q-FLOQUET THEORY AND ITS EXTENSIONS TO TIME SCALES PERIODIC IN SHIFTS

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q-FLOQUET THEORY AND ITS EXTENSIONS TO TIME SCALES PERIODIC IN SHIFTS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

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We have read the dissertation entitled **"q-FLOQUET THEORY AND ITS EXTENSIONS TO TIME SCALES PERIODIC IN SHIFTS"** completed by **HALİS CAN KOYUNCUOĞLU** under supervision of **Prof. Dr. Murat Adıvar** and we certify that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

q-FLOQUET THEORY AND ITS EXTENSIONS TO TIME SCALES PERIODIC IN SHIFTS

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This thesis proposes a Floquet theory for q-difference systems which are constructed on $\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z},\, q > 1\} \cup \{0\}$ by using multiplicative periodicity notion. The Floquet decomposition theorem is given by obtaining the solution of a matrix exponential equation. The existence of periodic solutions of both homogeneous and nonhomogeneous systems are investigated by providing the necessary and sufficient conditions. Additionally, by establishing a linkage between Floquet multipliers and Floquet exponents of a q-Floquet system, stability analysis is done. The obtained results for q-difference systems are unified on time scales by using new periodicity concept based on shift operators introduced in [11] (see also [12]). This approach enables us to discuss Floquet theory of dynamic systems on more general domains including nonadditive domains, such as $\{\pm n^2, n \in \mathbb{Z}\}$ and $\cup_{k=1}^{\infty} \left[3^{\pm k}, 2.3^{\pm k}\right] \cup \{0\}$. Given results provide a wide perspective for Floquet theory and they are the most general results that are obtained in the existing literature.

Keywords: Floquet theory, transition matrix, matrix exponential, multiplicative periodicity, time scales, shift operator, periodicity in shifts, Floquet multipliers, Floquet exponents, stability.

ÖZ

q-FLOQUET TEORİSİ VE KAYDIRMA OPERATÖRLERİNE GÖRE PERİYODİK ZAMAN SKALALARINDA GENELLEŞTİRİLMESİ

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Bu tezde $\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z}, q > 1\} \cup \{0\}$ tanım aralığında kurulan q-fark sistemlerinin Floquet teorisi çarpımsal periyodiklik kavramı kullanılarak incelenmiştir. Floquet ayrışma teoremi üstel matris fonksiyonu denkleminin çözümünün varlığı ispatlanarak verilmiştir. Homojen ve homojen olmayan q-Floquet fark sistemleri incelenerek, periyodik çözümün varlığı için gerek yeter koşullar gösterilmiştir. Ayrıca, Floquet çarpanları ve Floquet kuvvetleri arasında kurulan ilişkinin ışığında elde edilen sonuçlar kararlılık analizinde kullanılmıştır. Tezin kalan kısmında, q-Floquet teorisi zaman skalalarında kaydırma operatörlerine bağlı olarak tanımlanan yeni periyodiklik kavramıyla ([11] ve [12]) genelleştirilmiştir. Bu yaklaşım dinamik sistemlerin Floquet teorisinin toplamsallık koşulu aranmaksızın $\pm n^2, n \in \mathbb{Z}$ ve $\cup_{k=1}^{\infty} \left[3^{\pm k}, 2.3^{\pm k}\right] \cup \{0\}$ gibi daha genel tanım aralıklarında tartışılmasına imkan tanımıştır. Genelleştirilen sonuçlar Floquet teorisine daha geniş bir açıdan bakılmasını sağlayıp, literatürdeki şu ana kadar Floquet teorisi üzerine yapılmış çalışmalar içerisinde en genel olanlarıdır.

Anahtar Kelimeler: Floquet teorisi, dönüşüm matrisi, üstel matris fonksiyonu, çarpımsal periyodiklik, zaman skalası, kaydırma operatörü, kaydırma operatörüne göre periyodiklik, Floquet çarpanları, Floquet üstelleri, kararlılık.

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Nomenclature

Symbol	Response	Symbol	Response
$\mathbb{R}$	**Set of real numbers**	C_{rd}^{1}	**Set of differentiable functions**
$\mathbb{R}_{+}$	**Set of positive real numbers**		**with rd-continuous derivatives**
$\mathbb{Z}$	**Set of integers**	$\int_{t_0}^{t} f(\tau)\,\Delta\tau$	**Δ-integral of function f**
$h\mathbb{Z}$	$\{hn : n \in \mathbb{Z}, h > 0\}$	$\overset{\circ}{i}$	**Hilger purely imaginary number**
$\overline{q^{\mathbb{Z}}}$	$\{q^n : n \in \mathbb{Z},\ q > 1\} \cup \{0\}$	$\mathcal{R}^{+}$	**Set of positively regressive**
$\mathbb{N}_0$	**Set of natural numbers**		**functions**
$\mathbb{C}$	**Set of complex numbers**	$\xi_{\mu(t)}$	**Cylinder transform**
$\mathbb{Q}$	**Set of rational numbers**	$e_p(.,.)$	**Exponential function**
$\mathbb{T}$	**Time scale**	$e_A(.,.)$	**Matrix exponential function**
$d_q f$	**q-differential of function f**	δ_{+}	**Forward shift operator**
$D_q f$	**q-derivative of function f**	$\mathcal{D}_{+}$	**Domain of forward shift**
$\int f(t)\,d_q t$	**q-Integral of function f**		**operator**
$\Phi_A(., t_0)$	**Transition matrix**	δ_{-}	**Backward shift operator**
σ	**Forward jump operator**	$\mathcal{D}_{-}$	**Domain of backward shift**
ρ	**Backward jump operator**		**operator**
μ	**Step size function**	L	**Lyapunov transformation**
f^{Δ}	**Δ-derivative of function f**	M	**Monodromy operator**
$\mathcal{R}$	**Set of regressive functions**	γ	**Floquet exponent**
C_{rd}	**Set of rd-continuous functions**	λ	**Floquet multiplier**

Chapter 1

Introduction

In the literature, analysis of periodic equations and systems has taken great interest due to its tremendous application potential in engineering, biology, biomathematics, chemistry etc. (see [35], [42], [45], [57], [66] and [70]). As it is well known, equations and systems with periodic coefficients are always constructed on periodic domains. To define an ω-periodic function on a set $\mathbb{T}$, one need to guarantee that $\mathbb{T}$ is periodic with period $P \leq \omega$. In terms of conventional thinking, the domain $\mathbb{T}$ is periodic if and only if there exists a $P \in \mathbb{T}$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [53]). The set of reals $\mathbb{R}$, the set of integers $\mathbb{Z}$, $h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$, and $\bigcup\limits_{k\in\mathbb{Z}} [2k, 2k+1]$ are examples of periodic domains. Hereafter, these domains are called as additively periodic domains. There is a vast literature dealing with periodic solutions of various types of differential, difference and integral equations and their systems which are constructed on additively periodic domains (see [4], [49], [50], [72] and [73]).

Floquet theory is an important tool in the study of periodic systems for investigation of periodic solutions and stability theory of dynamic systems. Indeed, it is a century-old theory which was introduced by Gaston Floquet in 1883 in order to analyze the solutions of systems of linear differential equations with periodic coefficients (see [43]). Afterwards, this theory has been extended to difference equations/systems, integral equations, integro-differential equations, and partial differential equations (see [17], [18], [21], [23], [55] and [65]). The extension of Floquet theory to different type of equations became very useful tool for researchers in fields of mathematics and physics. By a short literature review, one may deduce that Floquet theory is very common in quantum physics, classical physics, chemistry, electronics, dynamic systems (see [31], [34], [40], [54], [56], [58] and [61]).

In recent years, the theory of time scales has taken prominent attention in the area of pure and applied mathematics. This popular theory was introduced by Stefan Hilger in 1988, in his Ph.D. thesis under the guidance of his advisor Bernd Aulbach (see [48]). The aim of Hilger's thesis was threefold: unification, extension and discretization. The motivation behind this theory and Hilger's subsequent works (see [46]-[48]) have opened a new way for mathematicians since the theory of time scales avoids separate studies for differential and difference systems by using similar arguments. For an excellent review on time scale calculus, we refer to pioneering works [24] and [25]. Consequently, the mentioned advantage of the time scale theory has motivated mathematicians especially who work on applied mathematics, and they have started to develop time scale analogues of existing theories such as: oscillation theory, stability theory, population dynamics, mathematical modelling, optimization, mathematical physics, integral equations, probability theory, Floquet theory, economics, boundary value problems, eigenvalue problems etc. (see [2], [3], [5], [8], [9], [10], [11], [14], [15], [16], [22], [26], [33], [36], [38], [37], [51], [60], [63], [69], [71], and [74]).

