frac{\mathbf{P}\{X_1 = k_i\}}{\mathbf{P}\{X_1 \geq k_i\}},$$

where $0 \leq k_1 \leq \cdots \leq k_n,\ F(k_n) < 1$ and for $n = 1$ the product in the last equality is equal to 1.

It immediately follows that the sequence $W(n)$ $(n \geq 1)$ forms a Markov chain. Formulaes for conditional densities are presented in the next Sects. 1 and 2.

Asymptotic properties of the ratio of weak records $W(n+m)/W(n)$ $(m \geq 1,\ n \to \infty)$ have been investigated in Dembińska and Stepanov (2006). With some motivation from the aforementioned paper we discuss in the present work the asymptotic behavior of the random spacings

$$\Delta(n, m) := W(n+m) - W(n) \qquad (m \geq 1,\ n \to \infty).$$

We will show that the convergence in distribution of $h(W(n))\Delta(n, m)$ $(m \geq 1, n \to \infty)$, where h is a positive measurable function, is closely related to the asymptotic behavior of the conditional excess $h(n)(X_n - n)|X_n \geq n$.

Another type of results (given in Sect. 4) concerns the almost sure behavior of $\Delta(n, m)$. In our study, we will show that $\Delta(n, m)$ does not tend to zero with probability one for any choice of discrete distribution function F with infinite upper endpoint. At the same time the sequence of spacings of weak record values can converge with probability one to infinity. This holds, in particular, for $F(n) = 1-n^{-\alpha}, \alpha > 0, n \in \mathbb{N}$.

In the next section, we present some preliminaries followed by results in Sect. 3, where the convergence in distribution of $\Delta(n, m), n \to \infty$ is discussed. Almost sure convergence is investigated in Sect. 4. Several illustrating examples are presented in Sect. 5. The proofs of all the results are given in Sect. 6 (the last section).

2 Preliminaries

We consider distribution functions F with infinite upper endpoint and support on non-negative integers. In order not to repeat this, we formulate the following assumption.

Assumption 2.1 The discrete distribution function F has support on $\mathbb{N} \cup \{0\}$ and the inequality $F(n) < 1$ holds for all $n \in \mathbb{N}$.

Let X_n $(n \geq 1)$ be a sequence of independent random variables with distribution function F satisfying Assumption 2.1 and $W(n)$ $(n \geq 1)$ be the corresponding random sequence of the weak record values. Set throughout the paper

$$p(n) := \mathbf{P}\{X_1 = n\}, \quad q(n) := \mathbf{P}\{X_1 \geq n\} = 1 - F(n-1) \quad (n \geq 0).$$

It follows easily that for $0 \leq k_n \leq \cdots \leq k_{n+m}, n \geq 1, m \geq 1$

$$\mathbf{P}\{W(n+m) = k_{n+m}, \ldots, W(n+1) = k_{n+1} | W(n) = k_n\} = \frac{p(k_{n+m})}{q(k_n)} \prod_{i=n+1}^{n+m-1} \frac{p(k_i)}{q(k_i)} \tag{1}$$

and

$$\mathbf{P}\{W(n+m)=j|W(n)=i\}=\frac{p(j)}{q(i)}\sum_{l_1=i}^{j}\frac{p(l_1)}{q(l_1)}\cdots\sum_{l_{m-1}=l_{m-2}}^{j}\frac{p(l_{m-1})}{q(l_{m-1})} \quad (0\leq i\leq j, m\geq 1), \tag{2}$$

where $\prod_{i=n+1}^{n} \frac{p(k_i)}{q(k_i)} = 1$ and the sum $\sum_{l_1=l}^{j} \frac{p(l_1)}{q(l_1)} \cdots \sum_{l_{m-1}=l_{m-2}}^{j} \frac{p(l_{m-1})}{q(l_{m-1})}$ is equal to 1 when $m = 1$ and to $\sum_{l_1=l}^{j} \frac{p(l_1)}{q(l_1)}$ when $m = 2$.

We will make use of both results in the proof of almost sure convergence. Related results obtained in Dembińska and Stepanov (2006) are collected in the two lemmas below.

In the following, $\stackrel{d}{\to}, \stackrel{p}{\to}, \stackrel{a.s}{\to}$ stand for convergence in distribution, convergence in probability, and almost sure convergence, respectively.

Lemma 2.1 *Let F be a distribution function satisfying Assumption 2.1.*

(a) *For any two integers i, k with $k \geq i$, we have*

$$\mathbf{P}\{W(n+1) \geq k | W(n) = i\} = \frac{q(k)}{q(i)}.$$

(b) *For any $i = 0, 1, \ldots$, the equality*

$$\sum_{n=1}^{\infty} \mathbf{P}\{W(n) = i\} = \frac{p(i)}{q(i+1)}$$

holds.

