RELIABILITY EVALUATION OF SYSTEMS WITH WEIGHTED COMPONENTS

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RELIABILITY EVALUATION OF SYSTEMS WITH WEIGHTED COMPONENTS

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M.S. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **"RELIABILITY EVALUATION OF SYSTEMS WITH WEIGHTED COMPONENTS"** completed by **Timur Aksoy** under supervision of **Assoc. Prof. Dr. Serkan Eryılmaz** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

RELIABILITY EVALUATION OF SYSTEMS WITH WEIGHTED COMPONENTS

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This thesis concerns the reliability evaluation of weighted systems and also their non-weighted (usual) models which have been studied extensively in the literature. All studied weighted systems have nonidentical, independent components which can take arbitrary weights.

Exact reliability formulas for weighted k-out-of-n and weighted consecutive k-out-of-n systems which already exist in literature are reviewed and explained explicitly. Chapters 3 and 4 introduce the adjustments of usual combined k-out-of-n and consecutive k-out-of-n systems, and k-within-consecutive-m-out-of-n systems to weighted models. Chapter 3 proposes an exact reliability formula and equivalent usual models of the weighted combined systems. Two lower bounds and an upper bound are presented for the reliability of k-within-consecutive-m-out-of-n system in Chapter 4.

The first lower bound and the upper bound perform well for weighted models. A second lower bound is obtained with the same method for the usual systems. The results show that both lower bounds are improvements for usual models as well. The second lower bound performs better than other bounds for the usual systems in the literature in some cases and can be a good approximation for all values of system variables.

Keywords: Reliability analysis, weighted components, k-out-of-n system, consecutive k-out-of-n system, k-within-consecutive-m-out-of-n system.

ÖZ

AĞIRLIKLI BİLEŞENLİ SİSTEMLERDE GÜVENİLİRLİK ANALİZİ

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Bu tez literatürde sıkça çalışılmış sistemlerin hem ağırlıklı modellerinin hem de ağırlıksız (olağan) durumlarının güvenilirlik analizlerini kapsar. Çalışılan tüm ağırlıklı modellerin bileşenleri farklı, bağımsızdır ve farklı ağırlıklara sahiptirler.

Literatürde zaten bulunan ağırlıklı n'den-k'lı ve ardıl n'den-k'lı sistemler incelenip detaylı bir biçimde anlatılmıştır. 3. ve 4. bölümler olağan bileşik n'den-k'lı ve ardıl n'den-k'lı sistemler ve n'den-ardıl-m-içinde-k'lı sistemleri, ağırlıklı modellere uyarlar. 3. bölüm ağırlıklı bileşik sistemlerin kesin güvenilirlik formülünü ve eşdeğer olağan modellerini açıklar. 4. bölümde n'den-ardıl-m-içinde-k'lı sistemlerin güvenilirliği için iki alt sınır ve bir üst sınır sunulmuştur.

İlk alt sınır ve üst sınır ağırlıklı modellerde iyi sonuçlar verir. Aynı metodla olağan modeller için ikinci bir alt sınır bulunmuştur. Sonuçlar, iki alt sınırın olağan sistemler için de daha gelişmiş olduğunu gösterir. İkinci alt sınır, olağan sistemlerde literatürde bulunan diğer sınırlara göre bazı durumlarda daha keskindir ve tüm sistem değerleri için güvenilirliğin yaklaşık değeri olarak kullanılabilir.

Anahtar Kelimeler: Güvenilirlik analizi, ağırlıklı bileşenler, n'den-k'lı sistem, ardıl n'den-k'lı sistem, n'den-ardıl-m-içinde-k'lı sistem.

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Chapter 1

Introduction

Reliability is most commonly defined as the survival probability of a system consisting of one or more components arranged in a specific structure. The structure may be a network of components where connections between components represent the path an electrical, mechanical or logical signal may travel from the source to the sink. In these structures, if all paths between the source and the sink are cut then the system fails. There are also other configurations such as general k-out-of-n systems where components are ordered in a line or circle but connections between components do not usually have a physical meaning.

The system reliability evaluation is based on reliabilities of individual components, their specific configuration and survival (failure) criteria. The system's reliability is always a function of the individual reliabilities and this function depends on the configuration and operation rules of the system. Series systems have components arranged in series and it fails when only one component fails. This structure has the lowest survival probability with respect to configurations. A parallel system fails iff all components fail, having the highest reliability and redundancy with respect to configurations. Most structures studied in the literature are combination of serial and parallel subsystems with system reliability falling in between these lowest and highest bounds. In reliability analysis, our main interest is to find the relation between component and system reliabilities.

The reliability evaluation may be time dependent (dynamic) and time independent (static). In time dependent reliability evaluation, individual components survival time is random varible with a specific cumulative distribution function $F_i(t)$ and system's survival time is a function of these random variables hence its distribution can be derived from components' distributions. In this paper, time will be constant $t = t_c$ and the reliability of components up to that fixed time will be considered. Therefore reliability of the system and components will be time-independent Bernoulli random variables with $F_i(t_c) = p_i$ and $\overline{F_i}(t_c) = 1 - p_i$ representing the reliability and unreliability of component i.

Redundancy is built into non-weighted general k-out-of-n and consecutive k-out-of-n models models where components are ordered in a line or cicle but they do not constitute a serial system in the reliability sense. The issue here is the failure (survival) criteria and serial connections between components do not have a physical interpretation, they only specify the order of the components. General k-out-of-n systems' criterion is the number of working(failed) components in the system therefore the order or locations of components do not matter. In consecutive k-out-of-n models, components have to be adjacent to each other to operate or fail the system. Therefore the indices of failed or working components have to be consecutive for the survival or failure of the system. The order of components in relation to others may represent physical proximity or logical connections between components. These systems can also be represented as combination of serial and parallel arrays.

Both of these systems have been very popular subjects in the literature. The earliest reliability computations for k-out-of-n systems were proposed by Barlow and Heidtmann (1984), Jain and Gopal (1985), Risse (1987) and Sarje and Prasad (1989). Many others have generalized the subject and applied it to different cases and problems. Consecutive k-out-of-n systems were introduced by Chiang and Niu (1981), and the earliest efficient algorithms for its reliability were proposed by Derman et al. (1982), Hwang (1982), Shanthikumar (1982), Ge and Wang (1990). Consecutive k-out-of-n systems are still very popular in the literature. Recent discussions on the topic are in the works of Navarro and Eryılmaz (2007), Eryılmaz (2007), Eryılmaz(2009). They have also been extended to two dimensions by Salvia and Lasher (1990) and Boehme et al. (1992). Yamamoto and Miyakawa (1995) have provided exact reliability and Koutras et al. (1997) have provided bounds for rectangular two-dimensional consecutive models.

The k-out-of-n systems where all components have weight 1 are special cases of weighted systems. Weights can be assigned arbitrarily in the weighted models and as described in the next chapter, they may be regarded as a measure of components' importance. Weights are most commonly applied to general k-out-of-n and consecutive k-out-of-n models in literature. This paper focuses on weighted k-out-of-n, weighted consecutive k-out-of-n, combined weighted and weighted k-within-m-out-of-n systems. Some of these models can be represented as a coherent network structure with physical connections. Whereas in usual general k-out-of-n and consecutive k-out-of-n systems, the component importances are only determined by their order, weights make some components more critical and some even irrelevant in weighted systems.

There are less publications in literature regarding weighted systems which were introduced by Wu and Chen (1994a). A method for finding reliability of weighted consecutive k-out-of-n systems was first proposed again by Wu and Chen (1994b) followed by Chang and Chen (1998) for the circular case. Chen and Yang (2005) have generalized the subject into two stage weighted k-out-of-n systems. Eryılmaz and Tütüncü (2008) provided recursive formulas for the reliability of weighted consecutive k-out-of-n systems consisting of Markov-dependent components. Weighted combined systems and k-within-consecutive-m-out-of-n systems are introduced for the first time in this paper. Their literature review and potential applications will be discussed in Chapters 3 and 4.

Advances in science and technology have made systems consisting of multiple components increasingly more complex. The management of all components in engineering systems becomes more difficult as the number of units rise and their structures become more sophisticated. Since absolute control over their operations may not be feasable, breakdowns of some parts may be an inevitable consequence. Failure of critical systems such as electrical power in hospitals, fluid transmission infrastructures and electronic equipment in space shuttles can lead to serious and unrecoverable damages. Therefore redundancy is required in critical situations to sustain the process under adverse circumstances.

Applications of reliability modeling can range from electrical, mechanical contexts to processes involving humans. The scale of applied area is usually the issue, some examples are arrays of solar cells producing power at a certain voltage at the micro-scale, arrays of batteries in serial-parallel arrangement constituting an emergency backup power supply for a hospital or a nuclear power plant in the mid-scale and power plants supplying power to a grid or communication systems in the Internet in the macro-scale. The specific application would depend on the configuration and survival (failure) criteria of the models.

Weighted systems have potential applications in systems where components make unequal contribution to the specific requirement. The most frequently referred applications are load versus capacity problems. An example of weighted k-out-of-n systems is a power plant with different power generators and total power supplied by the plant is the sum of capacities of working generators. If it cannot supply the grid with a desired capacity it would result in reduced voltage or blackouts. Units providing ventilation in a facility with different cooling capacities required to keep the temperature below certain degrees is another example (Samaniego, 2008).

Usual and weighted consecutive k-out-of-n systems has applications in monitoring systems, relayed communication and fluid transmission systems. An example of a weighted F system may be waste water flowing through treatment facilities that clean it. As the waste water travels between facilities, different amount of pollution is added to the sewage at each center and if a facility is failed, the pollution continues to accumulate until the next working facility. If pollution is below a fixed amount then a working facility can treat it completely but if it exceeds the threshold, then the waste water is completely contaminated and cannot be treated anywhere.

Chapter 2

Systems with Weighted Components

2.1 Introduction

Components in a system may have unequal contribution to systems performance. Wu and Chen (1994a) introduced a k-out-of-n system model considering unequal weights for the components. The system was called weighted k-out-of-n:F(G) system. A weighted k-out-of-n:F(G) system has n components ordered in a line or circle, each with its own positive integer weight, fails (works) iff the total weight of failed (working) components is at least k. Obviously these systems reduce to the usual k-out-of-n models if each component has weight 1.

In binary weighted systems, the components and the structure has only two states as in usual systems. The component variable x_i representing its state remains as a boolean variable, 1 indicating on and 0 indicating off or vice versa.

Define:

$$Z_i = w_i X_i$$

and

$$Y_i = w_i(1 - X_i)$$

for $i = 1, ..., n$ where w_i represents the weight associated with component i and X_i is a binary random variable representing the state of component i. Thus the reliability of a weighted system can be represented in terms of random variables $Y_1, Y_2, \ldots, Y_n$ or $Z_1, Z_2, \ldots, Z_n$.

In this paper, all studied weighted systems will be linear, binary and with independent and nonidentical components unless stated otherwise. All weights are assumed to be positive integer values.

2.2 Review of Boolean Algebra

Three basic boolean operations will be briefly reviewed. All boolean variables and functions take the value of either 0 and 1. Let $B = \{0, 1\}$ and B^n be set of all possible n-tuples of 0s and 1s which actually contains n Boolean variables belonging to set B. If the Boolean variable takes value only from the set B, then the function from B^n to B is called a boolean function of degree n.

The complement of a variable changes the value of its state to the other state. It is obtained by subtracting the variable from one.

$$\overline{x_1} = 1 - x_1$$

The boolean product operates like multiplication but the variables x_i's are idempotent i.e. $x_i^2 = x_i$ therefore larger terms absorb the smaller terms. Boolean product takes the minimum of the states of multiplied terms.

$$x_1x_2x_3 \wedge x_2x_3 = min(x_1x_2x_3, x_2x_3) = x_1x_2x_3$$

In boolean sum, the boolean product of complements of each term is subtracted from 1. The smaller terms absorb the larger terms. Boolean sum takes the maximum of the added boolean terms.

$$x_1x_2x_3 \vee x_2x_3 = max(x_1x_2x_3, x_2x_3) = 1 - (1 - x_1x_2x_3)(1 - x_2x_3)$$
$$= x_1x_2x_3 + x_2x_3 - x_1x_2x_3 = x_2x_3$$

Product $\prod$ and coproduct $\coprod$ have the same meanings as boolean product $\wedge$ and sum $\vee$ respectively but they operate on more than two terms. Therefore $\coprod_{j=1}^{n} x_j = 1 - \prod_{j=1}^{n}(1 - x_j) = 1 - (1 - x_1)(1 - x_2)\dots(1 - x_n)$.