The time scale variant of Floquet theory was first treated by Ahlbrandt and Ridenhour [14], and this study basicly focused on Floquet's theorem on mixed domains and Putzer representations of matrix logarithms. Adamec [1] criticized the approach in [14] and stated his concerned about suitability of using real exponential function instead of time scale exponential function. Later on, Jeffery J. DaCunha's work on Floquet theory gave a great contribution to applied mathematics. Liapunov stability and unified Floquet theory regarding Liapunov transformations were first handled by DaCunha in 2004 in his Ph.D. thesis. The study [37] not only improved the results of [14] but also extended the study of Floquet theory on time scales extensively. The highlights of the thesis are as follows:

1. Unification and extension of Liapunov's direct method.
2. Development of unified Floquet theory including Liapunov transformations.
3. Application of Floquet theory of homogeneous linear dynamic systems to nonhomogeneous linear dynamic systems.
4. By using monodromy operators, establishing a linkage between Floquet (characteristic) multipliers and Floquet exponents.
5. Stability analysis of time varying linear dynamic equations on time scales.

In [38], DaCunha and Davis improved the results of [37] by answering the following questions:

Q1 Does the solution of matrix exponential equation $e_R(t,\tau) = M$, exist for a nonsingular, constant, $n \times n$ matrix M?

Q2 Is the $n \times n$ matrix R in continuous, discrete and unified versions of Floquet's theorem necessarily constant? Can it be time varying?

Q3 If the matrix R was time varying, how would we make the stability analysis?

In all existing works on Floquet theory, dynamic systems are constructed on additively periodic domains. However, additive periodicity assumption is a strong restriction for the class domains on which periodic solutions of dynamic equations can be analyzed. For instance

$$\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z}\} \cup \{0\}, \, q > 1$$

is not an additively periodic domain. Since q-difference equations are the dynamic equations defined on $\overline{q^{\mathbb{Z}}}$, Floquet theories in previous studies including [38] and [37] are insufficient to investigate periodic solutions of systems q-difference equations. A q-difference equation is an equation including a q-derivative D_q, given by

$$D_q(f)(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad t \in q^{\mathbb{Z}} := \{q^n : q > 1 \,\text{is a constant and}\, n \in \mathbb{Z}\},$$

of its unknown function. Observe that the $q-$derivative $D_q(f)$ of a function f turns into ordinary derivative f' if we let $q \to 1$. The theory of q-difference equations is a useful tool for the discretization of differential equations used for modeling continuous processes [41], [59], [62]. Pulita [68] concluded that "in the p-adic context, q-difference equations are not simply a discretization of solutions of differential equations, but they are actually equal". One may also refer to [20] for further discussion about the equivalence between q-difference equations and differential equations. There is a vast literature on the existence of periodic solutions of differential equations, unlike the existence of periodic solutions of q-difference equations. Thus, it is of importance to study the existence of periodic solutions of q-difference equations. That is, investigation of periodicity on additively periodic domains rules out very important domains that are not additively periodic.

In order to improve and extend the results of the above mentioned works related to Floquet theory, the definition of a periodic domain should be revisited. The idea behind this need can be presented briefly as follows:

1. Once a starting point $t \in \mathbb{T}$ is determined, a periodic domain $\mathbb{T}$ should contain an element at each backward or forward step with size p.
2. Addition is not always the way to step forward and backward on a domain, for instance, $2^{\pm 1}t$ lead to one unit backward and forward steps over the domain $\{2^n : n \in \mathbb{Z}\} \cup \{0\}$.
3. In some way, we should characterize the backward and forward steps on a domain without using only addition.

Periodicity notion and Floquet theory on the domain

$$q^{\mathbb{N}_0} = \{q^n : q > 1 \text{ and } n = 0, 1, 2, \ldots\}$$

have been studied in [28] and [32]. In [28] and [32], an ω-periodic function f on $q^{\mathbb{N}_0}$ is defined to be the one satisfying

$$f(q^\omega t) = \frac{1}{q^\omega} f(t) \text{ for all } t \in q^{\mathbb{N}_0} \text{ and a fixed } \omega \in \{1, 2, \ldots\}.$$

According to this periodicity definition the function $g(t) = 1/t$ is q-periodic over the domain $q^{\mathbb{N}_0}$. Unlike the conventional periodic functions in the existing literature, the function $g(t) = 1/t$ does not repeat its values at each period $t, q^\omega t, (q^\omega)^2 t, \ldots$. The periodicity notion used in [28] and [32] resembles the periodicity on $\mathbb{R}$ in geometric meaning. Their idea is based on the equality of areas lying below the graph of the function at each period. In parallel with conventional periodicity perception, we define a periodic function to be the one repeating its values at each forward/backward step on its domain with a certain size. For instance, according to our definition the function $h(t) = (-1)^{\frac{\ln t}{\ln q}}$ is a q^2-periodic function on $q^{\mathbb{Z}} = \{q > 1 : q^n, n \in \mathbb{Z}\}$ since

$$h\left(q^{\pm 2} t\right) = (-1)^{\frac{\ln t}{\ln q} \pm 2} = (-1)^{\frac{\ln t}{\ln q}} = h(t).$$

Obviously, the function $h(t)$ repeats the values -1 and 1 at each backward/forward step with the size q^2. Consequently, the use of new periodicity notion for $\overline{q^{\mathbb{Z}}}$ in Floquet theory provides not only a generalization but also an alternative approach to already existing literature in particular cases [28], [32].

The organization of the rest of the thesis is as follows: In the next chapter, some basics of quantum calculus and time scales calculus are outlined under the guidance of [24] and [52]. Additionally, the periodicity notion for quantum calculus and the new periodicity concept on time scales are introduced according to pioneering work of Adıvar [12]. In Chapter 3, q-Floquet theory is established with respect to Lyapunov transformations and Chapter 4 is devoted to generalization and unification of results by using periodicity in shifts (see [6]).

Chapter 2

Preliminaries

2.1 A brief introduction to quantum calculus

In this section, some definitions and results about the quantum calculus constructed on $q^{\mathbb{Z}}$, $q > 1$, are presented according to books [24] and[52]. Most of the definitions, theorems and examples can be found in [24] and [52].

Definition 1. Let f be an arbitrary function defined on $q^{\mathbb{Z}}, q > 1$. The q-differential of f at a point q^m, $m \in \mathbb{Z}$, is given by

$$d_q f\left(q^m\right) := f\left(q^{m+1}\right) - f\left(q^m\right).$$

Notice that, symmetry property is lost for production of two functions in quantum differentials. That is

$$d_q\left(f\left(q^m\right) g\left(q^m\right)\right) = f\left(q^{m+1}\right) d_q g\left(q^m\right) + g\left(q^m\right) d_q f\left(q^m\right),$$

and

$$d_q\left(g\left(q^m\right) f\left(q^m\right)\right) = g\left(q^{m+1}\right) d_q f\left(q^m\right) + f\left(q^m\right) d_q g\left(q^m\right)$$

may not be equal all the time.

Definition 2. The q-derivative of a function f defined on $q^{\mathbb{Z}}, q > 1$ at a point

q^m, $m \in \mathbb{Z}$ is given by

$$D_q f(q^m) = \frac{d_q f(q^m)}{d_q q^m} = \frac{f(q^{m+1}) - f(q^m)}{(q-1)\, q^m}.$$

Obviously, when $q \to 1$, $D_q f \to \frac{df}{dx}$ and like the ordinary differentiation operator, q-derivation operator is a linear operator.

The properties of q-derivation operator, D_q, are as follows:

Theorem 2.1 *Assume $f, g : q^{\mathbb{Z}} \to \mathbb{R}$ be two functions and α, β are constants. Then*

i. $D_q(fg)(q^m) = D_q f(q^m)\, g(q^m) + f(q^{m+1})\, D_q g(q^m)\,, m \in \mathbb{Z}$

ii. If $f(q^m)\, f(q^{m+1}) \neq 0$ *for all* $m \in \mathbb{Z}$, *then we have*

$$D_q\left(\frac{1}{f}\right)(q^m) = -\frac{D_q f(q^m)}{f(q^m)\, f(q^{m+1})}$$

iii. If $g(q^m)\, g(q^{m+1}) \neq 0$ *for all* $m \in \mathbb{Z}$, *then we have*

$$D_q\left(\frac{f}{g}\right)(q^m) = \frac{D_q f(q^m)\, g(q^m) - f(q^m)\, D_q g(q^m)}{g(q^m)\, g(q^{m+1})}.$$

The following definitions and results are given according to [24].