Lemma 2.2 *For any choice of discrete F satisfying Assumption 2.1*

$$W(n) \stackrel{a.s.}{\to} \infty \quad (n \to \infty).$$

Further, if for any constant $K > 1$

$$\lim_{n\to\infty} \frac{1 - F(Kn)}{1 - F(n)} = 0, \tag{3}$$

then

$$\frac{W(n+m)}{W(n)} \xrightarrow{p} 1 \quad (m \geq 1, n \to \infty).$$

We write in the following $X \sim G$ to indicate that the random variable X possesses the distribution function G. If also $Y \sim G$ we write alternatively $X \sim Y$ without mentioning explicitly the distribution function G. Denote by $\mathcal{NB}(m, p)$ and $Gamma(a, \lambda)$ the Negative Binomial and the Gamma distribution function, respectively. In our notation $\mathcal{NB}(m, p)$ $(m > 0, 0 < p < 1)$ possesses the probability mass function

$$\frac{\Gamma(m+k)}{\Gamma(m)\Gamma(k+1)} p^m [1-p]^k \quad (k \geq 0),$$

and $Gamma(a, \lambda)$ $(a > 0, \lambda > 0)$ possesses the density function

$$\frac{\lambda^a}{\Gamma(a)} x^{a-1} \exp(-\lambda x) \quad (x > 0),$$

where $\Gamma(\cdot)$ is the Gamma function.

3 Weak limit results

Let X_n $(n \geq 1)$ be independent random variables with discrete distribution F satisfying Assumption 2.1 and $X_{h,n}$ $(n \geq 1)$ be independent random variables (in the same probability space) with distribution function $F_{h,n}$ given by

$$F_{h,n}(x) := \mathbf{P}\{h(n)(X_1 - n) \leq x | X_1 \geq n\} \quad (x \geq 0),$$

where $h(t), t \in \mathbb{R}$ is a positive measurable function.

In this section we focus on the weak convergence of the scaled spacings $\Delta(n, m)$.

In view of Lemma 2.1 for any $x \geq 0, n > 1$, we have

$$\begin{aligned}
\mathbf{P}\{X_{h,n} \geq x\} &= \frac{\mathbf{P}\{X_1 \geq n + x/h(n)\}}{\mathbf{P}\{X_1 \geq n\}} \\
&= \mathbf{P}\{W(n+1) \geq n + x/h(n) | W(n) = n\} \\
&= \mathbf{P}\{h(n)(W(n+1) - n) \geq x | W(n) = n\} \\
&= \mathbf{P}\{h(n)(W(n+1) - W(n)) \geq x | W(n) = n\} \\
&= \mathbf{P}\{h(n)\Delta(n, 1) \geq x | W(n) = n\},
\end{aligned}$$

which shows that the scaled spacings $h(n)\Delta(n, 1)$ are closely related to the random sequence $X_{h,n}$.

We assume in the following that the convergence in distribution

$$X_{h,n} \xrightarrow{d} Y \quad (n \to \infty) \tag{4}$$

holds with $Y \geq 0$ almost surely. If $Y = 0$ or $Y = \infty$ (almost surely) we suppose that the above convergence holds in probability.

Convergence in (4) is closely related to the Gumbel max-domain of attraction of F. It is well-known from extreme value theory that if F is in the max-domain of attraction of the unit Gumbel distribution $\Lambda(x) = \exp(-\exp(-x))$, then there exists a positive measurable function w (see e.g. Falk et al. (2004) or Kotz and Nadarajah (2005)) such that

$$\lim_{u \uparrow x_F} \frac{1 - F(u + x/w(u))}{1 - F(u)} = \exp(-x) \quad (x \in \mathbb{R}), \tag{5}$$

where $x_F := \sup\{x : F(x) < 1\}$. We write for short $F \in GMDA(w)$.

Consequently if $F \in GMDA(w)$, then (4) holds with

$$h(n) = w(n) \quad \textit{and} \quad Y \sim Gamma(1, 1). \tag{6}$$

Note in passing that the scaling function w can be defined asymptotically (see e.g., Embrechts et al. (1997)) by

$$w(t) = \frac{1 + o(1)}{\mathbf{E}\{X_1 - t | X_1 > t\}} \quad (t \to \infty).$$

At this point, we begin presenting results of our work.

Theorem 3.1 *Let X_n $(n \geq 1)$ be independent random variables with common distribution function F satisfying Assumption 2.1. Suppose that there exist h and Y such that (4) holds. Then*

$$(h(W(n))\Delta(n, 1), \ldots, h(W(n + m))\Delta(n + m, 1)) \xrightarrow{d} (\mathcal{Y}_0, \ldots, \mathcal{Y}_m) \quad (n \to \infty), \tag{7}$$

where $\mathcal{Y}_0, \ldots, \mathcal{Y}_m$ are independent random variables with $\mathcal{Y}_i \sim Y, 0 \leq i \leq m$.

The following corollary is immediate (recall (6)):

Corollary 3.1 *Under the assumptions and the notation of Theorem 3.1 if $F \in GMDA(w)$, then (7) holds with $h(n) = w(n), n \geq 1$ and $Y \sim Gamma(1, 1)$.*

In the above corollary the limiting random variable Y possesses a continuous distribution function. We consider next a case when the limiting random variable Y possesses a discrete distribution. Let us define a random variable Y_β as $Y_\beta \sim \mathcal{NB}(1, 1 - \beta)$, if $\beta \in (0, 1)$ and $Y_\beta = 0$, $Y_\beta = \infty$ (almost surely) for $\beta = 0$ and $\beta = 1$, respectively.