With these boolean laws, we can turn a boolean expression into an ordinary algebra expression which is required for deriving structure and reliability functions of systems.

2.3 Weighted k-out-of-n:G Systems

Weighted k-out-of-n:G system survives iff the total weight of working components is at least a fixed threshold k. The reliability function of this system can be derived in several ways. Components will be assumed to be independent but not necessarily identical. The states of components will be $x_i = 0$ if component i is failed and $x_i = 1$ if component i survives. We can write all path sets of the system $P_1, P_2 \dots P_l$ which belong to set of subsets $\{J : W_j \geq k\}$ where $J = \{i_1, i_2, \dots i_s\} \subseteq \{1, 2, \dots, n\}$ and $W_j = \sum_{j=1}^{s} w_{i_j}$. Thus, a path set includes any combination of components in the system whose total weights are at least k. Therefore, if all components in the set survive, the system survives. In contrast to minimal path sets, a path set may be a subset of another path set, a condition not allowed in minimal path sets. If the system has any chance of survival, the union of all path sets would include all components from 1 to n. Therefore the set of all path subsets include the all combinations of working components which would operate the system. If we set all components in each path set as working and ones in its complement as failed, we generate a distinct event from each path subset. The union of all these events would yield the reliability of the system since it includes all possible component states for system's survival. Due to mutual exclusiveness, we can add the probabilities of these events to derive the reliability function of Weighted k-out-of-n:G system in an unsimplified form:

$$P(\sum_{i=1}^{n} Z_i \geq k) = P(\sum_{i=1}^{n} Y_i < W - k) = h(\mathbf{p}) = \sum_{j=1}^{l} \left[\prod_{i \in P_j} p_i \prod_{s \in {P_j}^c} (1 - p_s) \right]$$

where $W = \sum_{i=1}^{n} w_i$ and l is the number of path sets.

Minimal path sets $MP_1, MP_2 \ldots MP_r$ can be found by removing all path sets containing another path set as a subset. According to Barlow and Proschan (1975), all variables in each minimal path set are multiplied with boolean products, we obtain minimal path structures for each set. If minimal path structures of each set are added with boolean sum, the boolean structure function of the system is obtained. Therefore,

$$\phi(\mathbf{x}) = \max_{1 \leq i \leq r} \min_{j \in MP_i} x_j$$

means

$$\phi(\mathbf{x}) = \bigvee_{i=1}^{r} \bigwedge_{j \in MP_i} x_j = \coprod_{i=1}^{r} \prod_{j \in MP_i} x_j$$

For deriving the probability function, the boolean terms have to be converted into ordinary algebra terms using boolean laws for sums and products and then x_i's are replaced with p_i's by taking their expectations. However this method involves cancellation of many terms hence not very efficent. Since the minimal path sets are known, Inclusion Exclusion (IE) and Sum of Disjoint Products (SDP) methods can also be applied. For these methods the reliability of the system is written as the union of survival events of every minimal path structure:

$$P(E_j) = P(\prod_{i \in MP_j} X_i = 1) = \prod_{i \in MP_j} p_i$$

$$h(\mathbf{p}) = P(\bigcup_{j=1}^{r} E_j)$$

IE method requires joint probabilities of all combinations of E_j for $j = 1, \ldots, r$ to be written and they are added and subtracted according to IE principle. In other words, all products of different combinations of E_j have to be found and then their probabilities are computed with law of boolean products. This involves many terms growing very large as number of minimal path sets grow large. SDP method sums the products of each minimal path structure with the complements of preceding minimal path structures. It involves less number of terms than IE method but extensive boolean algebra operations may be still required for cancelling variables because minimal path structures may have many components in common.

The most efficient method proposed by Wu and Chen (1994a) involves a recursive algorithm for computing the exact reliability. The reliability function of the G system is derived using pivotial decomposition. According to Barlow and Proschan (1975), the pivotial decomposition is described as follows,

Lemma 2.1

$$h(\mathbf{p}) = p_i h(1_i, \mathbf{p}) + (1 - p_i) h(0_i, \mathbf{p}) \textit{ for } i = 1, \ldots, n. \tag{2.1}$$

If $R^G(j, i, \mathbf{p})$ is the reliability of Weighted j-out-of-i:G system then

$$R^G(j, i, \mathbf{p}) = p_i R^G(j, i, 1_i, \mathbf{p}) + (1 - p_i) R^G(j, i, 0_i, \mathbf{p}) \text{ for } i = 1, \ldots, n.$$

$R_G(j, i, 1_i, \mathbf{p})$ is equal to $R^G(j - w_i, i - 1, \mathbf{p})$ because given that i^{th} component is working, the rest of the system requires working components weighing at least $j - w_i$ to survive, thus it is $j - w_i$-out-of-$i - 1$:G system. $R_G(j, i, 0_i, \mathbf{p})$ is equal to $R_G(j, i - 1, \mathbf{p})$ because given that i^{th} component is failed, the rest of the system still requires working components weighing at least j to survive, thus it is j-out-of-$i - 1$:G system. Therefore the recursive formula is:

$$R^G(j, i, \mathbf{p}) = p_i R^G(j - w_i, i - 1, \mathbf{p}) + (1 - p_i) R^G(j, i - 1, \mathbf{p}) \text{for } i = 1, \ldots, n.$$

with the initial conditions,

$$R^G(j, i, \mathbf{p}) = 1, \text{for } j \leq 0 \text{ and } i = 0, 1, \ldots, n.$$
$$R^G(j, 0, \mathbf{p}) = 0, \text{for } j > 0.$$

Considering $R^G(j, 0, \mathbf{p})$ for $j \leq 0$, since the weight of 0 components is always 0, which is greater than or equal to j hence $R^G(j, 0, \mathbf{p})$ is 1. If we replace p_i's with x_i's in the formula, we can also obtain the structure function of the system. A table with at most n+1 rows and k+1 columns is needed to record reliabilities of all called functions. Therefore this algorithm requires $O((n+1)(k+1))$ or $O(n \cdot k)$ computing time and space to evaluate the reliability of n components.

Note that the reliability of weighted k-out-of-n:G system can also be represented as

$$R^G(k, n, \mathbf{p}) = P(\sum_{i=1}^{n} Y_i \geq k) = P(\sum_{i=1}^{n} w_i X_i \geq k) \tag{2.2}$$

and the recursive relation (2.1) can be obtained from (2.2) by conditioning on Y_n.

2.4 Weighted k-out-of-n:F Systems

Weighted k-out-of-n:F System survives iff the total weight of failed components is less than a fixed threshold k. The weighted k-out-of-n:F System is the dual of the Weighted k-out-of-n:G system with same configuration. A cut set of the system contains any combination of components whose total weight is at least k. Therefore path sets of the G system are the cut sets of the F system. This holds for minimal path and cut sets as well.

The states of components will be $x_i = 0$ if component i is failed and $x_i = 1$ if component i survives. Therefore if each cut set generates the event that all components in the cut set are failed and components in its complement are good, the sum of probabilities of events derived from all cut sets yield the unreliability of the system. Since reliability is the complement of the unreliability, the reliability function is:

$$P(\sum_{i=1}^{n} Y_i < k) = P(\sum_{i=1}^{n} Z_i \geq W - k) = h'(\mathbf{p}) = 1 - \sum_{j=1}^{l} \left[\prod_{i \in C_j} (1 - p_i) \prod_{s \in {C_j}^c} p_s \right]$$

where $W = \sum_{i=1}^{n} w_i$ and l is the number of cut sets.

These equations coincide with the definition of duality $h^D(\mathbf{p}) = 1 - h(\mathbf{1} - \mathbf{p}) =$

$h'(\mathbf{p})$ where $\mathbf{p}$ is the probability vector of the system and $\mathbf{1}$ is a vector of ones of size n.

Alternatively to derive the structure function, minimal cut sets $MC_1, MC_2 \ldots MC_s$ are obtained by removing all cut sets containing other cut sets as a subset. The coproduct of all components in a minimal cut set yield the minimal cut structure of that set. The product of all minimal cut structures yield the boolean structure function of the system. After it is converted to ordinary algebra terms using boolean laws, the expectation would yield the reliability of the system. So,

$$\phi(\mathbf{x}) = \min_{1 \leq i \leq s} \max_{j \in MC_i} x_j$$

means

$$\phi(\mathbf{x}) = \bigwedge_{i=1}^{s} \bigvee_{j \in MC_i} x_j = \prod_{i=1}^{s} \coprod_{j \in MC_i} x_j$$

Using Wu and Chen's method, the reliability function of the F system is derived similarly. By the duality of G and F systems we only replace p_i's by $1-p_i$'s and take the complement of initial condition values. If initial condition values did not change, the result would yield the unreliability of F system. Complementing the initial values takes the complement of the whole function which yields the reliability of the F system. The reliability function of weighted j-out-of-i:F system $R^F(j, i, \mathbf{p})$ is obtained by pivotial decomposition as in G system:

$$R^F(j, i, \mathbf{p}) = (1 - p_i) R^F(j, i, 0_i, \mathbf{p}) + p_i R^F(j, i, 1_i, \mathbf{p}) \text{for } i = 1, \ldots, n.$$

$R_F(j, i, 0_i, \mathbf{p})$ is equal to $R^F(j - w_i, i - 1, \mathbf{p})$ because given that i^{th} component is failed the rest of the system requires failed components weighing at least $j - w_i$ to fail, thus it is $j - w_i$-out-of-$i-1$:F system. $R_F(j, i, 1_i, \mathbf{p})$ is equal to $R_F(j, i-1, \mathbf{p})$ because given that i^{th} component is working, the rest of the system still requires failed components weighing at least j to fail, thus it is j-out-of-$i - 1$:F system. Therefore the recursive formula is:

$$R^F(j, i, \mathbf{p}) = (1 - p_i) R^F(j - w_i, i - 1, \mathbf{p}) + p_i R^F(j, i - 1, \mathbf{p}) \text{for } i = 1, \ldots, n.$$

with initial conditions,

$$R^F(j, i, \mathbf{p}) = 0, \text{for } j \leq 0 \text{ and } i = 0, 1, \ldots, n.$$
$$R^F(j, 0, \mathbf{p}) = 1, \text{for } j > 0.$$

We took complement of initial condition values, since weight argument less than or equal to 0 would imply that the total weights of the failed components have exceeded the threshold hence the unreliability is 1. Likewise if there are no components left and weight argument is greater than 0, it would imply that total weights of failed components were below the threshold, hence the unreliability is 0. If we replace p_i's with x_i's in the formula, we obtain the structure function of the F system.

2.5 Equivalent Coherent Systems and Component Importance in G Systems

Samaniego and Shaked (2008) stated that weighted systems can have equivalent coherent systems with a specific structure. First minimal path and cut sets for the weighted system are derived then any coherent system with same minimal path and cut sets is considered as an equivalent coherent system. In other words, a coherent system with the same structure or reliability function as the weighted system is its equivalent. The components not included in any of the minimal path or cut sets are irrelevant components thus their variables are cancelled out in the reliability and structure functions. The position of a component in a coherent structure would be determined by the weights of components and the threshold. If the locations of components in a coherent structure solely determine their structural importances, the weights of components solely determine their structural importances in weighted k-out-of-n systems. The position of a component has no significance in these weighted systems. We will consider weighted k-out-of-n:G systems for the evaluation of component importance. Similar arguments can be extended to F systems. We can derive the reliability and structural importances of each component from reliability and structure functions. As we have stated before, the reliability function is obtained just by taking the expectation of the structure function. The reliability importance of a components also takes account the reliabilities of other components and uses the reliability function of the system:

$$R_\phi(i, \mathbf{p}_{\{\mathbf{i}\}^\mathbf{c}}) = \frac{\partial}{\partial p_i} h(\mathbf{p})$$

The partial derivative implies that the reliability importance measures the effect of a change in reliability of a component to the reliability of the whole system.