Definition 3 (q-Exponential function)**.** Let p be a regressive function on $q^{\mathbb{Z}}$, i.e.

$$1 + (q-1)\, q^m p(q^m) \neq 0,\ \forall m \in \mathbb{Z}.$$

Then $e_p(., q^{m_0})\, y_0$ is unique solution of the initial value problem

$$D_q y(q^m) = p(q^m) y(q^m),\ \ y(q^{m_0}) = y_0$$

for $m_0 \in \mathbb{Z}$ with $m \in [m_0, \infty) \cap \mathbb{Z}$. Furthermore, the explicit form of q-exponential function $e_p(q^m, q^{m_0})$ is given by

$$e_p(q^m, q^{m_0}) := \prod_{\tau\in[m_0,m)_{\mathbb{Z}}} [1 + (q-1)\, q^\tau p\,(q^\tau)]\,,$$

where $[m_0, m)_{\mathbb{Z}} := [m_0, m) \cap \mathbb{Z}$.

The following theorem summarizes some basic properties of q-exponential functions.

Theorem 2.2 *Let p and g are regressive functions on $q^{\mathbb{Z}}$. Then*

i. $e_0(q^m, q^{m_0}) \equiv 1$ *and* $e_p(q^m, q^m) \equiv 1$

ii. $e_p(q^{m+1}, q^{m_0}) = (1 + (q-1)\, q^m p(q^m))e_p(q^m, q^{m_0})$

iii. $e_p(q^m, q^{m_0})e_g(q^m, q^{m_0}) = e_{p\oplus g}(q^m, q^{m_0}),$ *where* $p \oplus g := p + g + pg\,(q-1)\, q^m$

iv. $e_p(q^m, q^{m_0}) = \frac{1}{e_p(q^{m_0}, q^m)} = e_{\ominus p}(q^{m_0}, q^m),$ *where* $\ominus p(q^m) = -\frac{p(q^m)}{1+(q-1)q^m p(q^m)}$

v. $e_p(q^m, q^{m_0})e_p(q^{m_0}, q^\tau) = e_p(q^m, q^\tau)$.

Definition 4. A matrix function $A : q^{\mathbb{Z}} \to \mathbb{R}^{n\times n}$ is said to be regressive matrix function if $I + (q-1)\, q^m A(q^m)$ is invertible for all $m \in \mathbb{Z}$.

Definition 5 (q-Matrix exponential function)**.** Let $m_0 \in \mathbb{Z}$ be fixed and $A : q^{\mathbb{Z}} \to \mathbb{R}^{n\times n}$ be a regressive matrix function. The unique matrix solution of the matrix system

$$D_q Y(q^m) = A(q^m)Y(q^m),\ Y(q^{m_0}) = I, \tag{2.1}$$

is called q-matrix exponential function and it is represented by $e_A\,(., q^{m_0})$ for $m \geq m_0$. Moreover, the matrix function $e_A\,(q^m, q^{m_0})$ can be evaluated as

$$e_A\,(q^m, q^{m_0}) = \prod_{\tau\in[m_0,m)_{\mathbb{Z}}} [I + (q-1)\, q^\tau A\,(q^\tau)]\,,$$

(see also [28]).

In the next result, some properties of the matrix exponential of the system (2.1) are listed.

Theorem 2.3 *If A is a regressive matrix function on $q^{\mathbb{Z}}$, then we have*

i. $e_0(q^m, q^{m_0}) \equiv I$ *and* $e_A(q^m, q^m) \equiv I$, *where* 0 *and* I *indicate the zero matrix and the identity matrix, respectively*

ii. $e_A(q^{m+1}, q^{m_0}) = (I + (q-1)\, q^m A(q^m))\, e_A(q^m, q^{m_0})$

iii. $e_A(q^m, q^{m_0}) = e_A^{-1}(q^{m_0}, q^m)$

iv. $e_A(q^m, q^{m_0})\, e_A(q^m, q^\tau) = e_A(q^m, q^\tau)$.

Theorem 3, Theorem 2.2 and Theorem 2.3 are the special cases of theorems [24, Theorem 1.20], [24, Theorem 2.36] and [24, Theorem 5.21] when the time scale $\mathbb{T}$ *is chosen to be* $\overline{q^{\mathbb{Z}}}, q > 1$, *respectively.*

The definitions of q-antiderivative and q-integral, introduced in Chapter 18 and Chapter 19 of [52], are as follows:

Definition 6 (q-Antiderivative)**.** Let f be a function defined on $q^{\mathbb{Z}}, q > 1$. The function F is called q-antiderivative of the function f and denoted by

$$\int f(t)\, d_q t$$

if $D_q F(q^m) = f(q^m)$ for all $m \in \mathbb{Z}$.

Remark. Similar to the conventional calculus, q-antiderivative is not unique. It is well known that uniqueness of the antiderivative of a function on set of reals is valid up to adding a constant. However, in quantum calculus having a zero derivative, $D_q\varphi(q^m) = 0$, does not mean that φ is a constant function. Obviously, any nonconstant function φ satisfying $\varphi(q^{m+1}) = \varphi(q^m)$ for $m \in \mathbb{Z}$ has a zero q-derivative.

Definition 7 (q-Integral)**.** The indefinite integral of a function f defined on $q^{\mathbb{Z}}, q > 1$ is given by the geometric series expansion

$$\int f(t)\, d_q t = (q-1)\, q^m \sum_{\tau=0}^{\infty} q^\tau f\left(q^{m+\tau}\right).$$

Additionally, the definite integral from q^m to q^n of the function f is defined by

$$\int_{q^m}^{q^n} f(s)\, d_q s := (q-1) \sum_{\tau=m}^{n-1} q^\tau f(q^\tau).$$

The following definition is the q-discrete analogue of the definition of transition matrix on arbitrary time scales given in [39], [38].

Definition 8 (Transition matrix)**.** Let $m_0 \in \mathbb{Z}$ be fixed and $A : q^{\mathbb{Z}} \to \mathbb{R}^{n\times n}$ be a regressive matrix function. The solution of the q-difference matrix system

$$D_q Y(q^m) = A(q^m) Y(q^m), Y(q^{m_0}) = Y_0$$

for $m \geq m_0$ is expressed by the equality

$$Y(q^m) = \Phi_A(q^m, q^{m_0})\, Y_0,$$

where $\Phi_A(q^m, q^{m_0})$ is called q-transition matrix of the system and given by

$$\begin{aligned}\Phi_A(q^m, q^{m_0}) = I &+ \int_{q^{m_0}}^{q^m} A(\tau_1)\, d_q\tau_1 + \int_{q^{m_0}}^{q^m} A(\tau_1) \int_{q^{m_0}}^{q^{\tau_1}} A(\tau_2)\, d_q\tau_2 d_q\tau_1 + \dots \\ &+ \int_{q^{m_0}}^{q^m} A(\tau_1) \int_{q^{m_0}}^{q^{\tau_1}} A(\tau_2) \dots \int_{q^{m_0}}^{q^{\tau_{i-1}}} A(\tau_i)\, d_q\tau_i \dots d_q\tau_1 + \dots . \end{aligned} \tag{2.2}$$

2.1.1 Periodicity notion on quantum calculus

In order to discuss periodic solutions of q-difference equations and systems which are constructed on the domain $q^{\mathbb{Z}}$, $q > 1$, one needs to provide periodicity notion on quantum calculus. By employing results of [12] in the special case $\mathbb{T} = \overline{q^{\mathbb{Z}}}, q > 1$, periodicity on quantum calculus is discussed in this part.

Definition 9 (Periodic functions on $q^{\mathbb{Z}}$)**.** Let f be a real valued function defined on $q^{\mathbb{Z}}$, $q > 1$. We say that f is periodic if there exists a $T \in [1, \infty)_{\mathbb{Z}}$ such that

$$f\left(q^{m\pm T}\right) = f(q^m) \text{ for all } m \in \mathbb{Z}. \tag{2.3}$$

The number q^T is called the period of f, if it is the smallest number satisfying (2.3).

Example 2.1. The function $h(q^m) = (-1)^m$ on $q^{\mathbb{Z}}$ is a q^2 periodic function since

$$h(q^{m\pm 2}) = (-1)^{m\pm 2} = (-1)^{\frac{\ln q^{m\pm 2}}{\ln q}} = (-1)^{\frac{\ln q^m}{\ln q}\pm 2} = (-1)^{\frac{\ln q^m}{\ln q}} = (-1)^m = h(q^m),$$

for all $m \in \mathbb{Z}$.

Definition 10 (Multiplicatively periodic functions on q-calculus)**.** A real valued function f on $q^{\mathbb{Z}}$, $q > 1$ is called multiplicatively periodic function if there exists a $T \in [1, \infty)_{\mathbb{Z}}$ such that

$$f\left(q^{m\pm T}\right) q^{\pm T} = f(q^m) \text{ for all } m \in \mathbb{Z}. \tag{2.4}$$

The smallest number q^T satisfying (2.4) is called the period of the function f.