Lemma 3.1 *Let X_n $(n \geq 1)$ be independent random variables with common distribution function F satisfying Assumption 2.1. If*

$$\lim_{n\to\infty} \frac{1-F(n+1)}{1-F(n)} = \beta \in [0,1], \tag{8}$$

then (4) holds with $h(n) = 1$ and $Y \sim Y_\beta$. Furthermore, (7) is satisfied.

By imposing some asymptotic condition on the scaling function h, we are able to generalize 3.1. More precisely, we assume that

$$\lim_{n\to\infty} nh(n) = \infty \tag{9}$$

and

$$\lim_{n\to\infty} \frac{h(a_{n+1})}{h(a_n)} = 1, \tag{10}$$

where a_n is a non-decreasing sequence of integers such that $\lim_{n\to\infty} a_{n+1}/a_n = 1$. Clearly the above conditions hold if $\lim_{n\to\infty} h(n) = c \in (0, \infty)$. For such instance our next theorem holds with the same assumptions as in Theorem 3.1.

Theorem 3.2 *Let X_n $(n \geq 1)$ be independent random variables with common distribution function F satisfying Assumption 2.1. Suppose that there exist h and Y such that conditions (4), (9), and (10) are satisfied. If $\mathbf{P}\{Y < \infty\} = 1$, then for given integers $k_1, k_2, \ldots, k_m$ and $m \geq 2$*

$$(h(W(n))\Delta(n,k_1), \ldots, h(W(n))\Delta(n+k_{m-1},k_m)) \xrightarrow{d} (\mathcal{Z}_{k_1}, \ldots, \mathcal{Z}_{k_m}) \quad (n\to\infty), \tag{11}$$

where $\mathcal{Z}_{k_i} \sim \sum_{j=1}^{k_i} \mathcal{Y}_j$ and $\mathcal{Y}_1, \ldots, \mathcal{Y}_m$ are independent random variables with the same distributions as Y.

The corollary below follows from Lemma 3.1, Theorem 3.1, and Theorem 3.2.

Corollary 3.2 *Let X_n $(n \geq 1)$ be independent random variables with common distribution function F satisfying Assumption 2.1.*

(a) *If $F \in GMDA(w)$ and $h(n) = w(n), n \geq 1$ is such that (10) holds, then (11) also holds with $Z_{k_i} \sim Gamma(k_i, 1), 1 \leq i \leq m$.*
(b) *If (8) holds, then (11) is valid with $h(n) = 1, n \geq 1$ and $\mathcal{Z}_{k_i} \sim \mathcal{NB}(k_i, 1-\beta), 1 \leq i \leq m$. Furthermore, $\mathcal{Z}_{k_i} = 0$ or $\mathcal{Z}_{k_i} = \infty$ if $\beta = 0$ or $\beta = 1$, respectively.*

We discuss next the convergence of the expectations of $\Delta(n,m)$.

Theorem 3.3 *Let X_n $(n \geq 1)$ be independent random variables with common distribution function F satisfying Assumption 2.1. Suppose that (8) holds with $\beta \in [0,1)$. If, in addition,*

$$\frac{1-F(n+1)}{1-F(n)} \leq \beta_* \in [\beta, 1) \tag{12}$$

is valid for all large enough n, then for any $m \geq 1$

$$\lim_{n\to\infty} \mathbf{E}\{\Delta(n,m)\} = \frac{\beta m}{1-\beta}.$$

Remark (a) Condition (9) in Theorem 3.2 can be dropped if $W(n+m)/W(n) \stackrel{a.s}{\to} 1$ holds for any integer m.

(b) The convergence in (10) holds, for example, if h is bounded and regular varying at infinity with index $\alpha \in \mathbb{R}$. We refer the reader for details on regular variation to Resnick (1987) or Embrechts et al. (1997). A key result for regularly varying functions is the uniform convergence theorem, see e.g., Theorem A 3.2 in Embrechts et al. (1997).

(c) If $\lim_{n\to\infty} h(n) = \infty$ and the Assumption 2.1 holds, then for a given positive real x we have

$$\lim_{n\to\infty} \frac{1-F(n+x/h(n))}{1-F(n)} = 1.$$

The above asymptotics implies $X_{h,n} \stackrel{p}{\to} 0$ as $n \to \infty$. This case is therefore not interesting.

4 Almost sure convergence

In this section we focus on the almost sure convergence of $\Delta(n,m)$, $m \geq 1$. It is interesting that $\Delta(n,m)$ cannot converge to 0 with probability 1 for any F satisfying Assumption 2.1, despite the fact that $\Delta(n,m) \stackrel{p}{\to} 0$ in the case when (8) holds with $\beta = 0$. In view of Theorem 3.2, $\beta = 1$ implies $\Delta(n,m) \stackrel{p}{\to} \infty$ as $n \to \infty$. Imposing some additional conditions, we strengthen this convergence to almost sure convergence; see Theorem 4.3 below.