The structural importance does not take account the reliabilities of components and but just their locations in an equivalent coherent structure. The structural importance assumes that all components have equal probability of failure and survival, therefore if the set the probabilities of all other components to 1/2 in the reliability importance function we obtain the structural importance of the component. Another way to find the structural importance is by finding the critical path set of the component. A critical path set for component i includes components that the system is down when they are up and system is up when in addition the component i is up. We can generate its corresponding binary vector of size n-1 indicating the states of other components where the components in the set are 1 and others are 0. At these states of n-1 components, the systems survival depends only on the component i. So the critical path vectors of component i consist of all $\mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}}$ such that

$$\phi(1_i, \mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}}) - \phi(0_i, \mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}}) = 1$$

The structural importance of i equals the number of its critical path vectors divided by the number permutations of n-1 component states. Therefore:

$$I_\phi(i) = \frac{1}{2^{n-1}} \sum_{\mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}}} \left[\phi(1_i, \mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}} - \phi(0_i, \mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}})\right]$$

where $\sum_{\mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}}} \left[\phi(1_i, \mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}} - \phi(0_i, \mathbf{x}_{\{\mathbf{i}\}^\mathbf{c}})\right]$ yields the number of critical path vectors (or equivalently critical path sets) of component i.

If we delete component i from all of minimal path sets including component i, then we obtain critical path sets for component i. Irrelevant components can also be included in the critical path set of component i since they do not effect the state of the system. Thus the number of critical path sets of component i are equal to or greater than the number of minimal path sets including component i. It is also obvious that irrelevant components' structural and reliability importances are 0 since their variables and reliabilities are not included in structure or reliability functions.

Because reliability and structure importances are a function of the structure function which in turn is a function of weights, importances can be considered as functions of weights. Samaniego and Shaked (2008) have proved that reliability and structure importances of a component are increasing function of its weight w_i if all other weights and the threshold are fixed. This is obvious from the fact that as a component's weight increases it is more likely to effect the state of the system. However importances of other components are not necessarily decreasing functions of w_i.

Samaniego and Shaked (2008) also stated that if a component's weight exceeds the threshold then the reliability and structural importances of other components are minimized with respect to w_i when other weights are kept fixed. We can further prove that if a component i's weight is increased from below the threshold to the threshold while keeping others fixed, the importances of all other components sharing the same critical path set with i will decrease since the number of their critical path vectors will decrease. Any component with weight below the threshold has to be in a critical path set of at least one component regardless of whether it is relevant or irrelevant. Thus the importance of at least one component will decrease if w_i increases to or above the threshold value. Increasing component i's weight any further above the threshold will not affect importances of other components.

2.6 Weighted Consecutive k-out-of-n:F Systems

A linear weighted consecutive k-out-of-n:F system consists of n components ordered in a line. The system fails iff the total weight of consecutively failed components is at least k. Usual consecutive k-out-of-n:F system is a special case of the weighted one, where each component has the weight 1.

Wu and Chen (1994b) have proposed an efficient algorithm for evaluation of the exact reliability. First the components that form a minimal cut are found by scanning components from the beginning to the end. The components in a cut set have to be consecutive therefore the weights of components are added in sequence from the beginning to the end until the total weight reaches k. Once the threshold k is reached or exceeded, a cut set will be obtained. Samaniego and Shaked (2008) have expressed cut sets C_i of the form $\{i, \ldots, n(i)\}$ where $1 \leq i \leq n(i) \leq n$ as:

$$\sum_{j=i}^{n(i)-1} w_j < k \leq \sum_{j=i}^{n(i)} w_j$$

The algorithm would stop adding weights when n(i) is reached and starts subtracting weights of components from the beginning of the cut set from the total weight sequentially. If the weight drops below k then a minimal cut has been obtained and the components at the beginning and the end of the minimal cut are recorded. The process repeats the same arguments starting from the second component of the minimal cut set. Once the minimal cut sets are derived, they are arranged in increasing order with respect to their components from 1 to r. Components in each set are also arranged in increasing order. Therefore $First(MC_i)$ denotes the component with smallest index and $Last(MC_i)$ indicates the component with the largest index in the set MC_i.

The methods discussed in previous sections can be applied to find the unreliability function of the system, namely multiplying the minimal cuts, inclusion exclusion method and sum of disjoint products method. Wu and Chen (1994) have employed the sum of disjoint products method which requires O(n) computing time. If $x_i = 1$ when component i is failed and $x_i = 0$ when component i survives, a minimal cut structure is expressed as the product of all component variables in the set in contrast to coproduct in the previous section where states of components were reversed. The unreliability of the system would be expressed as the failure of at least one of the minimal cut structures:

$$F(k,n,\mathbf{p}) = E(\coprod_{i=1}^{r} \prod_{m\in MC_i} X_m) = E(\prod_{m\in MC_1} X_m \vee \prod_{m\in MC_2} X_m \vee \ldots \vee \prod_{m\in MC_r} X_m)$$

Expectation of binary random variables in this equation would yield the unreliabilities of components and the system. Let S_i denote the event that all components in the minimal cut set MC_i are failed or the minimal cut structure fails, i.e. the product of all variables in the set is 1. If any S_i for $i = 1,\ldots,r$ occurs, system fails. Therefore,

$$P(S_i) = E(\prod_{m\in MC_i} X_m) = P(\prod_{m\in MC_i} X_m = 1)$$
$$F(k,n,\mathbf{p}) = P(S_1 \cup S_2 \cup \ldots \cup S_r)$$

According to the sum of disjoint products method, the unreliability from component 1 to last component of j^{th} minimal cut set equals the probability that at least one of all preceding minimal cut structures fail plus the joint probability none of the preceding minimal cut structures fail and the j^{th} minimal cut structure fails:

$$F(k, Last(MC(j)),\mathbf{p}) = P(S_1 \cup S_2 \cup \ldots \cup S_{j-1}) + P(\overline{S_1} \cap \overline{S_2} \cap \ldots \cap \overline{S_{j-1}} \cap S_j) \tag{2.3}$$

which is same as

$$\begin{aligned} F(k, Last(MC(j)),\mathbf{p}) = P(&\coprod_{i=1}^{j-1} \prod_{m\in MC_i} X_m = 1) \\ &+ P((1-\prod_{m\in MC_1} X_m)(1-\prod_{m\in MC_2} X_m)\ldots \\ &\cdot (1-\prod_{m\in MC_{j-1}} X_m)\cdot(\prod_{m\in MC_j} X_m) = 1) \end{aligned}$$

The probabilities of two events on the r.h.s. of (2.3) can be added by their mutual exclusiveness. The first term on the r.h.s. is the unreliablity of the system from component 1 to the last component of the $j-1^{th}$ minimal cut set. The second term is the joint probability that the system up to last component of the $j-1^{th}$ minimal cut set is working and all components in j^{th} minimal cut set are failed. If $j-1^{th}$ and j^{th} minimal cut sets have no components in common then the probability of S_j can be multiplied due to independence with the reliability of the system from component 1 to $Last(MC_{j-1})$,

$$F(k, Last(MC(j)), \mathbf{p}) = F(k, Last(MC(j-1)), \mathbf{p}) + R(k, Last(MC(j-1)), \mathbf{p})P(S_j)$$

If last two minimal cut sets have components in common, then all components in the last set must be failed and at least one component not element of the last set but element of the preceding set must be working. Therefore,

$$\begin{aligned} F(k, Last(MC(j), \mathbf{p})) = {} & P(\coprod_{i=1}^{j-1} \prod_{m \in MC_i} X_m = 1) \\ & + P((1 - \prod_{m \in MC_1} X_m)(1 - \prod_{m \in MC_2} X_m) \ldots \\ & \cdot (1 - \prod_{m \in MC_{j-1}, m \notin MC_j} X_m) \cdot (\prod_{m \in MC_j} X_m) = 1) \end{aligned}$$

There are $\|MC_{j-1} - MC_j\|$ mutually exclusive events for the different locations of last working component p (working component with the highest index) which is in the set MC_{j-1} but not in MC_j. The reliability of the system up to that component p is simply the probability that all minimal cut structures which do not include component p work. Since all minimal cut structures including component p work, their reliability is 1. All components following component p must be failed. Thus the probability of each event can be found by independence of p, failed components after p and the reliability of the system before p which we already know. The formula for this algorithm is written explicitly by Wu and Chen (1994b). This recursive formula is efficient requiring O(n) computing time since it requires at most one entry for each $F(k, i, \mathbf{p})$ for every $i = 1, \ldots, n$. There is another recursive method starting from the end of the structure that conditions on the last working component and this method will be used in the next chapter.

Chapter 3

Combined Systems with Weighted Components

3.1 Introduction

Reliability of systems with multiple failure and survival criteria have been studied in literature recently. If we combine the criteria of both weighted f-out-of-n structure and weighted consecutive k-out-of-n structure in one system, we obtain the following combined models:

A linear weighted $(n, f, k, \mathbf{w})$:F system consists of n components ordered in a line and the system fails iff the total weight of failed components is at least f or the total weight of the failed consecutive components is at least k. Thus the system works iff total weight of failed components is less than f AND the total weight of failed consecutive components is less than k. Therefore this system survives iff both corresponding weighted f-out-of-n:F and weighted consecutive k-out-of-n:F structures survive.

A linear weighted $(n, f, k, \mathbf{w})$:G system consists of n components ordered in a line and the system survives iff the total weight of working components is at least f OR the total weight of the working consecutive components is at least k.

Thus the system fails iff total weight of working components is less than f AND the total weight of working consecutive components is less than k. Therefore this system survives when either one of weighted f-out-of-n:G or weighted consecutive k-out-of-n:G structures work. Thus, failure of the system requires that both corresponding weighted f-out-of-n:G and the weighted consecutive k-out-of-n:G structure fail. The system's unreliability will be found by the probability that both failure criteria occur.

A linear weighted-$\langle n, f, k, \mathbf{w} \rangle$:F system consists of n components ordered in a line and the system fails iff the total weight of failed components is at least f AND the total weight of the failed consecutive components is at least k. Thus the system works if total weight of failed components is less than f OR the total weight of failed consecutive components is less than k. Therefore if either one of weighted f-out-of-n:F OR weighted consecutive k-out-of-n:F structures work, the system survives . If both of these structures fail, the system fails.

Tung (1982) has first introduced (n, f, k) systems in reliability context, Sun and Liao (1990) and Cheng et al. (1999) have presented formulas for special and general cases respectively. Zuo et al. (2000) and Gera (2004) have also studied F and G combined systems respectively. Recently Eryılmaz (2008) has derived lifetime distributions of combined systems with exchangable components. Cui et al. (2005) have proposed both recursive methods and Markov Chain imbedding technique for exact reliability of both (n, f, k):F(G) and $\langle n, f, k \rangle$:F(G) models without weights. In this chapter, exact recursive reliability formulas for $(n, f, k, \mathbf{w})$:F(G) systems with weighted, independent components are presented which can also be used for usual models of these systems. Our method is as efficient as the formula presented by Zuo et al. (2000) for usual systems.

The algorithms explained in this chapter can also be extended to weighted $\langle n, f, k \rangle$:F(G) to obtain the weighted version of the formula found by Cui et al. (2005). In other words, if all component weights are 1, weighted unreliability formula for $\langle n, f, k \rangle$:F system found by method presented here would reduce to the related formula in Cui et al. (2005), therefore it will not be discussed in this thesis.

These models can be used in practical problems having failure (survival) criteria involving both weighted consecutive components and all weighted components together.