Example 2.2. The function $f(t) = \frac{1}{t}$ is a multiplicatively q-periodic function on $q^{\mathbb{Z}}$.

One can deduce the following result using Theorem 2 in [12]:

Theorem 2.4 *Let $f : q^{\mathbb{Z}} \to \mathbb{R}$ be a multiplicatively periodic function with period q^T, $T \in [1, \infty)_{\mathbb{Z}}$. Then*

$$\int_1^{q^m} f(s) d_q s = \int_{q^{\pm T}}^{q^{m\pm T}} f(s) d_q s.$$

2.2 A brief introduction to time scale calculus

This part is devoted to some basic concepts of time scale calculus. The definitions, results and examples given in this section can be found in [19], [24], and [25].

2.2.1 Basic calculus on time scales

Any arbitrary, nonempty closed subset of real numbers is called a time scale. Any closed interval and the sets $\mathbb{R}, \mathbb{Z}, h\mathbb{Z}, \mathbb{N}_0, q^{\mathbb{Z}} \cup \{0\}$ and Cantor set are examples of time scales. It should be emphasized that $\mathbb{Q}, \mathbb{R}\backslash\mathbb{Q}, \mathbb{C}$ are not time scales. Henceforth, a time scale will be denoted by $\mathbb{T}$. The basic operators on time scales which enables moving forward and backward steps on $\mathbb{T}$ are defined as follows:

Definition 11. Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\},$$

and

$$\rho(t) := \sup \{s \in \mathbb{T} : s < t\},$$

respectively. The step size function (graininess function) $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Moreover, any point on a time scale can be classified by using forward and backward jump operators. The following table introduces several types of points on a time scale.

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

Table 2.1: Classification of points on a time scale

Example 2.3. The next table shows forward&backward jump operators and step size functions for the most well known time scales: .

In order to define differentiation on an arbitrary time scale, we have to define

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$h\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$, $q>1$
$\rho(t)$	t	$t-1$	$t-h$	t/q
$\sigma(t)$	t	$t+1$	$t+h$	qt
$\mu(t)$	0	1	h	$(q-1)t$

Table 2.2: Basic operations on $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$, $q>1$

the set $\mathbb{T}^{\kappa}$ by using $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 12 (Hilger derivative of a function)**.** Let f be a real valued function on $\mathbb{T}$ and $t \in \mathbb{T}^{\kappa}$. Then $f^{\Delta}(t)$ is defined to be a number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U : (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U.$$

Furthemore, f is called Δ-differentiable on $\mathbb{T}^{\kappa}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

Now, we present the result of [24] which provides some important properties regarding Δ-derivative of a function f on an arbitrary time scale.

Theorem 2.5 *Assume f is a real valued function defined on a time scale $\mathbb{T}$ and $t \in \mathbb{T}^{\kappa}$. Then we have the following results:*

i. If f is differentiable at t, then f is continuous at t

ii. If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

iii. If t is right dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number

iv. If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$$

Example 2.4. The following table shows Δ-derivative of a function defined on the time scales $\mathbb{R}, \mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$, $q > 1$.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$, $q > 1$
$f^{\Delta}(t)$	$f'(t) = \lim\limits_{h \to 0} \frac{f(t+h)-f(t)}{h}$	$\Delta f(t) = f(t+1) - f(t)$	$D_q f(t) = \frac{f(qt)-f(t)}{(q-1)t}$

Table 2.3: Continuous and discrete counterparts of Δ-derivative

Theorem 2.6 *Let f and g are Δ-differentiable functions at $t \in \mathbb{T}^{\kappa}$. Then, we have*

i. $(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t)$, *for any constant α and β*

ii. $(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t)$

iii. If $f(t) f(\sigma(t)) \neq 0$, *then* $\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}$

iv. If $g(t) g(\sigma(t)) \neq 0$, *then* $\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}$.

Regularity and rd-continuity notions for functions defined on arbitrary time scales are introduced in the next definition:

Definition 13. Let f be a real valued function defined on a time scale $\mathbb{T}$. f is said to be regular if its right and left sided limits exist as finite numbers at all right dense and left dense points in $\mathbb{T}$. Moreover, f is called rd-continuous if it is continuous at all right dense points in $\mathbb{T}$ and its left sided limits exist as finite numbers at all left dense points in $\mathbb{T}$. Hereafter, C_{rd} and C^1_{rd} stand for the class of rd-continuous functions and differentiable functions with rd-continuous derivatives, respectively.

Theorem 2.7 *Suppose the function f is defined on a time scale $\mathbb{T}$. Then we have the following implications:*

i. Continuity $\Rightarrow$ rd-continuity

ii. Rd-continuity $\Rightarrow$ regularity

iii. f^{σ} is regulated (rd-continuous) when f is regulated (rd-continuous).

Theorem 2.8 *The sufficient condition for a function to have an antiderivative is being rd-continuous. In particular if $t_0 \in \mathbb{Y}$ the antiderivative F of an rd-continuous function f is defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\,\Delta\tau \text{ for } t \in \mathbb{T}.$$

Example 2.5. The following table shows integration on time scales $\mathbb{R}$, $\mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$:

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$, $q < 1$
$\int_0^b f(t)\Delta t$	$\int_0^b f(t)dt$	$\sum_{t=0}^{b-1} f(t)$, $(0 < b)$	$\int_0^b f(t)d_qt = (1-q)\,b\sum_{j=0}^{\infty} q^j f(q^j b)$

Table 2.4: Continuous and discrete counterparts of integration on particular time scales

2.2.2 Hilger's complex plane and time scale exponential function

Definition 14. Hilger complex numbers are defined by the set

$$\mathbb{C}_\mu \ : \ = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{\mu(t)} \right\}.$$

Moreover, Hilger real axis, Hilger alternating axis and the Hilger imaginary circle are given by the sets

$$\begin{aligned}
\mathbb{R}_\mu &:= \left\{ z \in \mathbb{C}_\mu : z \in \mathbb{R} \text{ and } z > -\frac{1}{\mu(t)} \right\}, \\
\mathbb{A}_\mu &:= \left\{ z \in \mathbb{C}_\mu : z \in \mathbb{R} \text{ and } z < -\frac{1}{\mu(t)} \right\}, \\
\mathbb{I}_\mu &:= \left\{ z \in \mathbb{C}_\mu : \left| z + \frac{1}{\mu(t)} \right| = \frac{1}{\mu(t)} \right\},
\end{aligned}$$

respectively. Notice that when $\mu(t) = 0$ we have $\mathbb{C}_0 = \mathbb{C}$, $\mathbb{R}_0 = \mathbb{R}$, $\mathbb{I}_0 = i\mathbb{R}$ and $\mathbb{A}_0 = \emptyset$.

The following figure illustrates the Hilger's complex plane: $Re_\mu(z), Im_\mu(z)$ and

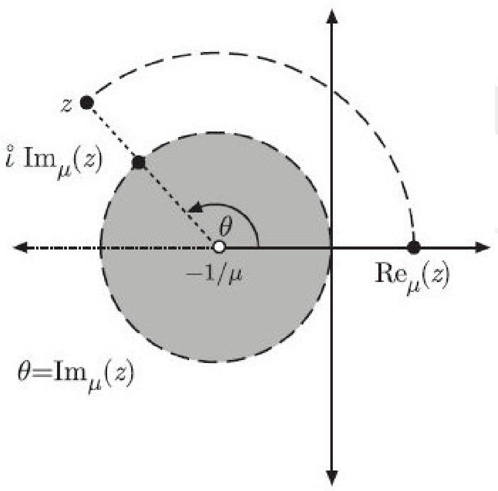

Figure 2.1: Hilger's complex plane

.

$\mathring{\imath}$ are called Hilger real part of z, Hilger imaginary part of z and Hilger purely imaginary number defined by

$$Re_\mu(z) := \frac{|z\mu(t) + 1| - 1}{\mu(t)},$$

$$Im_\mu(z) := \frac{Arg(z\mu(t) + 1)}{\mu(t)},$$

and

$$\mathring{\imath}\omega = \frac{e^{i\omega\mu(t)} - 1}{\mu(t)} \text{ for } -\frac{\pi}{\mu(t)} < \omega \leq \frac{\pi}{\mu(t)},$$

respectively.

Definition 15. The circle plus, $\oplus$, on $\mathbb{C}_\mu$ is defined by

$$z \oplus w = z + w + zw\mu\left(t\right).$$

Notice that $(\mathbb{C}_\mu, \oplus)$ is an Abelian group. Moreover, the inverse of z with respect to the operation $\oplus$ is represented by $\ominus z$ given by

$$\ominus z := -\frac{z}{1 + z\mu\left(t\right)},$$

and the operation $\ominus$ satisfies

i. $z \ominus w = z \oplus (\ominus w)$

ii. $z \ominus z = 0$

iii. $z \ominus w = \frac{z-w}{1+w\mu(t)}$.