Theorem 4.1 *Let* X_n $(n \geq 1)$ *be a sequence of independent random variables with common distribution function* F *satisfying Assumption* 2.1. *Then for any* $m \geq 1$

$$\mathbf{P}\{\Delta(n,m) > 0 \; i.o.\} = 1.$$

We get immediately:

Corollary 4.1 *For any choice of discrete* F *satisfying Assumption* 2.1, *the random sequence* $\Delta(n,m)$ *can not converge to zero with probability one.*

Corollary 4.2 *Let* X_n $(n \geq 1)$ *be a sequence of independent random variables with common distribution function* F *satisfying Assumption* 2.1, *and let* (8) *hold with* $\beta \in (0,1]$. *Then for any* $m \geq 1, k \geq 0$

$$\mathbf{P}\{\Delta(n,m) > k \; i.o.\} = 1.$$

In the next theorem we exclude the case $\beta = 1$.

Theorem 4.2 *Under the assumptions and notation of 4.2, if* $\beta \in (0, 1)$ *then for any* $m \geq 1$

$$\mathbf{P}\{\Delta(n, m) = 0 \; i.o.\} = 1.$$

It follows from Corollary 3.2 or Theorem 4.2 that for $\beta \in (0, 1)$ the sequence $\Delta(n, m)$ cannot converge to infinity with probability 1.

In the previous section we showed that if $\beta = 1$, then $\Delta(n, m) \stackrel{p}{\to} \infty$ as $n \to \infty$. In the next theorem we strengthen this to almost sure convergence.

Theorem 4.3 *Let* X_n $(n \geq 1)$ *be a sequence of independent random variables with common distribution function* F *satisfying Assumption 2.1. If*

$$\sum_{n=1}^{\infty} \left(\frac{p(n)}{q(n)} \right)^2 < \infty, \tag{13}$$

then for any $m \geq 1$

$$\Delta(n, m) \stackrel{a.s.}{\to} \infty \quad (n \to \infty).$$

5 Examples

5.1 Let us consider the case when F is the geometric distribution with parameter $p \in (0, 1)$. We have

$$\lim_{n \to \infty} \frac{1 - F(n + 1)}{1 - F(n)} = 1 - p \in (0, 1),$$

hence Theorem 3.2 implies (set $h(n) = 1, n \geq 1$)

$$\Delta(n, m) \stackrel{d}{\to} Y \sim \mathcal{NB}(m, p) \quad (n \to \infty, m \geq 1).$$

The above convergence can be confirmed directly since $\Delta(n, m)$ is a negative binomial random variable with parameters m and p, and the distribution function does not depend on n at all.

5.2 Define a discrete distribution function F such that for all n large we have $F(n) = 1 - n^{\alpha} C^{-n^{\delta}}$, $\alpha \in \mathbb{R}$, $\delta \in (0, \infty)$, $C \in (1, \infty)$. It follows that

$$\lim_{n \to \infty} \frac{1 - F(n + 1)}{1 - F(n)} = \beta,$$

with $\beta = 1$ if $\delta < 1$, and $\beta = 1/C$ or $\beta = 0$ if $\delta = 1$ or $\delta > 1$, respectively. If $\beta = 1/C$, then Corollary 3.2 (part (b)) yields

$$\Delta(n, m) \xrightarrow{d} Y \sim \mathcal{NB}\left(m, 1 - \frac{1}{C}\right) \quad (n \to \infty).$$

By Theorem 3.3 we obtain

$$\lim_{n\to\infty} \mathbf{E}\{\Delta(n, m)\} = \mathbf{E}\{Y\} = \frac{m}{C - 1}.$$

For $\beta = 0$ or $\beta = 1$, the convergence in probability to 0 or ∞ follows. In the latter case (corresponding to $\delta < 1$) we can not say if the convergence holds almost surely because (13) is not valid.

5.3 Let us consider the case when F is a Poisson distribution with positive parameter λ. It is well-known that F does not belong to the max-domain of attraction of any univariate max-stable distribution function. We have

$$\frac{1 - F(n)}{1 - F(n - 1)} = 1 - \frac{1}{1 + \frac{\lambda}{n+1} + \frac{\lambda^2}{(n+1)(n+2)} + \cdots} \leq \lambda/(n + 1) \to 0 \quad (n \to \infty).$$

Hence Corollary 3.2 (part (b)) implies $\Delta(n, m) \xrightarrow{p} 0$ as $n \to \infty$ for any $m \geq 1$.