3.2 Definition of Symbols

Let X_i denote the state of the ith component ($X_i = 1(0)$ if component i is failed (working)), $i = 1, 2, ..., n$. Suppose that component i is associated with a weight $w_i \in \mathbb{Z}^+, i = 1, 2, ..., n$ and the components are independent and the reliability of the ith component is $p_i = P\{X_i = 0\}$ ($q_i = 1 - p_i$), $i = 1, 2, ..., n$. Define the vectors $\mathbf{Y} = (Y_1, ..., Y_n), \mathbf{Z} = (Z_1, ..., Z_n)$, $\underline{\theta}^{(\mathbf{1})} = (\theta_1^{(1)}, ..., \theta_n^{(1)})$, and $\underline{\theta}^{(\mathbf{0})} = (\theta_1^{(0)}, ..., \theta_n^{(0)})$, where

$$Y_i = w_i \cdot X_i,\ i = 1, 2, ..., n,$$

$$Z_i = w_i \cdot (1 - X_i),\ i = 1, 2, ..., n,$$

and

$$\theta_{i+1}^{(1)} = \begin{cases} \theta_i^{(1)} + Y_{i+1} & \text{if } Y_{i+1} \neq 0 \\ 0 & \text{if } Y_{i+1} = 0 \end{cases},\ i = 0, 1, ..., n-1,$$

and

$$\theta_{i+1}^{(0)} = \begin{cases} \theta_i^{(0)} + Z_{i+1} & \text{if } Z_{i+1} \neq 0 \\ 0 & \text{if } Z_{i+1} = 0 \end{cases},\ i = 0, 1, ..., n-1,$$

with the convention $\theta_0^{(1)} = \theta_0^{(0)} = 0$. It is clear that the random variables $\sum_{i=1}^n Y_i$ and $\sum_{i=1}^n Z_i$ represent the total weight of the failed and working components, respectively.

It can be easily seen that $\theta_i^{(1)}, \theta_i^{(0)} \in \left\{0, w_i, w_{i-1} + w_i, w_{i-2} + w_{i-1} + w_i, ..., \sum_{j=1}^i w_j\right\}$, $i = 1, 2, ..., n$, $w_0 \equiv 0$.

Example 3.1. Let the states of $n = 10$ components be 1001100010 with the weight vector $\mathbf{w} = (1, 2, 1, 3, 2, 3, 3, 2, 2, 3)$, where "1" and "0" represent failure and working states, respectively. Then the linear weighted-$(10, 6, 7, \mathbf{w})$:F system is in a failure state while the linear weighted-$(10, 6, 7, \mathbf{w})$:G system is in a working state. Then it is also true that $\theta^{(1)} = (1, 0, 0, 3, 5, 0, 0, 0, 2, 0)$.

3.3 (n, f, k):F Systems with Weighted Components

The reliability of the weighted $(n, f, k, \mathbf{w})$:F system can be represented in terms $\mathbf{Y}$, and $\underline{\theta}^{(\mathbf{1})}$. They will be used to represent two structures that need to work for survival of the system, namely, weighted consecutive k-out-of-n:F and weighted f-out-of-n:F structures. More explicitly, the survival of the linear weighted-$(n, f, k, \mathbf{w})$:F system requires that the sum of Y_i's must be less than f, and all the elements of the sequence $\left\{\theta_i^{(1)}, 1 \leq i \leq n\right\}$ which measure the total weight of consecutive failed components, must be less than k. Thus the reliability of linear weighted $(n, f, k, \mathbf{w})$:F system is represented by the following probability.

$$R_{\mathbf{w}}(n, f, k\text{:F}) = P\left\{\theta_1^{(1)} < k, ..., \theta_n^{(1)} < k \wedge \sum_{i=1}^{n} Y_i < f\right\}.$$

Theorem 3.1 *The reliability of linear weighted-$(n, f, k, \mathbf{w})$:F system is given as follows.*

For $f \leq k$

$$R_{\mathbf{w}}(n, f, k\text{:}F) = R_{\mathbf{w}}(n, f\text{:}F)$$

where $R_{\mathbf{w}}(n, f\text{:}F)$ is the reliability of weighted f-out-of-n:F System.

For $f > k$

if $k \leq w_n$ then

$$R_{\mathbf{w}}(n, f, k\text{:}F) = R_{\mathbf{w}}(n-1, f, k\text{:}F)p_n,$$

if $w_{n-j+1} + ... + w_n < k \leq w_{n-j} + ... + w_n, j = 1, ..., n-1$ then

$$R_{\mathbf{w}}(n, f, k\text{:}F) = \sum_{m=1}^{j+1}\left[R_{\mathbf{w}}(n-m, f - \sum_{i=n-m+2}^{n} w_i, k\text{:}F)p_{n-m+1} \prod_{t=n-m+2}^{n} q_t\right],$$

and if $w_1 + ... + w_n < k$ then $R_{\mathbf{w}}(n, f, k\text{:}F) = 1$.

Proof. We will start with the case $f > k$. First the reliability of the weighted consecutive k-out-of-n:F structure will be considered. Maximum number of failed components j that can follow the last working component p is found. By conditioning on possible locations for p, we create mutually exclusive events whose probabilities can be added. Since $k < f$, the weighted f-out-of-n structure also survives.

If $w_{n-j+1} + ... + w_n < k \leq w_{n-j} + ... + w_n, j = 1, ..., n-1$ then

$$R_{\mathbf{w}}(n, f, k\text{:F}) = P(\theta_1^{(1)} < k, ..., \theta_{n-1}^{(1)} < k, X_n = 0 \wedge \sum_{i=1}^{n-1} Y_i < f)$$

$$+ P(\theta_1^{(1)} < k, ..., \theta_{n-2}^{(1)} < k, X_{n-1} = 0, X_n = 1 \wedge \sum_{i=1}^{n-2} Y_i < f - w_n) + \dots$$

$$+ P(\theta_1^{(1)} < k, ..., \theta_{n-j-1}^{(1)} < k, X_{n-j} = 0, X_{n-j+1} = 1, \dots X_n = 1$$

$$\wedge \sum_{i=1}^{n-j-1} Y_i < f - w_n - \dots - w_{n-j+1})$$

Therefore,
if $w_{n-j+1} + ... + w_n < k \leq w_{n-j} + ... + w_n, j = 1, ..., n-1$ then

$$R_{\mathbf{w}}(n, f, k\text{:F}) = R_{\mathbf{w}}(n-1, f, k\text{:F})p_n + R_{\mathbf{w}}(n-2, f - w_n, k\text{:F})p_{n-1}q_n + \dots$$

$$+ R_{\mathbf{w}}(n-j-1, f - w_n - \dots - w_{n-j+1}, k\text{:F})p_{n-j}q_{n-j+1} \dots q_n.$$

Regarding the initial condition, if total weight of components left in the system is less than k, total weight of failed components must be less than both f and k, thus none of the failure criteria can occur so the system's survival would be certain.

If $f \leq k$ then the survival of the second structure $\sum_{i=1}^{n-2} Y_i < f$ would imply that all $\theta^{(1)}$'s are also less than f, thus less than k. Therefore if the weighted f-out-of-n:F structure works, it implies that weighted consecutive k-out-of-n:F structure also works so the system is reduced to a weighted f-out-of-n:F structure. □

In the following example we illustrate the computation of $R_{\mathbf{w}}(n, f, k\text{:F})$ using Theorem 3.1. We choose $n = 5$ so that the computations can be done by hand.

Example 3.2. Let the system consists of $n = 5$ components with the weight vector $\mathbf{w} = (2, 3, 2, 1, 3)$. For $f = 4$ and $k = 3$ we have

$$R_{\mathbf{w}}(5, 4, 3\text{:F}) = R_{\mathbf{w}}(4, 4, 3\text{:F})p_5 \ (k \leq w_5),$$

$$R_{\mathbf{w}}(4, 4, 3\text{:F}) = R_{\mathbf{w}}(3, 4, 3\text{:F})p_4 + R_{\mathbf{w}}(2, 3, 3\text{:F})p_3q_4 \ (w_4 < k \leq w_3 + w_4),$$

$$R_{\mathbf{w}}(3, 4, 3\text{:F}) = R_{\mathbf{w}}(2, 4, 3\text{:F})p_3 + R_{\mathbf{w}}(1, 2, 3\text{:F})p_2q_3 \ (w_3 < k \leq w_2 + w_3),$$

$$R_{\mathbf{w}}(1, 2, 3\text{:F}) = p_1, R_{\mathbf{w}}(2, 3, 3\text{:F}) = p_2, R_{\mathbf{w}}(2, 4, 3\text{:F}) = p_2.$$

Thus

$$R_{\mathbf{w}}(5, 4, 3\text{:F}) = p_2p_3p_5 + p_1p_2q_3p_4p_5.$$

If all weights are positive integers, the computation time and space needed for these formulas is $O(n \cdot f)$ since a table with at most n+1 columns and f+1 rows can be constructed to record the values of functions called within recursive formulas for every value of f and n.

To compare our formula to one found by Zuo et al. (2000), both formulas for the usual case of combined F model (where all weights are 1) will be given. Zuo et al.'s (2000) formula for (i, f, k):F model:

$$\begin{aligned} R(i, f, k\text{:F}) &= p_iR(i-1, f, k\text{:F}) + q_iR(i-1, f-1, k\text{:F}) \\ &+ [1 - R(i-k-1, f-k, k\text{:F})]p_{i-k} \prod_{t=i-k+1}^{i} q_t \end{aligned}$$

Our formula for the usual case of (i, f, k):F model:

$$\begin{aligned} R(i, f, k\text{:F}) &= \sum_{m=1}^{k} [R(i-m, f-m+1, k\text{:F})p_{i-m+1} \prod_{t=i-m+2}^{i} q_t] \\ &= R(i-1, f, k\text{:F})p_i + R(i-2, f-1, k\text{:F})p_{i-1}q_i + \ldots \\ &+ R(i-k, f-k+1, k\text{:F})p_{i-k+1}q_{i-k+2} \ldots q_i. \end{aligned}$$

Zuo et al.'s (2000) formula has 3 and our formula has k terms but they have the same complexity requiring $O(n \cdot f)$ computing time and space. Therefore, our formula can be an alternative to Zuo et al.'s (2000) formula.

3.4 (n, f, k):G Systems with Weighted Components

The unreliability of the weighted $(n, f, k, \mathbf{w})$:G system can be represented in terms $\mathbf{Z}$, and $\underline{\theta}^{(\mathbf{0})}$. They will be used to represent two structures that need to fail for failure of the system, namely, weighted consecutive k-out-of-n:G and weighted f-out-of-n:G structures. More explicitly, the failure of the linear weighted $(n, f, k, \mathbf{w})$:G system requires that the sum of Z_i's must be less than f, and all the elements of the sequence $\left\{\theta_i^{(0)}, i \geq 1\right\}$ which measure the total weight of consecutive working components, must be less than k. Thus the unreliability of linear weighted $(n, f, k, \mathbf{w})$:G can be represented as

$$F_{\mathbf{w}}(n, f, k\text{:G}) = P\left\{\theta_1^{(0)} < k, ..., \theta_n^{(0)} < k \wedge \sum_{i=1}^{n} Z_i < f\right\}.$$

Theorem 3.2 *The unreliability of linear weighted-$(n, f, k, \mathbf{w})$:G system is given as follows.*

For $f \leq k$ we have

$$F_{\mathbf{w}}(n, f, k\text{:}G) = F_{\mathbf{w}}(n, f\text{:}G)$$

where $F_{\mathbf{w}}(n, f\text{:}G)$ is the unreliability of weighted f-out-of-n:G System.