Observe that for any complex number z in $\mathbb{C}_\mu$ can be decomposed as

$$z = Re_\mu\left(z\right) \oplus \mathring{i} Im_\mu\left(z\right).$$

Definition 16. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if

$$1 + \mu\left(t\right) p\left(t\right) \neq 0 \text{ for all } t \in \mathbb{T}^\kappa.$$

Hereafter, we use the notation $\mathcal{R}$ to represent the set of all regressive and rd-continuous functions defined on $\mathbb{T}$.

Definition 17. Let $p \in \mathcal{R}$. Then the time scale exponential function $e_p\left(t, s\right)$ is defined by

$$e_p\left(t, s\right) = \exp\left(\int_s^t \xi_{\mu(\tau)}\left(p\left(\tau\right)\right) \Delta\tau\right) \text{ for } s, t \in \mathbb{T},$$

where $\xi_{\mu(t)}$ is the cylinder transform given by

$$\xi_{\mu(t)}\left(z\right) = \log\left(1 + z\mu\left(t\right)\right).$$

Furthermore, for a fixed $t_0 \in \mathbb{T}$ the time scale exponential function is defined as a solution of the following regressive initial value problem

$$y^{\Delta} = p(t)\, y,\ \ y(t_0) = 1$$

on $\mathbb{T}$.

In the following result, basic properties of time scale exponential function are given.

Theorem 2.9 *Let $p, q \in \mathcal{R}$. Then*

i. $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$

ii. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$

iii. $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$

iv. $e_p(t,s)e_p(s,r) = e_p(t,r)$

v. $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Example 2.6. The following table demonstrates some exponential functions over some particular time scales.

$\mathbb{T}$	$e_\alpha(t,t_0)$
$\mathbb{R}$	$e^{\alpha(t-t_0)}$
$\mathbb{Z}$	$(1+\alpha)^{t-t_0}$
$h\mathbb{Z}$	$(1+h\alpha)^{(t-t_0)/h}$
$q^{\mathbb{N}_0}$	$\prod_{s\in[t_0,t)_{q^{\mathbb{N}_0}}} [1+(q-1)\alpha s]\,,\ t > t_0$
$\frac{1}{n}\mathbb{Z}$	$\left(1+\frac{\alpha}{n}\right)^{n(t-t_0)}$

Table 2.5: Exponential functions on some particular time scales

2.2.3 Regressive matrices and time scale matrix exponential

Definition 18. Let A be an $n \times n$ matrix function so that $A : \mathbb{T} \to \mathbb{R}^{n \times n}$. A is said to be rd-continuous if each entry of A is rd-continuous on $\mathbb{T}$. The notation C_{rd} is also used for the representation of rd-continuous matrix functions.

Definition 19. Let A be an $n \times n$ matrix function on $\mathbb{T}$. A is said to be regressive on $\mathbb{T}$ if $I + \mu(t) A(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$. The notation $\mathcal{R}$ represents the class of regressive rd-continuous matrix functions on $\mathbb{T}$ like the scalar case.

Definition 20. Suppose A is an $n \times n$ regressive matrix valued function on $\mathbb{T}$ and $t_0 \in \mathbb{T}$. Then the matrix exponential function, $e_A(., t_0)$ is defined as the unique matrix valued solution of the following matrix initial value problem

$$Y^{\Delta} = A(t) Y, \; Y(t_0) = I,$$

where I indicates the the identity matrix.

Example 2.7. If A is chosen to be a constant $n \times n$ matrix, then we have the matrix exponentials

$$e_A(t, t_0) = e^{A(t-t_0)}$$

on $\mathbb{T} = \mathbb{R}$ and

$$e_A(t, t_0) = (I + A)^{t-t_0}$$

on $\mathbb{T} = \mathbb{Z}$ provided $A \neq -I$.

The following theorem summarizes the basic properties of time scale matrix exponential functions:

Theorem 2.10 *Let A be an $n \times n$ regressive matrix function on $\mathbb{T}$. Then, we have*

i. $e_0(t, s) \equiv I$ *and* $e_A(t, t) \equiv I$*, where* 0 *denotes zero matrix*

ii. $e_A(\sigma(t), s) = (I + \mu(t) A(t)) e_A(t, s)$

iii. $e_A(t,s) = e_A^{-1}(s,t)$

iv. $e_A(t,s)\,e_A(s,r) = e_A(t,r)$.

For an arbitrary matrix A, the solution of the system

$$x^{\Delta}(t) = A(t)x(t), x(t_0) = x_0$$

is given by the equality

$$x(t) = \Phi_A(t,t_0)\,x_0,$$

where $\Phi_A(t,t_0)$, called the transition matrix for the system (4.1), is given by

$$\begin{aligned}\Phi_A(t,t_0) = I &+ \int_{t_0}^{t} A(\tau_1)\,\Delta\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\,\Delta\tau_2\Delta\tau_1 + \dots \\ &+ \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\dots\int_{t_0}^{\tau_{i-1}} A(\tau_i)\,\Delta\tau_i\dots\Delta\tau_1 + \dots \end{aligned} \tag{2.5}$$

As it is discussed in [39], the matrix exponential $e_A(t,t_0)$ is not always identical to $\Phi_A(t,t_0)$. We have $e_A(t,t_0) \equiv \Phi_A(t,t_0)$ only if the matrix A satisfies the equality

$$A(t)\int_{s}^{t} A(\tau)\,\Delta\tau = \int_{s}^{t} A(\tau)\,\Delta\tau A(t).$$

2.2.4 The new periodicity concept on time scales

In this part, shift operators and new periodicity notion based on shift operators on time scales are introduced according to studies [11] and [12]. Given results and examples can be directly found in [12].

Definition 21 (Shift operators)**.** Let $\mathbb{T}^*$ be a nonempty subset of $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_{\pm} : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties

i. Shift operators $\delta_{\pm}$ are strictly increasing in their second arguments, if

$$(T,t),(T,u)\in\mathcal{D}_{\pm}:=\{(s,t)\in[t_0,\infty)_{\mathbb{T}}\times\mathbb{T}^*:\delta_{\pm}(s,t)\in\mathbb{T}^*\},$$

then

$$T\leq t<u \text{ implies } \delta_{\pm}(T,t)<\delta_{\pm}(T,u)$$

ii. If $(T_1,u),(T_2,u)\in\mathcal{D}_-$ with $T_1<T_2$, then $\delta_-(T_1,u)>\delta_-(T_2,u)$ and if $(T_1,u),(T_2,u)\in\mathcal{D}_+$ with $T_1<T_2$, then $\delta_+(T_1,u)<\delta_+(T_2,u)$

iii. If $t\in[t_0,\infty)_{\mathbb{T}}$, then $(t,t_0)\in\mathcal{D}_+$ and $\delta_+(t,t_0)=t$. Moreover, if $t\in\mathbb{T}^*$, then $(t_0,t)\in\mathcal{D}_+$ and $\delta_+(t_0,t)=t$

iv. (a) If $(s,t)\in\mathcal{D}_+$, then $(s,\delta_+(s,t))\in\mathcal{D}_-$ and $\delta_-(s,\delta_+(s,t))=t$

(b) If $(s,t)\in\mathcal{D}_-$, then $(s,\delta_-(s,t))\in\mathcal{D}_+$ and $\delta_+(s,\delta_-(s,t))=t$

v. (a) If $(s,t)\in\mathcal{D}_+$ and $(u,\delta_+(s,t))\in\mathcal{D}_-$, then $(s,\delta_-(u,t))\in\mathcal{D}_+$ and $\delta_-(u,\delta_+(s,t))=\delta_+(s,\delta_-(u,t))$

(b) If $(s,t)\in\mathcal{D}_-$ and $(u,\delta_-(s,t))\in\mathcal{D}_+$, then $(s,\delta_+(u,t))\in\mathcal{D}_-$ and $\delta_+(u,\delta_-(s,t))=\delta_-(s,\delta_+(u,t))$.

The operators δ_+ and δ_- are called as forward and backward shift operators corresponding the initial point t_0. Moreover, the sets $\mathcal{D}_+$ and $\mathcal{D}_-$ are the domains of the forward and backward shift operators, respectively.