5.4 Let F be such that $p(n) = \exp(-a(n-1)^b) - \exp(-an^b)$, $n \geq 1$ with $a > 0$, $b \in (0, 1]$. Setting

$$w(t) := abt^{b-1} \quad (t > 0)$$

we obtain as $t \to \infty$

$$\frac{1 - F(t + x/w(t))}{1 - F(t)} \sim \exp\left(-at^b\left[\left(1 + \frac{x}{abt^b}\right)^b - 1\right]\right) \to \exp(-x) \quad (\forall x \in \mathbb{R}).$$

For $h(n) := w(n)$, $\forall n \geq 1$ we have

$$nh(n) = abn^b \to \infty \quad (n \to \infty).$$

Hence, Theorem 3.2 implies that for any integer m

$$ab(W(n))^{b-1}\Delta(n, m) \xrightarrow{d} Y \sim Gamma(m, 1) \quad (n \to \infty).$$

Also, observe that

$$\frac{1 - F(Kn)}{1 - F(n)} = \exp\left(-an^b\left[\left(\frac{[Kn]}{n}\right)^b - 1\right]\right) \to 0 \qquad (K > 1, n \to \infty),$$

where $[x]$ is the integral part of the number x. Condition (3) is fulfilled and

$$\frac{W(n+m)}{W(n)} \xrightarrow{p} 1 \quad (m \geq 1, n \to \infty).$$

5.5 Let $q(n) = n^{-\alpha}, n \geq 1, \alpha > 0$. We have

$$\lim_{n\to\infty} \frac{q(n+1)}{q(n)} = 1.$$

Consequently, $\Delta(n, 1) \xrightarrow{p} \infty$ as $n \to \infty$. Furthermore

$$\frac{p(n)}{q(n)} = 1 - \left(1 - \frac{1}{n+1}\right)^{\alpha} \sim \alpha/n \quad (n \to \infty);$$

hence condition (13) is fulfilled, and thus $\Delta(n, 1) \xrightarrow{a.s} \infty, n \to \infty$.

6 Proofs

Proof of Theorem 3.1 Applying the first part of Lemma 2.1 for $x \geq 0$, we obtain

$$\begin{aligned}
\mathbf{P}\{h(W(n))\Delta(n, 1) \geq x\} &= \mathbf{E}\{I\{\Delta(n, 1) \geq x/h(W(n))\}\} \\
&= \mathbf{E}\{I\{W(n+1) \geq W(n) + x/h(W(n))\}\} \\
&= \mathbf{E}\{\mathbf{E}\{I\{W(n+1) \geq W(n) + x/h(W(n))\}|W(n)\}\} \\
&= \mathbf{E}\left\{\frac{q(W(n) + x/h(W(n)))}{q(W(n))}\right\},
\end{aligned}$$

where $I\{A\}$ is the indicator of A. Choose $x \in \mathbb{R}$ a continuity point of the distribution function of Y. The convergence in (4) implies

$$\lim_{n\to\infty} \frac{q(n + x/h(n))}{q(n)} = \mathbf{P}\{Y \geq x\}.$$

In view of Lemma 2.2, $W(n) \to \infty$ almost surely as $n \to \infty$. Hence for any $k > 0$

$$\lim_{n\to\infty} \frac{q(W(n) + k/h(W(n)))}{q(W(n))} = \lim_{n\to\infty} \frac{q(n + k/h(n))}{q(n)} = \mathbf{P}\{Y \geq k\} \quad \text{a.s.} \quad (14)$$

The claim for $m = 0$ follows now from the bounded convergence theorem.

Next, let $k > 0$ be an integer and $0 \leq x_0 \leq x_1 \leq \cdots \leq x_k$ be continuity points of the distribution function of Y. By Lemma 2.1 and the Markov property of the random sequence $W(n)$, almost surely we have for $i = 0, 1, \dots, k$

$$\begin{aligned}
&\mathbf{P}\{h(W(n+i))\Delta(n+i,1) \geq x_i | W(n), \dots, W(n+i)\} \\
&= \mathbf{P}\{W(n+i+1) \geq x_i/h(W(n+i)) + W(n+i) | W(n), \dots, W(n+i)\} \\
&= \mathbf{P}\{W(n+i+1) \geq x_i/h(W(n+i)) + W(n+i) | W(n+i)\} \\
&= \frac{q(W(n+i) + x_i/h(W(n+i)))}{q(W(n+i))} \quad (n > 1).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\mathbf{P}\{h(W(n))\Delta(n,1) \geq x_0, \dots, h(W(n+k))\Delta(n+k,1) \geq x_k\} \\
&= \mathbf{E}\Bigg\{ I\,\{h(W(n))\Delta(n,1) \geq x_0, \dots, h(W(n+k-1))\Delta(n+k-1,1) \geq x_{k-1}\} \\
&\quad \times \left[\frac{q(W(n+k) + x_k/h(W(n+k)))}{q(W(n+k))} - \mathbf{P}\{Y \geq x_k\}\right]\Bigg\} \\
&\quad + \mathbf{P}\{Y \geq x_k\}\mathbf{P}\{\Delta(n,1) \geq x_0, \dots, h(W(n+k-1))\Delta(n+k-1,1) \geq x_{k-1}\}.
\end{aligned}$$

As above, for $n \to \infty$

$$\frac{q(W(n+k) + x_k/h(W(n+k)))}{q(W(n+k))} \xrightarrow{a.s} \mathbf{P}\{Y \geq x_k\}.$$

The claim follows now by passing to the limit and using induction with respect to k. □