For $f > k$,

if $k \leq w_n$ then

$$F_{\mathbf{w}}(n, f, k\text{:}F) = F_{\mathbf{w}}(n-1, f, k\text{:}F)q_n,$$

if $w_{n-j+1} + ... + w_n < k \leq w_{n-j} + ... + w_n, j = 1, ..., n-1$ then

$$F_{\mathbf{w}}(n, f, k\text{:}G) = \sum_{m=1}^{j+1}\left[F_{\mathbf{w}}(n-m, f - \sum_{i=n-m+2}^{n} w_i, k\text{:}F)q_{n-m+1} \prod_{t=n-m+2}^{n} p_t\right],$$

and if $w_1 + ... + w_n < k$ then $F_{\mathbf{w}}(n, f, k\text{:}G) = 1$.

Proof. We will start with the case $f > k$. First the unreliability of the weighted consecutive k-out-of-n:G structure will be considered. The maximum number of working components j, that can follow the last failed component q is found. By conditioning on possible locations for q, we would create a mutually exclusive event whose probabilities can be added. Since $k < f$, the weighted f-out-of-n:G structure also fails.

If $w_{n-j+1} + ... + w_n < k \leq w_{n-j} + ... + w_n, j = 1, ..., n-1$ then

$$R_{\mathbf{w}}(n, f, k\text{:G}) = P(\theta_1^{(0)} < k, ..., \theta_{n-1}^{(0)} < k, X_n = 1 \wedge \sum_{i=1}^{n-1} Z_i < f)$$
$$+ P(\theta_1^{(0)} < k, ..., \theta_{n-2}^{(0)} < k, X_{n-1} = 0, X_n = 1 \wedge \sum_{i=1}^{n-2} Z_i < f - w_n) + \dots$$
$$+ P(\theta_1^{(0)} < k, ..., \theta_{n-j-1}^{(0)} < k, X_{n-j} = 0, X_{n-j+1} = 1, \dots X_n = 1$$
$$\wedge \sum_{i=1}^{n-j-1} Z_i < f - w_n - \dots - w_{n-j+1})$$

Therefore,
if $w_{n-j+1} + ... + w_n < k \leq w_{n-j} + ... + w_n, j = 1, ..., n-1$ then

$$F_{\mathbf{w}}(n, f, k\text{:G}) = F_{\mathbf{w}}(n-1, f, k\text{:G})p_n + F_{\mathbf{w}}(n-2, f - w_n, k\text{:G})p_{n-1}q_n + \dots$$
$$+ F_{\mathbf{w}}(n-j-1, f - w_n - \dots - w_{n-j+1}, k\text{:G})p_{n-j}q_{n-j+1} \dots q_n.$$

Regarding the initial condition, if total weight of components left in the system is less than k, total weight of failed components must be less than both f and k, so none of the survival criteria can occur so the system's failure would be certain.

If $f \leq k$ then the failure of the second structure $\sum_{i=1}^{n-2} Z_i < f$ would imply that all $\theta^{(0)}$'s are also less than f, thus less then k. Therefore if the weighted f-out-of-n:G structure fails, it implies that the weighted consecutive k-out-of-n:G structure also fails so the system is reduced to a weighted f-out-of-n:F structure.

It is clear that if we replace p_i's in the unreliability function of the weighted $(n, f, k) : G$ system by q_i's, we obtain the reliability function of the weighted $(n, f, k) : F$ system. Since reliability is the complement of failure, this coincides with the definition of duality. Hence weighted $(n, f, k) : G$ and $(n, f, k) : F$ systems are duals of each other.

□

Example 3.3. For the system described in example 3.2,

$$F_{\mathbf{w}}(5, 4, 3\text{:G}) = F_{\mathbf{w}}(4, 4, 3\text{:G})q_5 \ (k \leq w_5),$$

$$F_{\mathbf{w}}(4, 4, 3\text{:G}) = F_{\mathbf{w}}(3, 4, 3\text{:G})q_4 + F_{\mathbf{w}}(2, 3, 3\text{:G})q_3 p_4 \ (w_4 < k \leq w_3 + w_4),$$

$$F_{\mathbf{w}}(3, 4, 3\text{:G}) = F_{\mathbf{w}}(2, 4, 3\text{:G})q_3 + F_{\mathbf{w}}(1, 2, 3\text{:G})q_2 p_3 \ (w_3 < k \leq w_2 + w_3),$$

$$F_{\mathbf{w}}(1, 2, 3\text{:G}) = q_1, F_{\mathbf{w}}(2, 3, 3\text{:F}) = q_2, F_{\mathbf{w}}(2, 4, 3\text{:G}) = q_2.$$

Thus

$$F_{\mathbf{w}}(5, 4, 3\text{:G}) = q_2 q_3 q_5 + q_1 q_2 p_3 q_4 q_5.$$

3.5 Equivalence of Usual Combined Systems and Weighted Combined Systems

If a non-weighted consecutive k-out-of-n:F(G) system has the same minimal cut (path) sets of a weighted consecutive k-out-of-n:F(G) they are considered as equivalent systems. This holds true for non-weighted and weighted f-out-of-n:F (G) systems as well. The minimal cuts (paths) of weighted consecutive k-out-of-n:F(G) and weighted-f-out-of-n:F(G) models will be found with the same methods described in Chapter 2 and their non-weighted equivalents (if they exist) whose minimal cuts (paths) are already known are found by the following theorem.

Theorem 3.3 *A consecutive-weighted-k-out-of-n:F (G) system is equivalent to a* *consecutive-k^*-out-of-n:F (G) system if*

$$\max_{1\le j\le n-k^*+2}\left(\sum_{i=j}^{j+k^*-2} w_i\right) < k \le \min_{1\le j\le n-k^*+1}\left(\sum_{i=j}^{j+k^*-1} w_i\right), \tag{3.1}$$

and a weighted-f-out-of-n:F (G) system is equivalent to a f^*-out-of-n:F (G) system if

$$\max_{\vec{i}_{f^*-1}\in C^{f^*-1}_{1,2,..,n}}\left(\sum_{j=1}^{f^*-1} w_{i_j}\right) < f \le \min_{\vec{i}_{f^*}\in C^{f^*}_{1,2,..,n}}\left(\sum_{j=1}^{f^*} w_{i_j}\right), \tag{3.2}$$

where $\vec{i}_m = (i_1, i_2, ..., i_m)$, and $C^m_{1,2,..,n}$ denotes the set of all m-combinations of $\{1, 2, ..., n\}$.

Proof A consecutive-weighted-k-out-of-n:F (G) system is equivalent to a consecutive-k^*-out-of-n:F (G) system if for all $1 \le j \le k^* - 1$,

$$w_{i-j+1} + ... + w_i < k, \quad i = j, j+1, ..., n, \tag{3.3}$$

and for all $k^* \le j \le n$

$$w_{i-j+1} + ... + w_i \ge k, \quad i = j, j+1, ..., n. \tag{3.4}$$

We observe that if (3.3) holds true for $j = k^* - 1$ then it is satisfied for all $1 \le j < k^* - 1$. Thus the condition (3.3) is equivalent to

$$w_{i-k^*+2} + ... + w_i < k, \quad i = k^* - 1, k^*, ..., n. \tag{3.5}$$

Similarly, if (3.4) holds true for $j = k^*$ then it is also true for all $k^* < j \le n$. That is, the condition (3.4) is equivalent to the condition

$$w_{i-k^*+1} + ... + w_i \ge k, \quad i = k^*, k^* + 1, ..., n. \tag{3.6}$$

Combining (3.5) and (3.6) we obtain (3.1). On the other hand, a weighted-k-out-of-n:F (G) system is equivalent to a f^*-out-of-n:F (G) system if for all $1 \le j \le f^* - 1$ the sum of any j weights is less than k, and for all $f^* \le j \le n$ the sum of any j weights is at least k. Repeating the above arguments for weighted-f-out-of-n:F (G) system we get (3.2).■

Corollary 3.4 *A linear weighted-$(n, f, k, \mathbf{w})$:F (G) system is equivalent to (n, f^*, k^*):F (G) system if both* (3.1) *and* (3.2) *hold true.*

To make the clear the above Theorem and its proof we provide the following simple illustration which includes all possible states of $n = 4$ components. In Table 3.1 we present the states of three weighted systems S1: weighted-7-out-of-4:F, S2: consecutive-weighted-5-out-of-4:F, and S3: weighted-$(4, 7, 5, \mathbf{w})$:F when $\mathbf{w}$ =$(2, 3, 3, 2)$.

States	**Y**	S1	S2	S3	States	**Y**	S1	S2	S3
0000	0000	0	0	0	0101	0302	0	0	0
1000	2000	0	0	0	0110	0330	0	1	1
0100	0300	0	0	0	0011	0032	0	1	1
0010	0030	0	0	0	1110	2330	1	1	1
0001	0002	0	0	0	0111	0332	1	1	1
1100	2300	0	1	1	1101	2302	1	1	1
1010	2030	0	0	0	1011	2032	1	1	1
1001	2002	0	0	0	1111	2332	1	1	1

Table 3.1. States of the systems S1, S2, and S3.

In view of Table 3.1, weighted-7-out-of-4:F system is equivalent to the 3-out-of-4:F system because the minimal cut and path sets of these two systems are same when $\mathbf{w}$ =$(2, 3, 3, 2)$. It can be easily checked that the condition (3.2) is satisfied for $f = 7, f^* = 3$, and $\mathbf{w}$ =$(2, 3, 3, 2)$. Similarly, consecutive-weighted-5-out-of-4:F system is equivalent to consecutive 2-out-of-4:F system. Thus weighted-$(4, 7, 5, \mathbf{w})$:F system is equivalent to $(4, 3, 2)$:F system when $\mathbf{w}$ =$(2, 3, 3, 2)$.

Example 3.4. For $\mathbf{w} = (2, 2, 2, 1, 2)$ a linear weighted-$(5, 5, 3, \mathbf{w})$:F (G) system is equivalent to $(5, 3, 2)$:F (G) system.

Example 3.5. For $\mathbf{w} = (2, 3, 2, 2, 3)$ a linear weighted-$(5, 9, 4, \mathbf{w})$:F (G) system is equivalent to $(5, 4, 2)$:F (G) system.

Chapter 4

The k-within-consecutive-m-out-of-n:F System

4.1 Introduction

A linear weighted k-within-consecutive-m-out-of-n:F systems consists of n components ordered in a line and fails iff the total weights of failed components among m consecutive components is at least k. If all components have weight 1, the model is reduced to a usual k-within-consecutive-m-out-of-n system. It reduces to weighted k-out-of-n:F system when $m = n$.

The usual k-within-consecutive-m-out-of-n:F system was first introduced by Griffith (1986), efficient lower and upper bounds were proposed by Sfakianikis et al. (1992) and Papastavridis and Koutras (1993). Sfakianikis et al. (1992) also presented exact reliability formula using combinatorics for the system with i.i.d. components for the case $k = 2$ and bounds based on improved Bonferroni inequalities for $k \geq 2$. Recursive equations for exact reliability for $k \geq 2$ was provided by Preuss (1997). This model was extended to rectangular or cylindirical two dimensional k-within-connected-$r \times s$-out-of-$m \times n$ systems having a very complicated reliability structure. Makri and Psikallis (1997) have provided a bound based on imporved Bonferroni Inequalities and Akiba and Yamamoto(2001) have presented an algorithm for exact reliability of the two-dimensional model.

Usual versions of this model have applications in quality control, radar and sliding window detection. When at least k items fall outside the limits among m consecutive items in Shewart control charts, the manufacturing process would be adjusted to correct the defects. In sliding window detection, if k ones are encoutered in an m-bit wide sliding window at an instance, an error would be detected (Papastavridis and Koutras (1993)).

The weighted version of this model can be used in overlapping local networks. Each window of m consecutive units in an array of n units with unequal capacities would be required to supply adjacent regions with a certain capacity where neighboring regions share m-1 units as their source. In the F system, if at least one region is not supplied with the desired load then the local failure would have a global impact. Power plants with different generation capacities might be an example of the sources of power for neighboring regions.

The lower and upper bounds presented in this chapter are based on bounds obtained for non-weighted models by Papastavridis and Koutras (1993).

4.2 Exact Reliability

The exact reliability computation of Weighted k-within-consecutive-m-out-of-n:F System is based on the method provided by Preuss(1997) for usual k-within-consecutive-m-out-of-n:F systems. When all component weights are 1, we obtain the formulas given by Preuss(1997).