Example 2.8. The following table shows the shift operators $\delta_{\pm}(s,t)$ on some time scales:

$\mathbb{T}$	t_0	$\mathbb{T}^*$	$\delta_-(s,t)$	$\delta+(s,t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t-s$	$t+s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t-s$	$t+s$
$q^{\mathbb{Z}}\cup\{0\},q>1$	1	$q^{\mathbb{Z}}$	$\frac{t}{s}$	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$(t^2-s^2)^{1/2}$	$(t^2+s^2)^{1/2}$

Table 2.6: Shift operators on particular time scales

Definition 22 (Periodicity in shifts)**.** Let $\mathbb{T}$ be a time scale with the shift operators $\delta_{\pm}$ associated with the initial point $t_0 \in \mathbb{T}^*$, then $\mathbb{T}$ is said to be periodic in shifts $\delta_{\pm}$, if there exists a $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p,t) \in \mathcal{D}_{\mp}$ for all $t \in \mathbb{T}^*$. P is called the period of $\mathbb{T}$ with respect to $\mathbb{T}^*$ if

$$P = \inf \{p \in (t_0, \infty)_{\mathbb{T}^*} : (p,t) \in \mathcal{D}_{\mp} \text{ for all } t \in \mathbb{T}^*\} > t_0.$$

Note that additive periodic time scales are unbounded sets. The following example indicates that a periodic time scale in shifts may be bounded.

Example 2.9. The time scale $\mathbb{T}=\left\{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\right\} \cup \{0,1\}$ is a bounded time scale which is periodic in shifts with respect to

$$\mathbb{T}^* = \left\{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\right\}.$$

and shift operators

$$\delta_{\pm}(P,t) = \frac{q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}{1+q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}, \quad P = \frac{q}{1+q}, \quad t_0 = \frac{1}{2}.$$

Definition 23 (Periodic function in shifts $\delta_{\pm}$)**.** Assume that $\mathbb{T}$ is a time scale P-periodic in shifts and f is a real valued function defined on $\mathbb{T}^*$. The function f is periodic in shifts $\delta_{\pm}$ if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T,t) \in \mathcal{D}_{\pm} \text{ and } f\left(\delta_{\pm}^{T}(t)\right) = f(t) \text{ for all } t \in \mathbb{T}^*, \tag{2.6}$$

where $\delta_{\pm}^{T}(t) = \delta_{\pm}(T,t)$. The possible smallest number T satisfying (2.6) is called the period of f.

Example 2.10. Let $\mathbb{T} = \mathbb{R}$ with initial point $t_0 = 1$, the function

$$f(t) = \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi\right), \quad t \in \mathbb{R}^* := \mathbb{R} - \{0\}$$

is 4-periodic in shifts $\delta_{\pm}$ since

$$
\begin{aligned}
f\left(\delta_{\pm}\left(4,t\right)\right) &= \begin{cases} f\left(t4^{\pm 1}\right) & \text{if } t \geq 0 \\ f\left(t/4^{\pm 1}\right) & \text{if } t < 0 \end{cases} \\
&= \sin\left(\frac{\ln|t| \pm 2\ln\left(1/2\right)}{\ln\left(1/2\right)}\pi\right) \\
&= \sin\left(\frac{\ln|t|}{\ln\left(1/2\right)}\pi \pm 2\pi\right) \\
&= \sin\left(\frac{\ln|t|}{\ln\left(1/2\right)}\pi\right) \\
&= f\left(t\right).
\end{aligned}
$$

Definition 24 (Δ-periodic function in shifts $\delta_{\pm}$)**.** Assume that $\mathbb{T}$ is a time scale P-periodic in shifts and f is a real valued function defined on $\mathbb{T}^{*}$. The function f is Δ-periodic function in shifts if there exists a $T \in [P,\infty)_{\mathbb{T}^{*}}$ such that

$$(T,t) \in \mathcal{D}_{\pm} \text{ for all } t \in \mathbb{T}^{*} \tag{2.7}$$

$$\text{the shifts } \delta_{\pm}^{T} \text{ are } \Delta\text{-differentiable with rd-continuous derivatives} \tag{2.8}$$

and

$$f\left(\delta_{\pm}^{T}\left(t\right)\right)\delta_{\pm}^{\Delta T}\left(t\right) = f\left(t\right) \tag{2.9}$$

for all $t \in \mathbb{T}^{*}$, where $\delta_{\pm}^{T}\left(t\right) = \delta_{\pm}\left(T,t\right)$. The possible smallest number T satisfying (2.7-2.9) is called the period of f.

The following result is useful for integration of functions which are Δ-periodic in shifts.

Theorem 2.11 *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period $P \in (t_0,\infty)_{\mathbb{T}^{*}}$ and f a Δ-periodic function in shifts $\delta_{\pm}$ with the period $T \in [P,\infty)_{\mathbb{T}^{*}}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_{\pm}^{T}(t_0)}^{\delta_{\pm}^{T}(t)} f(s)\Delta s.$$

Chapter 3

q-Floquet theory

In this chapter, q-analogue of Floquet theory for homogeneous and nonhomogeneous dynamic systems constructed on $q^{\mathbb{Z}}$ for $q > 1$ is established to give sufficient conditions for existence of periodic solutions by using Lyapunov transformation. Lyapunov transformation on q-domain can be defined similar to Definition 2.1 in [38] as follows:

Definition 25. A matrix valued function $L : q^{\mathbb{Z}} \to \mathbb{R}^{n\times n}$ is called Lyapunov transformation if $L(q^m)$ is invertible, has a bounded matrix norm, and for some $\eta \in \mathbb{R}_+$

$$|\det L(q^m)| \geq \eta \text{ for all } m \in \mathbb{Z}.$$

3.1 Homogeneous case

Consider the nonautonomous regressive q-difference system

$$D_q x(q^m) = A(q^m)\, x(q^m),\ \ x(q^{m_0}) = x_0,\ \ m, m_0 \in \mathbb{Z} \text{ and } m \geq m_0, \tag{3.1}$$

where the matrix function $A : q^{\mathbb{Z}} \to \mathbb{R}^{n\times n}$ is multiplicatively periodic with period q^T. Hereafter, the system (3.1) is called q-Floquet system.

In order to construct the matrix R which is a solution of the matrix exponential equation, the definition of real power of a matrix is essential.

Definition 26 (Real power of a matrix)**.** [38, Definition A.5] Given an $n \times n$ nonsingular matrix M with elementary divisors $\{(\lambda - \lambda_i)^{m_i}\}_{i=1}^{k}$ and any $r \in \mathbb{R}$, the real power of the matrix M is introduced by

$$M^r := \sum_{i=1}^{k} P_i(M)\,\lambda_i^r \left[\sum_{j=0}^{m_i-1} \frac{\Gamma(r+1)}{j!\Gamma(r-j+1)} \left(\frac{M-\lambda_i I}{\lambda_i}\right)^j\right], \tag{3.2}$$

where

$$P_i(\lambda) := a_i(\lambda)\, b_i(\lambda),$$

$$b_i(\lambda) := \Pi_{\substack{j \neq i \\ j=1}}^{k} (\lambda - \lambda_j),$$

$$\frac{1}{p(\lambda)} = \sum_{i=1}^{k} \frac{a_i(\lambda)}{(\lambda - \lambda_i)^{m_i}},$$

and $p(\lambda)$ is the characteristic polynomial of M.

It can be seen from Proposition A.3 in [38] that $M^{s+r} = M^s M^r$ for any $r, s \in \mathbb{R}$.

Theorem 3.1 *Let M be a nonsingular, $n \times n$ constant matrix and $R : q^{\mathbb{Z}} \to \mathbb{C}^{n\times n}$ be a matrix function. A solution of the matrix exponential equation*

$$e_R\left(q^{m_0+T}, q^{m_0}\right) = M, \;\; m_0, T \in \mathbb{Z},$$

is given by

$$R(q^m) = \frac{M^{(q^{-T}+1)} - I}{(q-1)\,q^m}, \;\; m \in \mathbb{Z}. \tag{3.3}$$

Proof. Let's construct the matrix exponential function $e_R(q^m, q^{m_0})$ as follows

$$e_R(q^m, q^{m_0}) := M^{(q^{-T+1}(m-m_0))}, \tag{3.4}$$

where real power of a nonsingular matrix M is given by Definition 26. In order to show that the function $e_R(q^m, q^{m_0})$ constructed in (3.4) is the matrix exponential one can observe that $e_R\left(q^{m_0}, q^{m_0}\right) = M^0 = I$. Then, differentiating (3.4) yields

$$D_q e_R\left(q^m, q^{m_0}\right) = R\left(q^m\right) e_R\left(q^m, q^{m_0}\right),$$

which can be seen from

$$\begin{aligned}
D_q e_R\left(q^m, q^{m_0}\right) &= \frac{e_R\left(q^{m+1}, q^{m_0}\right) - e_R\left(q^m, q^{m_0}\right)}{(q-1)\, q^m} \\
&= \frac{M^{\left(q^{-T+1}(m+1-m_0)\right)} - M^{\left(q^{-T+1}(m-m_0)\right)}}{(q-1)\, q^m} \\
&= \frac{M^{\left(q^{-T+1}\right)} - I}{(q-1)\, q^m} M^{\left(q^{-T+1}(m-m_0)\right)} \\
&= R\left(q^m\right) e_R\left(q^m, q^{m_0}\right).
\end{aligned}$$