Proof of Lemma 3.1 Let $Y \sim \mathcal{NB}(1, 1-\beta)$ and set $h(n) := 1, n \geq 1$. By the assumption, for any $k \geq 1$

$$\lim_{n\to\infty} \frac{1 - F(n+k)}{1 - F(n)} = \beta^k.$$

Since $\mathbf{P}\{Y \geq k\} = \beta^k$ the random sequence $X_{h,n}$ $(n \geq 1)$ converges in distribution as $n \to \infty$ to Y, provided that $\beta \in (0, 1)$. If $\beta = 0$, then

$$\lim_{n\to\infty} \mathbf{P}\{X_{h,n} \geq 1\} = \lim_{n\to\infty} \frac{1 - F(n+1)}{1 - F(n)} = 0.$$

Hence $X_{h,n} \xrightarrow{p} Y = 0$ as $n \to \infty$. In the same way, one can show that $X_{h,n} \xrightarrow{p} \infty$ when $\beta = 1$. □

Proof of Theorem 3.2 Condition (9) implies that for any $M, \varepsilon > 0$ the inequality

$$\frac{(n-1)}{n + M/h(n)} > 1 - \varepsilon$$

holds for all large n. For arbitrary $K > 1$ choose ε such that $(1-\varepsilon)K > 1$. Then, for M a continuity point of the distribution function of Y, the inequality

$$K(n-1) > n + M/h(n)$$

holds for all large n. Consequently,

$$\begin{aligned}\frac{1-F(K(n-1))}{1-F(n-1)} &= \frac{\mathbf{P}\{X_1 > K(n-1)\}}{\mathbf{P}\{X_1 \geq n\}}\\ &\leq \frac{\mathbf{P}\{X_1 \geq n + M/h(n)\}}{\mathbf{P}\{X_1 \geq n\}} \quad (n \to \infty)\\ &\to \mathbf{P}\{Y \geq M\} \to 0 \quad (M \to \infty).\end{aligned}$$

Then, Lemma 2.2 implies that for any $m \geq 1$

$$Z_n^* := \frac{W(n+m)}{W(n)} \xrightarrow{p} 1 \quad (n \to \infty).$$

The proof follows from Theorem 3.1 and Slutsky lemma (see e.g. Kallenberg (1997)) if we show that for any $m \geq 1$

$$Z_n := \frac{h(W(n+m))}{h(W(n))} \xrightarrow{p} 1 \quad (n \to \infty). \tag{15}$$

Let $Z_{n_k}(k \geq 1)$ be a subsequence of $Z_n(n \geq 1)$. By the subsequence principle for convergence in probability (p. 555 in Embrechts et al. (1997)) it suffices to show that there exists a subsequence $Z_{n_{k_j}}(j \geq 1)$ such that $Z_{n_{k_j}} \xrightarrow{a.s} 1$ as $j \to \infty$. By the convergence in probability proved above and the mentioned principle there exists a subsequence $Z^*_{n_{k_j}}(j \geq 1)$ such that

$$Z^*_{n_{k_j}} \xrightarrow{a.s} 1 \quad (j \to \infty).$$

Since as $n \to \infty$, we have $W(n) \xrightarrow{a.s} \infty$, $W_{n_{k_j}}$ is a non-decreasing subsequence (almost surely) and (10) holds, then

$$Z_{n_{k_j}} \xrightarrow{a.s} 1 \quad (j \to \infty).$$

The last implies (15) and completes the proof. □

Proof of Corollary 3.2 (a) It is well-known (cf. Resnick (1987)) that the assumption $F \in GMDA(w)$ implies

$$\lim_{n\to\infty} nw(n) = \infty$$

and

$$\lim_{n\to\infty} \frac{1 - F(n-1)}{1 - F(n)} = 1.$$

Since Y has a continuous distribution function, it follows that

$$\limsup_{n\to\infty} w(n) = 0.$$

Further, by the fact that the convergence in (5) holds uniformly for x in compact sets of $\mathbb{R}$, for any sequence $x_n, n \geq 1$ such that $\lim_{n\to\infty} x_n = x$, we have

$$\lim_{n\to\infty} \frac{1 - F(n + x_n/w(n))}{1 - F(n)} = \lim_{n\to\infty} \frac{1 - F(n + x/w(n))}{1 - F(n)} = \exp(-x).$$

Set now $x_n = x + w(n), n \geq 1$. By the asymptotic property of $w(n)$, we have $\lim_{n\to\infty} x_n = x$. Consequently,

$$\lim_{n\to\infty} \frac{1 - F(n + 1 + x/w(n))}{1 - F(n)} = \lim_{n\to\infty} \frac{1 - F(n + (x + w(n))/w(n))}{1 - F(n)} = \exp(-x), \quad \forall x \in \mathbb{R}.$$

Hence, we obtain

$$\lim_{n\to\infty} \frac{1 - F(n + x/w(n))}{1 - F(n)} = \lim_{n\to\infty} \frac{\mathbf{P}\{X_1 \geq n + x/w(n)\}}{\mathbf{P}\{X_1 \geq n\}} = \mathbf{P}\{Y \geq x\} = \exp(-x).$$

Thus (9) is satisfied with $h(n) = w(n), \forall n \geq 1$. The proof of the first statement follows now from Theorem 3.2 and (6).