4.2.1 Definition of Symbols

p_j: Survival probability of component j for $j = 1 \dots n$.
w_j: The weight of component j for $j = 1 \dots n$.
y_j: $y_j \in \{0, w_j\}$ for $j = 1 \dots n$.
Y_j: $Y_j = w_j$ if component j fails and $Y_j = 0$ if component j works for $j = 1 \dots n$.
A_i: Event that weighted k-within-m-out-of-i system consisting of components 1,..,i works for i=m,...,n.
$B_{n,m}$ The set of all m-element positive integer vectors $(y_{n-m+1}, y_{n-m+2}, \dots, y_n)$ such that $\sum_{j=1}^{m} y_{n-m+j} < k$.
R_n: The exact reliability of weighted k-within-consecutive-m-out-of-n:F System

4.2.2 Reliability Evaluation

$$R_n = P(A_n) = \sum_{(y_{n-m+1}, y_{n-m+2}, \dots, y_n) \in B_{n,m}} P(A_n | Y_{n-m+1} = y_{n-m+1}, \dots, Y_n = y_n) \cdot \prod_{j=n-m+1}^{n} (p_j I(y_j = 0) + (1 - p_j) I(y_j = w_j))$$

where

$$I(y_j = w_j) = \begin{cases} 1 & \text{if } y_j = w_j \\ 0 & \text{if } y_j \neq w_j \end{cases}$$

and probabilities $P(A_n | Y_{n-m+1} = y_{i-m+1}, \dots, Y_n = y_n)$ are computed by recursion:

for $j = m, \dots, n-1$

$$P(A_{j+1} | Y_{j-m+2} = y_{j-m+2}, \dots, Y_{j+1} = y_{j+1}) =$$

$$\begin{cases} 0 & \text{if } \sum_{i=1}^{m} y_{j-m+1+i} \geq k \\ P(A_j | Y_{j-m+1} = 0, Y_{j-m+2} = y_{j-m+1}, \dots, Y_j = y_j) p_j & \\ +P(A_j | Y_{j-m+1} = w_{j-m+1}, Y_{j-m+2} = y_{j-m+1}, \dots, Y_j = y_j) & \\ \cdot (1 - p_j) & \text{if } \sum_{i=1}^{m} y_{j-m+1+i} < k \end{cases}$$

with initial condition,

$$P(A_m|Y_1 = y_1, \ldots, Y_m = y_m) = \begin{cases} 0 & \text{if } \sum_{j=1}^{m} y_j \geq k \\ 1 & \text{if } \sum_{j=1}^{m} y_j < k \end{cases}$$

The complexity of this algorithm is $O(n \cdot \sum_{i=1}^{k-1} \binom{m}{i})$ which grows very fast for large n and m.

4.3 Lower bound

The lower bound in Papastavridis and Koutras (1993) has been improved and adjusted for weighted model. When all weights are 1, sharper lower bounds are obtained compared to Papastavridis and Koutras (1993) therefore our lower bounds are improvements for usual models as well. A second lower bound obtained by the same method and it is even sharper than the first lower bound but it is more complicated hence the formula for only the usual case is given.

4.3.1 Definition of Symbols

$R_W(k, m, n)$: The exact reliability of weighted k-within-consecutive-m-out-of-n-system.
X_i : The random variable of component i. $X_i = 1$ if it is failed and $X_i = 0$ if it is working for $i = 1 \ldots n$.
Y_i : $w_i \cdot X_i$.
$\phi_i(\mathbf{x})$ The state or structure function of k-within-m-out-of-i:F system consisting of components 1,2,....,i. $\phi_n(\mathbf{x})$ yields the state of the whole system for $i = 1 \ldots n$.
q_i, p_i : Unreliability and reliability of component i for $i = 1 \ldots n$.
$R(j, s, i)$: The reliability of j-out-of-s:F system consisting of components i-s+1,....,i for $i = 1 \ldots n$.
R_i^{LB1}, R_i^{LB2} : The first and second lower bounds for the reliability of linear weighted k-within-consecutive-m out of i:F system consisting of components 1,2,....,i, for $i = 1 \ldots n$.

LB^1 : The first lower bound for the reliability of the whole system. Same as R_n^{LB1}.

LB^2 : The second and sharpest lower bound for the reliability of the whole system. Same as R_n^{LB2}.

4.3.2 Association of Events

If two structure functions $\phi_1(\mathbf{X})$ and $\phi_2(\mathbf{X})$ are s-independent random variables we can simply multiply their reliabilities:

$$P(\phi_1(\mathbf{X}) = 1 \cap \phi_2(\mathbf{X}) = 1) = P(\phi_1(\mathbf{X}) = 1)P(\phi_2(\mathbf{X}) = 1)$$

According to Esary and Proschan (1970), if two random variables' covariance is greater than or equal to 0, then they are associated. This means that an increase in a random variable does not decrease the probability that the other random variable will increase. Two events are associated if given that one event has occurred the probability of other event does not decrease. If two structures share at least one component in common then the random variables of those structures are associated but not independent i.e. their covariance is greater than or equal to 0. Then survival events of these two structures are also associated but not independent i.e. survival probabilities of these two structures are increasing functions of each other. If $\phi_1(\mathbf{X})$ and $\phi_2(\mathbf{X})$ are associated binary random variables then:

$$P(\phi_1(\mathbf{X}) = 1 \cap \phi_2(\mathbf{X}) = 1) \geq P(\phi_1(\mathbf{X}) = 1)P(\phi_2(\mathbf{X}) = 1)$$

4.3.3 Reliability Evaluation

The system is decomposed into n-m+1 weighted k-out-of-m:F subsystems consisting of m consecutive components. Each consecutive substructure has m-1 components in common with the neighboring substructure. The survival event for every substructure is $\sum_{i=j}^{j+m-1} Y_i < k$ for $j = 1, ..., n-m+1$. When one event in this sequence does not occur i.e. when at least 1 of these substructures fail, the whole system fails. The survival of the system requires that all substructures survive i.e. $\sum_{i=j}^{j+m-1} Y_i < k$ for $j = 1, ..., n - m + 1$. Therefore we can express the survival probability of the whole system as:

$$\begin{aligned} P(\phi_n(\mathbf{x}) = 1) &= P(\sum_{i=1}^{m} Y_i < k \cap \sum_{i=2}^{m+1} Y_i < k \cap \ldots \cap \sum_{i=n-m+1}^{n} Y_i < k) \\ &= P(\bigcap_{j=1}^{n-m+1} \sum_{i=j}^{j+m-1} Y_i < k) \end{aligned}$$

According to Barlow and Proschan (1971), each substructure is a minimal cut structure since failure of one of them causes the system to fail. Since minimal cut structures are arranged in series, the intersection of their survival events yield the survival event of the system. Taking the joint probability of these events yields the reliability of the system.

A lower bound is obtained based on the minimal cut structure representation of the system. The simplest lower bound will be the product of reliabilities of every substructure i.e. weighted k-out-of-m:F subsystems but because these substructures in the sequence are m-dependent on each other, the error of the bound would be high. In this section, we reduced the dependencies between consecutive subsystems to attain a sharper lower bound. With pivotial decomposition and removing dependence between neighboring substructures a more accurate lower bound will be obtained. The substructures are decomposed by conditioning on the components at the ends and shared components, and an expression in terms of weighted k*-out-of-m*:F systems is obtained.

We begin with the reliability of the subsystem consisting of first m+1 components. Its reliability will be expressed in terms of the joint probability of survival events of first two minimal cut structures.

Lemma 4.1 *The exact reliability of weighted k-within-consecutive-m-out-of-m+1* *system with independent components is:*

$$\begin{aligned}P(\phi_{m+1}(\mathbf{x}) = 1) &= P(\sum_{i=1}^{m} Y_i < k \cap \sum_{i=2}^{m+1} Y_i < k) \\ &= [p_{m+1}R(k, m, m)] + q_{m+1}[p_1 R(k - w_{m+1}, m-1, m) \\ &+ q_1 R(k - max(w_1, w_{m+1}), m-1, m)]\end{aligned}$$

If all components have weight 1, this formula is reduced to

$$P(\phi_{m+1}(\mathbf{x}) = 1) = R(k, m, m)p_{m+1} + R(k, m-1, m)q_{m+1}$$

Proof. As two substructures with m-1 components in common, we use pivotial decomposition on the last component which is included only in the second substructure to generate two mutually exclusive events whose reliabilities can be added. The random variable Y_{m+1} would be removed from the survival event of the second substructure as the lemma of pivotial decomposition describes:

$$\begin{aligned}P(\sum_{i=1}^{m} Y_i < k \cap \sum_{i=2}^{m+1} Y_i < k) &= P(Y_{m+1} = 0, \sum_{i=1}^{m} Y_i < k, \sum_{i=2}^{m} Y_i < k) \\ &+ P(Y_{m+1} = w_{m+1}, \sum_{i=1}^{m} Y_i < k, \sum_{i=2}^{m} Y_i < k - w_{m+1})\end{aligned}$$

Since the events $Y_{m+1} = w_{m+1}$ and $Y_{m+1} = 0$ are independent from the rest of the events their probabilities can be multiplied. The event $\sum_{i=2}^{m} Y_i < k$ is absorbed by the event $\sum_{i=1}^{m} Y_i < k$ since the latter event's occurrence would imply that the former has occurred. Also the sequence of events $\sum_{i=j}^{m} Y_i < k$ for $j = 1, \cdots, m$ is monotone decreasing therefore the second event is the subset of the first event. Thus we can omit the event $\sum_{i=2}^{m} Y_i < k$.

Now we have to decompose the intersection of two events $\sum_{i=1}^{m} Y_i < k \cap \sum_{i=2}^{m} Y_i < k - w_{m+1}$ into independent terms. These two structures have m-1 components in common and the only random variable not included in the event $\sum_{i=2}^{m} Y_i < k - w_{m+1}$ is Y_1. After using pivotial decomposition component 1 and removing Y_1 from $\sum_{i=1}^{m} Y_i < k$, we have:

$$P(\sum_{i=1}^{m} Y_i < k, \sum_{i=2}^{m} Y_i < k - w_{m+1}) = P(Y_1 = 0, \sum_{i=2}^{m} Y_i < k, \sum_{i=2}^{m} Y_i < k - w_{m+1}) + P(Y_1 = w_1, \sum_{i=2}^{m} Y_i < k - w_1, \sum_{i=2}^{m} Y_i < k - w_{m+1}) \tag{4.1}$$

Since $Y_1 = 0$ and $Y_1 = w_1$ are independent from the rest of the events, we can multiply their probabilities. The event $\sum_{i=2}^{m} Y_i < k - w_{m+1}$ is absorbed by the event $\sum_{i=2}^{m} Y_i < k$ since the occurrence of first event implies that the second one has occurred. Also, sequence of events $\sum_{i=2}^{m} Y_i < j$ for $0 \leq j$ is monotone increasing thus the second event is subset of the first event.