This shows that (3.3) is a solution of $e_R\left(q^{m_0+T}, q^{m_0}\right) = M$. □

Corollary 3.2 *The matrices $R\left(q^m\right)$ and M defined in Theorem 3.1 have the same eigenvectors.*

Proof. Let $\{\lambda_i, v_i\}$, $i = 1, 2, ..., n$ be eigenpairs of M, i.e., $M v_i = \lambda_i v_i$ for all $i = 1, 2, ..., n$. Consider

$$R(q^m) v_i = \frac{M^{\left(q^{-T+1}\right)} - I}{(q-1)\, q^m} v_i = \left(\frac{\lambda_i^{\left(q^{-T+1}\right)} - 1}{(q-1)\, q^m}\right) v_i. \tag{3.5}$$

Substituting $\gamma_i(q^m) = \frac{\lambda_i^{\left(q^{-T+1}\right)} - 1}{(q-1) q^m}$ into (3.5), one can conclude that $R(q^m)$ has the eigenpairs $\{\gamma_i(q^m), v_i\}_{i=1}^n$. □

Lemma 3.3 *Let $P : q^{\mathbb{Z}} \to \mathbb{R}^{n\times n}$ be a regressive, multiplicatively periodic matrix valued function with period q^T such as*

$$P\left(q^m\right) = P\left(q^{m+T}\right) q^T, \;\; m \in \mathbb{Z}.$$

Then the transition matrix of the q-difference system

$$D_q Y\left(q^m\right) = P\left(q^m\right) Y\left(q^m\right), \;\; Y\left(q^{m_0}\right) = Y_0, \;\; m, m_0 \in \mathbb{Z} \text{ with } m \geq m_0, \tag{3.6}$$

is unique up to period q^T. That is

$$\Phi_P\left(q^m, q^{m_0}\right) = \Phi_P\left(q^{m+T}, q^{m_0+T}\right) \tag{3.7}$$

for all $m, m_0 \in \mathbb{Z}$.

Proof. By [39], the unique solution to (3.6) is $Y\left(q^m\right) = \Phi_P\left(q^m, q^{m_0}\right) Y_0$. Observe that

$$D_q Y\left(q^m\right) = D_q \Phi_P\left(q^m, q^{m_0}\right) Y_0 = P\left(q^m\right) \Phi_P\left(q^m, q^{m_0}\right) Y_0$$

and

$$Y\left(q^{m_0}\right) = \Phi_P\left(q^{m_0}, q^{m_0}\right) Y_0 = Y_0.$$

To verify (3.7), it should be shown that $\Phi_P\left(q^{m+T}, q^{m_0+T}\right) Y_0$ also solves (3.6). Taking the q-derivative of $\Phi_P\left(q^{m+T}, q^{m_0+T}\right) Y_0$ gives

$$\begin{aligned} D_q\left[\Phi_P\left(q^{m+T}, q^{m_0+T}\right) Y_0\right] &= P\left(q^{m+T}\right) q^T \Phi_P\left(q^{m+T}, q^{m_0+T}\right) Y_0 \\ &= P\left(q^m\right) \Phi_P\left(q^{m+T}, q^{m_0+T}\right) Y_0. \end{aligned}$$

On the other hand,

$$\Phi_P\left(q^{m+T}, q^{m_0+T}\right)_{m=m_0} Y_0 = \Phi_P\left(q^{m_0+T}, q^{m_0+T}\right) Y_0 = Y_0,$$

which means that $\Phi_P\left(q^{m+T}, q^{m_0+T}\right) Y_0$ solves (3.6). From the uniqueness of the solution of (3.6), we get (3.7). $\square$

The next result can be proved in a similar way.

Corollary 3.4 *Let $P : q^{\mathbb{Z}} \to \mathbb{R}^{n\times n}$ be a regressive, multiplicatively periodic matrix valued function with period q^T. Then*

$$e_P\left(q^m, q^{m_0}\right) = e_P\left(q^{m+T}, q^{m_0+T}\right) \text{ for } m, m_0 \in \mathbb{Z}. \tag{3.8}$$

Theorem 3.5 (Floquet decomposition) *Let A be a matrix valued function that is multiplicatively periodic with period q^T. The transition matrix for A in (3.1)* *can be decomposed in the following form*

$$\Phi_A(q^m, q^\tau) = L(q^m)\, e_R(q^m, q^\tau)\, L^{-1}(q^\tau)\,, \text{ for all } m, \tau \in \mathbb{Z}, \tag{3.9}$$

where $R : q^{\mathbb{Z}} \rightarrow \mathbb{C}^{n\times n}$ *as in (3.3) is multiplicatively* q^T*-periodic and* $L : q^{\mathbb{Z}} \rightarrow \mathbb{R}^{n\times n}$ *is defined by*

$$L(q^m) := \Phi_A(q^m, q^{m_0})\, e_R^{-1}(q^m, q^{m_0}) \text{ for } m, m_0 \in \mathbb{Z} \tag{3.10}$$

periodic with period q^T *and invertible.*

Proof. Let the matrix function R defined as in Theorem 3.1 with a constant matrix $M := \Phi_A\left(q^{m_0+T}, q^{m_0}\right)$. Then consequently, we have $e_R\left(q^{m_0+T}, q^{m_0}\right) = \Phi_A\left(q^{m_0+T}, q^{m_0}\right)$ and we define the Lyapunov transformation L as in (3.10). Obviously, L is defined on $q^{\mathbb{Z}}$ and invertible. The equality

$$\Phi_A(q^m, q^{m_0}) = L(q^m)\, e_R(q^m, q^{m_0}) \tag{3.11}$$

along with (3.10) implies

$$\begin{aligned}\Phi_A(q^{m_0}, q^m) &= e_R^{-1}(q^m, q^{m_0})\, L^{-1}(q^m)\\ &= e_R(q^{m_0}, q^m)\, L^{-1}(q^m)\,.\end{aligned} \tag{3.12}$$

Equation (3.9) can be obtained by combining (3.11) and (3.12). The periodicity of L can be shown by using (3.7) and (3.8) which yields

$$\begin{aligned}L\left(q^{m+T}\right) &= \Phi_A\left(q^{m+T}, q^{m_0}\right) e_R^{-1}\left(q^{m+T}, q^{m_0}\right)\\ &= \Phi_A\left(q^{m+T}, q^{m_0+T}\right) \Phi_A\left(q^{m_0+T}, q^{m_0}\right) e_R\left(q^{m_0}, q^{m+T}\right)\\ &= \Phi_A\left(q^{m+T}, q^{m_0+T}\right) \Phi_A\left(q^{m_0+T}, q^{m_0}\right) e_R\left(q^{m_0}, q^{m_0+T}\right) e_R\left(q^{m_0+T}, q^{m+T}\right)\\ &= \Phi_A\left(q^{m+T}, q^{m_0+T}\right) e_R\left(q^{m_0+T}, q^{m+T}\right)\\ &= \Phi_A\left(q^{m+T}, q^{m_0+T}\right) e_R^{-1}\left(q^{m+T}, q^{m_0+T}\right)\\ &= \Phi_A(q^m, q^{m_0})\, e_R^{-1}(q^m, q^{m_0})\\ &= L(q^m) \text{ for } m, m_0 \in \mathbb{Z}.\end{aligned}$$

□

Hereafter, (3.9) is refered as the *q-Floquet decomposition* for Φ_A. The following result can be proven similar to Theorem 3.7 in [38].

Theorem 3.6 *Let the transition matrix Φ_A of the system (3.1) is decomposed as $\Phi_A(q^m, q^{m_0}) = L(q^m)\, e_R(q^m, q^{m_0})$. Then, $x(q^m) = \Phi_A(q^m, q^{m_0})\, x_0$ is a solution of the q-Floquet system (3.1) if and only if the linear q-difference system*

$$D_q z(q^m) = R(q^m)\, z(q^m)\,,\ \ z(q^{m_0}) = x_0,\ \ m, m_0 \in \mathbb{Z} \text{ and } m \geq m_0$$

has a solution of the form $z(q^m) = L^{-1}(q^m)\, x(q^m)$.