(b) Clearly, $h(n) = 1, \forall n \geq 1$ satisfies both (9) and (11). The the proof of the second statement follows from Lemma 3.1 and Theorem 3.2. □

Proof of Theorem 3.3 Since $\Delta(n, m) = \Delta(n+m-1, 1) + \cdots + \Delta(n, 1)$, it is enough to prove the result for $m = 1$. By assumption (12) and the fact that $W(n) \overset{a.s}{\to} \infty, n \to \infty$ we obtain that

$$\frac{q(W(n) + k)}{q(W(n))} \leq \beta_*^k$$

holds almost surely as $n \to \infty$ for any $k \geq 1$. Further as in (14) (set $h(n) = 1, n \geq 1$)

$$\frac{q(W(n) + k)}{q(W(n))} \overset{a.s.}{\to} \mathbf{P}\{\mathcal{Z}_1 \geq k\} = \beta^k \quad (n \to \infty),$$

where $\mathcal{Z}_1 \sim \mathcal{NB}(1, 1-\beta)$. Since $\Delta(n, 1), n \geq 1$ is a non-negative random sequence we may write

$$\begin{aligned}\mathbf{E}\{\Delta(n,1)\} &= \int_0^\infty \mathbf{P}\{\Delta(n,1) \geq s\}\,ds \\ &= \int_0^\infty \mathbf{P}\{W(n+1) \geq s + W(n)\}\,ds \\ &= \int_0^\infty \mathbf{E}\left\{\frac{q(W(n)+s)}{q(W(n))}\right\} ds.\end{aligned}$$

The dominated convergence theorem implies thus

$$\lim_{n\to\infty} \mathbf{E}\{\Delta(n,1)\} = \int_0^\infty \mathbf{P}\{\mathcal{Z}_1 \geq s\}\,ds = \mathbf{E}\{\mathcal{Z}_1\} = \frac{\beta}{1-\beta},$$

and the result follows. □

Proof of Theorem 4.1 As remarked in Balakrishnan and Stepanov (2010), for a sequence of events $A_1, A_2, \ldots$ the equality

$$\mathbf{P}\{A_n\ i.o.\} = \alpha \in [0, 1]$$

holds iff

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbf{P}\{\overline{A}_n \ldots \overline{A}_{n+k-1} A_{n+k}\} = \alpha,$$

where $\overline{A}_i$ denotes the complement of the event A_i.

Choose $A_n = \{\Delta(n,1) > 0\}$ and $\overline{A}_n = \{\Delta(n,1) = 0\}$. Making use of (1), we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty} \mathbf{P}\{\Delta(n,1) = 0, \ldots, \Delta(n+k-1,1) = 0, \Delta(n+k,1) > 0\} \\ &\quad = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} \mathbf{P}\{W(n+1) = \cdots = W(n+k) = i, W(n+k+1) \\ &\quad = j \mid W(n) = i\}\mathbf{P}\{W(n) = i\} \\ &\quad = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \left(\frac{p(i)}{q(i)}\right)^k \frac{q(i+1)}{q(i)}\,\mathbf{P}\{W(n) = i\} = 1.\end{aligned}$$

Consequently, $\mathbf{P}\{\Delta(n,1) > 0 \; i.o.\} = 1$. Since $\Delta(n,m) \geq \Delta(n,1)$ $(m \geq 1)$ almost surely, the result follows. □

Proof of Corollary 4.2 The corollary can be proved in the same manner as Theorem 4.1. □

Proof of Theorem 4.2 In view of (8) for given $\varepsilon > 0$ we can find $I \in \mathbb{N}$ such that

$$\beta - \varepsilon < \frac{q(i+1)}{q(i)} < \beta + \varepsilon \quad (i \geq I).$$

By the argument given in the proof of Theorem 4.1,

$$P\{\Delta(n,m) = 0 \; i.o.\} = \lim_{n\to\infty} c_{n,m},$$

where

$$c_{n,m} = \sum_{k=0}^{\infty} P\{\Delta(n,m) > 0, \ldots, \Delta(n+k-1,m) > 0, \Delta(n+k,m) = 0\}.$$

Taking into account that

$$\begin{aligned}
&\{\Delta(n,m) > 0, \ldots, \Delta(n+k-1,m) > 0, \Delta(n+k,m) = 0\}\\
&\quad = \{W(n) < W(n+m), \ldots, W(n+k-m-1) < W(n+k-1),\\
&\quad W(n+k-1) < W(n+k) = \cdots = W(n+k+m)\},\\
&P\{\Delta(n,m) > 0, \ldots, \Delta(n+k-1,m) > 0, \Delta(n+k,m) = 0\}\\
&= \sum_{i_n=0}^{\infty} P\{\Delta(n,m) > 0, \ldots, \Delta(n+k-1,m) > 0, \Delta(n+k,m) = 0 \mid W(n) = i_n\}\\
&\qquad \times P\{W(n) = i_n\}
\end{aligned}$$