The events $\sum_{i=2}^{m} Y_i < k - w_1$ and $\sum_{i=2}^{m} Y_i < k - w_{m+1}$ are inequalities with same functions of random variables. Since the sequence of events $\sum_{i=2}^{m} Y_i < j$ for $0 \leq j$ is monotone increasing, the event with larger j would be absorbed. The event with smaller of the weights $k - w_1$, $k - w_{m+1}$, hence the larger of weights w_1, w_{m+1} would absorb the other event. Now we have:

$$P(\sum_{i=2}^{m} Y_i < k - w_1, \sum_{i=2}^{m} Y_i < k - w_{m+1}) = P(\sum_{i=2}^{m} Y_i < k - max(w_1, w_{m+1})) \tag{4.2}$$

After equation (4.2) is substituted into equation (4.1), we have:

$$P(\sum_{i=1}^{m} Y_i < k \cap \sum_{i=2}^{m+1} Y_i < k) = P(Y_{m+1} = 0)P(\sum_{i=1}^{m} Y_i < k) + P(Y_{m+1} = w_{m+1})[P(Y_1 = 0)P(\sum_{i=2}^{m} Y_i < k - w_{m+1}) + P(Y_1 = w_1)P(\sum_{i=2}^{m} Y_i < k - max(w_1, w_{m+1}))]$$

Now we have decomposed two dependent events into independent events. The probability of every individual event can be solved by reliability formulas we already know. □

Theorem 4.2 *The first lower bound of weighted k-within-consecutive-m-out-of-n system with independent components is calculated recursively as:*

$$\begin{aligned} R_W(k,m,i) &\geq p_i R_W(k,m,i-1) + q_i R_W(k,m,i-2)[(p_{i-m}R(k-w_i,m-1,i-1) \\ &+ q_{i-m}R(k-max(w_{i-m},w_i),m-1,i-1))] \\ &\geq p_i R_{i-1}^{LB1} + q_i R_{i-2}^{LB1}[(p_{i-m}R(k-w_i,m-1,i-1) \\ &+ q_{i-m}R(k-max(w_{i-m},w_i),m-1,i-1)] \\ &= R_i^{LB} \end{aligned}$$

and the first lower bound of a usual k-within-consecutive- m-out-of-n system with all component weights 1 and independent and nonidentical components is reduced to:

$$\begin{aligned} R_W(k,m,i) &\geq p_i R_W(k,m,i-1) + q_i R_W(k,m,i-2)R(k-1,m-1,i-1) \\ &\geq p_i R_{i-1}^{LB1} + q_i R_{i-2}^{LB1} R(k-1,m-1,i-1) \\ &= R_i^{LB} \end{aligned}$$

for $i = m+2, \ldots, n.$
and $LB^1 = R_n^{LB1}$
where we substitute the exact reliabilities for R_m^{LB1} *and* R_{m+1}^{LB1} *as provided in lemma 4.1.*

Proof. We will find the joint probability of the survival of third substructure and the survival of the existing structure $\phi_{m+1} = 1$. Every structure adds one component to the existing system which will grow and eventually reach ϕ_n All of the remaining substructures will be added with the same method. By pivotial decomposition on component m+2:

$$\begin{aligned} P(\phi_{m+1} = 1 \wedge \sum_{i=3}^{m+2} Y_i < k) &= P(Y_{m+2} = 0, \phi_{m+1} = 1) \\ &+ P(Y_{m+2} = w_{m+2}, \phi_{m+1} = 1, \sum_{i=3}^{m+1} Y_i < k - w_{m+2}) \end{aligned} \quad (4.3)$$

Since events $Y_{m+2} = w_{m+2}$ and $Y_{m+2} = 0$ are independent from the rest of the events, we can multiply their probabilities. Now we have to decompose

$\phi_{m+1} = 1 \cap \sum_{i=3}^{m+1} Y_i < k - w_{m+2}$. The survival of first m+1 component system $\phi_{m+1} = 1$ can be decomposed into $\phi_m = 1 \cap \sum_{i=2}^{m+1} Y_i < k$. Since the event $\phi_m = 1$ is $\sum_{i=1}^{m} Y_i < k$, we have:

$$P(\phi_{m+1} = 1 \cap \sum_{i=3}^{m+2} Y_i < k - w_{m+2}) = P(\sum_{i=1}^{m} Y_i < k \cap \sum_{i=2}^{m+1} Y_i < k \cap \sum_{i=3}^{m+2} Y_i < k - w_{m+2}))$$

On the right hand side, the event $\sum_{i=1}^{m} Y_i < k$ is dependent on the other two events since they have m-1 and m-2 random variables or components in common. They will not be decomposed further not to complicate the formula, therefore the dependency will remain. The events $\sum_{i=1}^{m} Y_i < k$ and $\sum_{i=2}^{m+1} Y_i < k \cap \sum_{i=3}^{m+2} Y_i < k - w_{m+2}$ are associated since the random variables $\sum_{i=1}^{m} Y_i$ and $\sum_{i=2}^{m+1} Y_i$, $\sum_{i=3}^{m+1} Y_i$ are associated but not independent. This means that occurrence of one event increases the probability of the other. This makes sense since if random variable $\sum_{i=1}^{m} Y_i$ increases, then random variables $\sum_{i=2}^{m+1} Y_i$ and $\sum_{i=3}^{m+1} Y_i$ are more likely to increase. The product of probabilities of two associated but not independent events underestimates their joint probability therefore,

$$\begin{aligned} &P(\sum_{i=1}^{m} Y_i < k \bigcap (\sum_{i=2}^{m+1} Y_i < k \cap \sum_{i=3}^{m+2} Y_i < k - w_{m+2})) \\ &\geq P(\sum_{i=1}^{m} Y_i < k) \cdot P(\sum_{i=2}^{m+1} Y_i < k \cap \sum_{i=3}^{m+2} Y_i < k - w_{m+2}) \end{aligned} \tag{4.4}$$

The joint probability of events $\sum_{i=2}^{m+1} Y_i < k$ and $\sum_{i=3}^{m+2} Y_i < k - w_{m+2}$ will be solved by using the same method in Lemma 4.1 i.e. by pivotial decomposition on the component 2 and absorption by subsets of events.

$$\begin{aligned} &P(\sum_{i=1}^{m} Y_i < k) \cdot P(\sum_{i=2}^{m+1} Y_i < k \cap \sum_{i=3}^{m+2} Y_i < k - w_{m+2}) \\ &= P(\sum_{i=1}^{m} Y_i < k) \cdot [P(Y_2 = 0) P(\sum_{i=2}^{m+1} Y_i < k - w_{m+2}) \\ &+ P(Y_2 = w_2) P(\sum_{i=3}^{m+1} Y_i < k - max(w_2, w_{m+2}))] \\ &= R(k, m, m) \cdot [p_2 R(k - w_{m+2}, m+1, m-1) \\ &+ q_2 R(k - max(w_2, w_{m+2}), m-1, m+1)] \end{aligned} \tag{4.5}$$

Then equation (4.5) is substituted into equations (4.4) and (4.3), and if we repeat this operation until R_n^{LB}, it will yield LB. □

The exact reliability of weighted k-within-consecutive-m-out-of-m+2 system with independent components can also be derived with the same method in Lemma 4.1. However the formula would be too long for arbitrary weights therefore the exact reliability of this system will be given for the case $w_j = 1$ for $j = 1, \ldots, n$ with independent and nonidentical components:

Lemma 4.3 *The exact reliability of a usual k-within-consecutive-m-out-of-m+2 system with independent and nonidentical components is:*

$$\begin{aligned}
P(\phi_{m+2}(\mathbf{x}) = 1) &= P(\sum_{i=1}^{m} X_i < k \cap \sum_{i=2}^{m+1} X_i < k \cap \sum_{i=3}^{m+2} X_i < k) \\
&= p_{m+2}[R(k, m, m)p_{m+1} + R(k, m-1, m)q_{m+1}] \\
&+ q_{m+2}[R(k-2, m-2, m)(q_1(q_{m+1} + q_2 p_{m+1})) \\
&+ R(k-1, m-2, m)(q_1 p_2 p_{m+1}) + R(k-1, m-1, m+1)p_1]
\end{aligned}$$

Sfakianakis et al. (1992) have also provided reliability formulas for k-within-consecutive-m-out-of-m+2 systems with i.i.d. components. The Lemma 4.3 can also be used with undientical components and can be adapted to weighted versions of this model.

A sharper lower bound can be obtained for the case $w_j = 1$ for $j = 1, \ldots, n$ using Theorem 4.2 and Lemma 4.3. The results of the second lower bound will be compared to 4.2 and exact reliability for usual systems nonidentical components.

Theorem 4.4 *The second lower bound of usual k-within-m-out-of-n system with* *independent and nonidentical components is calculated recursively as:*

$$
\begin{aligned}
R_W(k,m,i) \geq\ & p_i R_W(k,m,i-1) + q_i R_W(k,m,i-3)[R(k-2,m-2,i-2) \\
& (q_{i-m-1}(q_{i-1}+q_{i-m}p_{i-1})) + R(k-1,m-2,i-2)(q_{i-m-1}p_{i-m}p_{i-1}) \\
& + R(k-1,m-1,i-1)p_{i-m-1}] \\
& \geq p_i R_{i-1}^{LB2} + q_i R_{i-3}^{LB2}[R(k-2,m-2,i-2)(q_{i-m-1} \\
& \cdot (q_{i-1}+q_{i-m}p_{i-1})) + R(k-1,m-2,i-2)(q_{i-m-1}p_{i-m}p_{i-1}) \\
& + R(k-1,m-1,i-1)p_{i-m-1}] \\
& = R_i^{LB2}
\end{aligned}
$$

for $i = m+3,\ldots,n$.
and $LB^2 = R_n^{LB2}$
where we substitute the exact reliabilities for R_m^{LB2}, R_{m+1}^{LB2} *and* R_{m+2}^{LB2} *which are provided in lemmas 4.1 and 4.3.*

The computational complexity of both lower bounds is $O(n \cdot m \cdot k)$ which is considerably smaller than the complexity of exact reliability formula for large n and m.

4.4 Upper bound

The upper bound is adjusted to weighted models from Papastavridis and Koutras (1993) which uses conditional probability. When all component weights are 1, it reduces to the upper bound in Papastavridis and Koutras (1993).

4.4.1 Definition of Symbols

p_j: Survival probability of component j.
Y_j: $Y_j = w_j$ if component j fails and $Y_j = 0$ if component j works.
A_i: Event that weighted k-within-m-out-of-i system consisting of components $1,\ldots,i$ works for $i = m,\ldots,n$.

E_i: Event that component i fails and total weight of failed components in subsytem consisting of components i-m+1,..,i-1 is less than k
but more than or equal to $k - w_i$ for $i = m, \ldots, n$.
G_i: Event that total weight of failed components in subsystem consisting of components i-m+1,..,i-1 is less than k for $i = m, \ldots, n$.
I_i^1: The set consisting of components $max(1, i-2m+2), \ldots, i-m$ for i=m+1,...,n.
I_i^2: The set consisting of components i-m+1,...,i-1 for $i = m+1, \ldots, n$.
MC_m^i: The minimal cut set containing components from both sets of I_i^1 and I_i^2 for m=1,...,r and $i = m+1, \ldots, n$.
SC_i: The set of components in I_i^1 with the highest weight (and highest index if more than one components have highest weight)
in each set MC_m^i for $m = 1, \ldots, r$ and $i = m+1, \ldots, n$.
C_i: The event that all components in the set SC_i work for $i = m+1, \ldots, n$.
$R(k, m, i : F)$: The reliability of weighted k-out-of-m:F system consisting of components i-m+1,...,i for $i = m, \ldots, n$.
UB: The upper bound of reliability of weighted k-within-m-out-of-n:F system.

4.4.2 Reliability Evaluation

The sequence of events A_i is monotone decreasing hence it has the following property for every i: $A_i \supseteq A_{i+1}$. It is clear that events in this sequence are highly associated. To remove this dependence, each event in the sequence has to be conditioned on the preceding event.

Lemma 4.5 *By the chain rule of the conditional probability we can write,*

$$P(A_n) = P(A_m) \cdot P(A_{m+1}|A_m) \cdot \ldots \cdot P(A_n|A_{n-1}) \tag{4.6}$$

Since $P(A_i) = 1 - P(A'_i)$ it is also true that $P(A_i|A_{i-1}) = 1 - P(A'_i|A_{i-1})$. Therefore the equation (4.6) can be rewritten as:

$$P(A_n) = P(A_m) \cdot (1 - P(A'_{m+1}|A_m)) \cdot \ldots \cdot (1 - P(A'_n|A_{n-1})) \tag{4.7}$$

To find $P(A'_n|A_{n-1})$ we need to express joint probability of A'_{i+1} and A_i in terms of independent events. The joint probability of these two events can be expressed as the probability of survival of i-m substructures and failure of the last one:

$$P(A_i'|A_{i-1}) = P(\sum_{j=1}^{m} Y_j < k \cap \ldots \cap \sum_{j=i-m}^{i-1} Y_j < k \cap \sum_{j=i-m+1}^{i} Y_j \geq k)$$

$\sum_{j=i-m}^{i-1} Y_j < k$ represents the survival event of $i - m^{th}$ substructure (may also be referred as minimal cut structure). Thus all substructures up to component i-1 have to survive and the last substructure has to fail. For this to happen, the last component i has to fail and the event $k - w_i \leq \sum_{j=i-m+1}^{i-1} Y_j < k$ i.e. E_i has to occur. So we have,

$$P(A_i'|A_{i-1}) = P(\sum_{j=1}^{m} Y_j < k \cap \ldots \cap \sum_{j=i-m}^{i-1} Y_j < k \\ \cap (k - w_i \leq \sum_{j=i-m+1}^{i-1} Y_j < k) \cap Y_i = w_n)$$

The event E_i is dependent on survival of m-1 preceding substructures from $i - 2m + 2$ to $i - m$ since they share common Y_j's. To remove this dependency between events, some components between $max(1, i-2m+2)$ and i-m (inclusive) may need to be turned on so when two independent events A_{i-m} and E_i occur, the event $A_i' \cap A_{i-1}$ occurs.

$$P(A_i'|A_{i-1}) = P(A_{i-m} \cap \sum_{j=i-2m+2}^{i-m+1} Y_j < k \cap \ldots \cap \sum_{j=i-m}^{i-1} Y_j < k \cap E_i)$$

Since the system up to component i-m works, the total weight of failed components between $max(1, i-2m+2)$ and i-m, i.e. in the set I_i^1, is less than k. Due to E_i, the total weight of failed components between 1-m+1 and i-1, i.e. in the set I_i^2 is also less than k. However the total weight of failed components within m consecutive components from both sets can still be more than k, if this is the case some of the substructures from i-2m+2 to i-m would fail. For all substructures up to component i-1 to work, some components in I_i^1 have to be turned on so that no substructures including components from both sets of I_i^1 and I_i^2 fail.

Let $MC_1^i, \ldots MC_r^i$ be all minimal cut sets including including components from both sets of I_i^1 and I_i^2. The component with the largest weight from I_i^1

would be turned on in each set MC_m^i for m=1,...r. If there is more than one such components with largest weight, then the one with largest index would be turned on so that none of the minimal cut structures of these sets fail by turning on as few components as possible. By taking exactly one component from every minimal cut set, we obtain the set SC_i which consists of all components which should be set as working so that events A_{i-m} and E_i imply that the system up to component i-1 survives and fails at i.

$$\begin{aligned} SC_i = \ & \{j \in I_i^1 \cap MC_m^i : w_j \geq w_s \text{ and if } w_j = w_s \text{ then } j > s \\ & \forall s \in MC_m \cap I_i^1 - \{j\} \text{ for } m = 1, \ldots, r\} \end{aligned}$$

Let C_i be the event that all components in set SC_i are working. Intersection of events E_i, A_{i-m} and C_i implies that the system up to component i-1 survives and fails at component i. Since E_i is independent from A_{i-m} and C_i:

$$P(A_i'|A_{i-1}C_i) = \frac{P(A_{i-m} \cap C_i \cap E_i)}{P(A_{i-1}C_i)} = \frac{P(A_{i-m} \cap C_i)P(E_i)}{P(A_{i-1}C_i)}$$

and

$$P(A_{i-1}) = P(A_{i-m} \cap \sum_{j=i-2m+2}^{i-m+1} Y_j < k \cap \ldots \sum_{j=i-m}^{i-1} Y_j < k \cap G_i)$$

If two independent events G_i, i.e. $\sum_{j=i-m+1}^{i-1} Y_j < k$, and A_{i-m} occur and some substructures from i-2m+2 to i-m may still fail. The event C_i implies that no minimal cut structures containing components from both I_i^1 and I_i^2 fail thus all substructures from i-2m+2 to i-m survive. Therefore,

$$P(A_{i-1} \cap C_i) = P(A_{i-m} \cap C_i \cap G_i)$$

Since G_i is independent from C_i and A_{i-m},

$$P(A_i'|A_{i-1}C_i) = \frac{P(A_{i-m} \cap C_i)P(E_i)}{P(A_{i-m} \cap C_i)P(G_i)} = \frac{P(E_i)}{P(G_i)}$$

We also need following properties described and proved in the paper of Papastavrdis and Koudras(1993) to prove the next theorem,

Lemma 4.6 *Let A, B, C be any events.*

$$P(A|B) \geq P(A|BC)P(C|B)$$

Lemma 4.7

$$P(C_i|A_{i-1}) \geq P(C_i)$$

Theorem 4.8 *The upper bound of reliability of weighted k-within-m-out-of-n:F system is:*

$$\begin{aligned} UB =& R(m, m, k : F) \cdot \prod_{i=m+1}^{n} (1 - (1 - p_i) \\ & \cdot [R(k, m-1, i-1 : F) - R(k - w_i, m-1, i-1 : F)] \cdot \frac{P(C_i)}{P(G_i)}) \end{aligned}$$

Proof. By using the lemmas 4.6 and 4.7, we can write,

$$P(A_i'|A_{i-1}) \geq P(A_i'|A_{i-1}C_i)P(C_i|A_{i-1}) \geq \frac{P(E_i)P(C_i)}{P(G_i)}$$

and

$$P(E_i) = (1 - p_i)\,[R(k, m-1, i-1 : F) - R(k - w_i, m-1, i-1 : F)]$$

If we substitute these equations into equation (4.7), the result follows. □

The complexity of the upper bound algorithm is $O(n \cdot m \cdot k)$ which is much less than computing time required for large n and m in exact reliability algorithm.

4.5 Statement of Results

The analytical solutions found in this chapter are computed with Student Version of MATLAB 7.7 and the values are compared. The results are obtained for different values of n, m and k in the following tables. In Table 4.1, each element of probability vector is given as $p_i^{(1)} = 1 - 2^{-i}$. The component weight vectors are $(1, 1, 2, 3, 2)$ for $n = 5$, $(2, 1, 2, 2, 3, 2, 3, 1, 2, 1)$ for $n = 10$ and $(2, 1, 3, 4, 2, 1, 3, 4, 2, 3, 1, 3, 2, 3, 1)$ for $n = 15$.

TABLE 4.1

Lower and Upper Bounds for Reliability of

weighted k-within-consecutive-m-out-of-n:F System for $\mathbf{p}^{(1)}$

n	m	k	LB1	Exact R	UB
5	2	2	0.6953	0.6953	0.7450
5	3	3	0.8609	0.8629	0.8918
5	4	3	0.8565	0.8565	0.8599
10	3	3	0.7707	0.7709	0.7895
10	5	3	0.7489	0.7492	0.7534
10	7	4	0.8747	0.8750	0.8750
10	8	5	0.9482	0.9482	0.9482
15	5	3	0.6955	0.6956	0.7012
15	7	5	0.8875	0.8878	0.8883
15	10	8	0.9920	0.9920	0.9920
15	12	10	0.9985	0.9985	0.9985

The elements of probability vector for the Table 4.2 are

$$p_i^{(2)} = \begin{cases} 0.7 & \text{if } i \text{ is odd} \\ 0.75 & \text{if } i \text{ is even} \end{cases}$$

The next table uses the same weight vectors in Table 4.1.

TABLE 4.2

Lower and Upper Bounds for Reliability of weighted k-within-consecutive-m-out-of-n:F System for $\mathbf{p}^{(2)}$

n	m	k	LB1	Exact	UB
5	2	2	0.3399	0.3399	0.3836
5	3	3	0.5852	0.6077	0.6740
5	4	3	0.5683	0.5683	0.5847
10	3	3	0.2325	0.2860	0.4495
10	5	3	0.1708	0.2352	0.3384
10	7	4	0.2570	0.3176	0.3512
10	8	7	0.7065	0.7065	0.7134
15	5	3	0.0362	0.0611	0.1459
15	7	5	0.1088	0.2028	0.3618
15	10	8	0.3133	0.4308	0.4986
15	12	10	0.4867	0.5645	0.5934

The next table compares usual systems so all component weights are 1. It uses the same probability vectors in Table 4.2.

TABLE 4.3

Lower Bounds for Reliability of usual k-within-consecutive-m-out-of-n:F System for $\mathbf{p}^{(2)}$

n	m	k	LB1	LB2	Exact
5	2	2	0.7639	0.7639	0.7639
5	3	2	0.6413	0.6602	0.6602
10	3	2	0.3581	0.3758	0.3980
10	5	3	0.6135	0.6363	0.6689
10	7	4	0.7903	0.8064	0.8202
10	8	5	0.9242	0.9312	0.9312
15	5	3	0.4241	0.4539	0.5122
15	6	2	0.0621	0.0715	0.1086
15	7	4	0.6167	0.6447	0.7022
15	10	6	0.8993	0.9076	0.9224
15	12	7	0.9452	0.9494	0.9545

In Table 4.4, our second lower bound is compared to the upper bound UB^B for unreliability of usual k-within-consecutive-m-out-of-n:F systems found by Sfakianakis et al. (1992) based on Bonferroni inequalities since an upper bound for unreliability is actually a lower bound for reliability. The Bonferroni upper bound for unreliability is the sharpest bound found by Sfakianakis et al. (1992) in most cases.

TABLE 4.4

Upper Bounds for Unreliability of usual k-within-consecutive-m-out-of-n:F System with i.i.d. components

n	m	k	p	Exact	$1 - LB2$	UB^B
10	7	2	0.25	0.999	0.999	0.1000
10	7	5	0.5	0.379	0.428	0.393
10	7	2	0.75	0.718	0.737	0.728
15	12	8	0.25	0.916	0.963	0.926
15	12	9	0.25	0.772	0.868	0.781
15	7	3	0.5	0.976	0.993	1.000
15	10	7	0.5	0.333	0.420	0.421
15	10	8	0.5	0.131	0.163	0.156
15	7	3	0.75	0.559	0.640	0.736
15	7	5	0.75	0.580	0.656	0.680
15	10	3	0.75	0.679	0.744	0.755
15	10	5	0.75	0.196	0.200	0.196
15	12	2	0.75	0.911	0.918	0.924
15	12	3	0.75	0.728	0.754	0.747
20	10	8	0.5	0.204	0.271	0.279
20	10	9	0.5	0.049	0.066	0.058
20	12	10	0.5	0.067	0.089	0.086

The first lower bound and the upper bound perform very well in case of high component and system reliabilities. Both bounds can be used as approximations in case of high reliabilities. But in Table 4.2, when component and system reliabilities drop, the error of both bounds increase considerably. Especially the first lower bound deviates more from the exact reliability compared to the Table 4.1.

In the Table 4.3, the second lower bound is superior to the first lower bound in all systems except the first one. The second lower bound is sharper when system's reliability is high. The percentage difference between the first and second lower bounds increase in low system reliabilities. Thus, the second lower bound is a good approximator in all cases.

There is no direct way to compare our second lower bound to Bonferroni lower bound of Sfakianakis et al. (1992). Our bound tends to perform better in low system reliabilities whereas Bonferroni lower bound outperforms in high reliabilities.

Chapter 5

Conclusions

The exact reliability in Chapter 3 and bounds in Chapter 4 are efficient methods for evaluating the reliability of such complex weighted systems introduced in this thesis. The assumptions are also few, components can have arbitrary weights and reliabilities. The exact reliabilities in Chapter 3 are efficient and can be used for usual models as well. Conversion of weighted combined systems to usual combined systems is useful for conserving time and space. The first lower bound in Chapter 4 performs well in high reliabilities for both weighted and usual models. The upper bound adjusted to the weighted models also preserved its good performance as in usual systems. The second lower bound outperforms the first one in usual models and tends to be sharper than the Bonferroni lower bound in low reliabilities. The methods described in this thesis can also be extended to more complicated structures. All programs needed for computation of formulas presented in this thesis are available upon request.

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