and applying (1), we obtain

$$\begin{aligned}
c_{n,m} = &\sum_{k=0}^{\infty}\sum_{i_n=0}^{\infty} \frac{\mathbf{P}\{W(n)=i_n\}}{q(i_n)} \sum_{i_{n+1}=i_n}^{\infty} \frac{p(i_{n+1})}{q(i_{n+1})} \cdots \sum_{i_{n+m-1}=i_{n+m-2}}^{\infty} \frac{p(i_{n+m-1})}{q(i_{n+m-1})}\\
&\sum_{i_{n+m}=\max\{i_n+1, i_{n+m-1}\}}^{\infty} \frac{p(i_{n+m})}{q(i_{n+m})} \cdots \sum_{i_{n+k-1}=\max\{i_{n+k-m-1}+1, i_{n+k-2}\}}^{\infty} \frac{p(i_{n+k-1})}{q(i_{n+k-1})}\\
&\times \sum_{i_{n+k}=i_{n+k-1}+1}^{\infty} \frac{p^{m+1}(i_{n+k})}{q^m(i_{n+k})}.
\end{aligned}$$

Clearly,

$$
\begin{aligned}
c_{n,m} \geq & \sum_{k=0}^{\infty} \sum_{i_n=I}^{\infty} \frac{\mathbf{P}\{W(n)=i_n\}}{q(i_n)} \sum_{i_{n+1}=i_n}^{\infty} \frac{p(i_{n+1})}{q(i_{n+1})} \cdots \sum_{i_{n+m-1}=i_{n+m-2}}^{\infty} \frac{p(i_{n+m-1})}{q(i_{n+m-1})} \\
& \sum_{i_{n+m}=\max\{i_n+1, i_{n+m-1}\}}^{\infty} \frac{p(i_{n+m})}{q(i_{n+m})} \cdots \sum_{i_{n+k-1}=\max\{i_{n+k-m-1}+1, i_{n+k-2}\}}^{\infty} \frac{p(i_{n+k-1})}{q(i_{n+k-1})} \\
& \times \sum_{i_{n+k}=i_{n+k-1}+1}^{\infty} \frac{p^{m+1}(i_{n+k})}{q^m(i_{n+k})}.
\end{aligned}
$$

Since $i_n, i_{n+1}, \ldots, i_{n+k} \geq I$, we get

$$
\sum_{i_{n+k}=i_{n+k-1}+1}^{\infty} \frac{p^{m+1}(i_{n+k})}{q^m(i_{n+k})} \geq (1-\beta-\varepsilon)^m q(i_{n+k-1}+1).
$$

Taking into account that

$$
\frac{q(\max\{i_{n+k-m-j}+1, i_{n+k-j-1}\})}{q(i_{n+k-j-1})} \geq \beta - \varepsilon \quad (1 \leq j \leq k-m),
$$

we find the lower bound for $c_{n,m}$:

$$
\begin{aligned}
c_{n,m} &\geq \sum_{k=0}^{\infty} \sum_{i_n=I}^{\infty} (1-\beta-\varepsilon)^m (\beta-\varepsilon)^k \mathbf{P}\{W(n)=i_n\} \\
&= \frac{(1-\beta-\varepsilon)^m}{1-\beta+\varepsilon} \mathbf{P}\{W(n) \geq I\}.
\end{aligned}
$$

Since $W(n) \stackrel{a.s}{\to} \infty$ as $n \to \infty$ (see Lemma 2.2), we get

$$
P\{\Delta(n,m)=0 \; i.o.\} \geq (1-\beta)^{m-1}.
$$

The result follows now from Kolmogorov's zero-one law since $\{\Delta(n,m)=0 \; i.o.\}$ is a tail event. □

Proof of Theorem 4.3 For any positive integer x and large enough integer J we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbf{P}\{\Delta(k,1)<x\} &= \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\mathbf{P}\{\Delta(k,1)<x|W(k)=j\}\mathbf{P}\{W(k)=j\}\\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\mathbf{P}\{W(k+1)<x+j|W(k)=j\}\mathbf{P}\{W(k)=j\}\\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}[1-\mathbf{P}\{W(k+1)\geq x+j|W(k)=j\}]\mathbf{P}\{W(k)=j\}\\ &= O(1)+\sum_{j=J}^{\infty}\left[1-\frac{q(x+j)}{q(j)}\right]\sum_{k=1}^{\infty}\mathbf{P}\{W(k)=j\}.\end{aligned}$$

Condition (13) implies $\beta=1$. Then for all large enough n and $\varepsilon>0$

$$\frac{q(n)}{q(n+1)}<\frac{1}{1-\varepsilon}$$

and

$$\sum_{j=J}^{\infty}\left[1-\frac{q(x+j)}{q(j)}\right]\sum_{k=1}^{\infty}\mathbf{P}\{W(k)=j\}<\frac{1}{1-\varepsilon}\sum_{k=0}^{x-1}\sum_{j=0}^{\infty}\frac{p(j)}{q(j)}\frac{p(j+k)}{q(j+k)}.$$

The strong convergence for $\Delta(n,1)$ comes now from the following observation. Let $u_n>0$ and $\sum_{n=1}^{\infty}u_n^2<\infty$, then $\sum_{n=1}^{\infty}u_nu_{n+k}<\infty$ $(k\geq 0)$. Further, the random sequence $\Delta(n,m)$ is monotone non-decreasing in m, and thus the proof follows. □

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