Theorem 3.7 *The q-Floquet system (3.1) has a q^T-periodic solution with a nonzero inital state $x(q^{m_0}) = x_0$ if and only if at least one of the eigenvalues of*

$$e_R\left(q^{m_0+T}, q^{m_0}\right) = \Phi_A\left(q^{m_0+T}, q^{m_0}\right),\ \ m_0 \in \mathbb{Z}$$

is 1*.*

Proof. Let $x(q^m)$ be a q^T-periodic solution of (3.1) corresponding to nonzero initial state $x(q^{m_0}) = x_0$. Then, the decomposition of the solution x can be written as

$$x(q^m) = \Phi_A(q^m, q^{m_0})\, x_0 = L(q^m)\, e_R(q^m, q^{m_0})\, L^{-1}(q^{m_0})\, x_0 \text{ for } m, m_0 \in \mathbb{Z},$$

by employing Theorem 3.5. Furthermore, we have

$$x\left(q^{m+T}\right) = L\left(q^{m+T}\right) e_R\left(q^{m+T}, q^{m_0}\right) L^{-1}(q^{m_0})\, x_0,$$

and using q^T-periodicity of x and L, one can obtain

$$e_R(q^m, q^{m_0})\, L^{-1}(q^{m_0})\, x_0 = e_R\left(q^{m+T}, q^{m_0}\right) L^{-1}(q^{m_0})\, x_0,$$

which gives the equalities

$$\begin{aligned} e_R\left(q^m, q^{m_0}\right) L^{-1}\left(q^{m_0}\right) x_0 &= e_R\left(q^{m+T}, q^{m_0+T}\right) e_R\left(q^{m_0+T}, q^{m_0}\right) L^{-1}\left(q^{m_0}\right) x_0 \\ &= e_R\left(q^m, q^{m_0}\right) e_R\left(q^{m_0+T}, q^{m_0}\right) L^{-1}\left(q^{m_0}\right) x_0, \end{aligned}$$

where we use Corollary 3.4 in the last equation. Hence

$$L^{-1}\left(q^{m_0}\right) x_0 = e_R\left(q^{m_0+T}, q^{m_0}\right) L^{-1}\left(q^{m_0}\right) x_0,$$

for $m, m_0 \in \mathbb{Z}$ and, we deduce that $L^{-1}\left(q^{m_0}\right) x_0$ is an eigenvector of the matrix $e_R\left(q^{m_0+T}, q^{m_0}\right)$ corresponding to the eigenvalue 1.

Conversely, let us assume that 1 is an eigenvalue of $e_R\left(q^{m_0+T}, q^{m_0}\right)$ with corresponding eigenvector $z_{0.}$ This means z_0 is real valued and nonzero. Let us define the function $z(q^m) = e_R\left(q^m, q^{m_0}\right) z_0$. By using Corollary 3.4, one may show the q^T-periodicity of z as follows:

$$\begin{aligned} z\left(q^{m+T}\right) &= e_R\left(q^{m+T}, q^{m_0}\right) z_0 \\ &= e_R\left(q^{m+T}, q^{m_0+T}\right) e_R\left(q^{m_0+T}, q^{m_0}\right) z_0 \\ &= e_R\left(q^{m+T}, q^{m_0+T}\right) z_0 \\ &= e_R\left(q^m, q^{m_0}\right) z_0 \\ &= z\left(q^m\right). \end{aligned}$$

Fixing $x_0 := L\left(q^{m_0}\right) z_0$ and employing Theorem 3.5, the nontrivial q^T-periodic solution x of q-Floquet system (3.1) is obtained as follows

$$\begin{aligned} x\left(q^m\right) = \Phi_A\left(q^m, q^{m_0}\right) x_0 &= L\left(q^m\right) e_R\left(q^m, q^{m_0}\right) L^{-1}\left(q^{m_0}\right) x_0 \\ &= L\left(q^m\right) e_R\left(q^m, q^{m_0}\right) z_0 = L\left(q^m\right) z\left(q^m\right). \end{aligned}$$

This completes the proof. □

Example 3.1. Consider the following linear homogeneous 2×2 diagonal q-difference initial value problem

$$D_q x = A\left(q^m\right) x\left(q^m\right), \; x(1) = x_0, \; m \in \mathbb{Z}, \tag{3.13}$$

where

$$x(q^m) = \begin{bmatrix} x_1(q^m) \\ x_2(q^m) \end{bmatrix},$$

$$x(1) = \begin{bmatrix} x_{1_0} \\ x_{2_0} \end{bmatrix},$$

and

$$A(q^m) = \begin{bmatrix} c_1 q^{-m} & 0 \\ 0 & c_2 q^{-m} \end{bmatrix}$$

with positive reals c_1 and c_2. Then, the matrix function is multiplicatively q-periodic and the transition matrix for the system (3.13) can be given as

$$\Phi_A(q^m, 1) = \begin{bmatrix} [1 + (q-1)c_1]^m & 0 \\ 0 & [1 + (q-1)c_2]^m \end{bmatrix}$$

where we also use the explicit form of q- exponential function (see Definition 3). Now as we did in Theorem 3.1, we have

$$\Phi_A(q, 1) = e_R(q, 1) = \begin{bmatrix} 1 + (q-1)c_1 & 0 \\ 0 & 1 + (q-1)c_2 \end{bmatrix} = M.$$

Then the matrix function $R(q^m)$ in Floquet decomposition is given by

$$\begin{aligned} R(q^m) &= \frac{M - I}{(q-1)q^m} \\ &= \frac{1}{(q-1)q^m} \begin{bmatrix} (q-1)c_1 & 0 \\ 0 & (q-1)c_2 \end{bmatrix} \\ &= \begin{bmatrix} q^{-m}c_1 & 0 \\ 0 & q^{-m}c_2 \end{bmatrix}. \end{aligned}$$

Furthermore, by (3.4) we have $e_R(q^m, 1) = M^m$ and the matrix function L is obtained as

$$\begin{aligned} L\left(q^m\right) &= \Phi_A\left(q^m,1\right) e_R^{-1}\left(q^m,1\right) \\ &= \begin{bmatrix} [1+(q-1)c_1]^m & 0 \\ 0 & [1+(q-1)c_2]^m \end{bmatrix} \begin{bmatrix} [1+(q-1)c_1]^{-m} & 0 \\ 0 & [1+(q-1)c_2]^{-m} \end{bmatrix} \\ &= I. \end{aligned}$$

3.2 Nonhomogeneous case

This part of the thesis concerns with the existence of a periodic solution of the following nonhomogeneous regressive initial value problem

$$D_q x\left(q^m\right) = A\left(q^m\right) x\left(q^m\right) + f\left(q^m\right),\ x\left(q^{m_0}\right) = x_0 \text{ for } m, m_0 \in \mathbb{Z}, \tag{3.14}$$

where $A : q^{\mathbb{Z}} \to \mathbb{R}^{n\times n}$, $f : q^{\mathbb{Z}} \to \mathbb{R}^n$ and f is regressive. Hereafter, A and f are supposed to be multiplicatively periodic with the period q^T.

Lemma 3.8 *A solution $x\left(q^m\right)$ of (3.14) is q^T-periodic if and only if $x\left(q^{m_0+T}\right) = x\left(q^{m_0}\right)$.*

Proof. Assume that $x\left(q^m\right)$ q^T-periodic and define $z\left(q^m\right)$ as

$$z\left(q^m\right) = x\left(q^{m+T}\right) - x\left(q^m\right),\ m \in \mathbb{Z},$$

and obviously $z\left(q^{m_0}\right) = x\left(q^{m_0+T}\right) - x\left(q^{m_0}\right) = 0$. Additionally, taking the q-derivative of both sides of (3.14) yields

$$\begin{aligned} D_q z\left(q^m\right) &= D_q\left[x\left(q^{m+T}\right) - x\left(q^m\right)\right] \\ &= D_q\left(x\left(q^{m+T}\right)\right) - D_q\left(x\left(q^m\right)\right) \\ &= A\left(q^{m+T}\right) x\left(q^{m+T}\right) q^T + f\left(q^{m+T}\right) q^T - A\left(q^m\right) x\left(q^m\right) - f\left(q^m\right). \end{aligned}$$

Since A and f are multiplicatively periodic with period q^T, we have

$$\begin{aligned} D_q z\left(q^m\right) &= A\left(q^m\right) x\left(q^{m+T}\right) + f\left(q^m\right) - A\left(q^m\right) x\left(q^m\right) - f\left(q^m\right) \\ &= A\left(q^m\right)\left[x\left(q^{m+T}\right) - x\left(q^m\right)\right] \\ &= A\left(q^m\right) z\left(q^m\right). \end{aligned}$$

By the uniqueness of solutions, one can conclude that $z\left(q^m\right) \equiv 0$ and $x\left(q^{m+T}\right) = x\left(q^m\right)$ for all $m \in \mathbb{Z}$. □

Theorem 3.9 *The solution of (3.14) is q^T-periodic for any inital point $q^{m_0}, m_0 \in \mathbb{Z}$ and corresponding initial state $x\left(q^{m_0}\right) = x_0$ if and only if the multiplicatively q^T-periodic homogeneous initial value problem*

$$D_q z\left(q^m\right) = A\left(q^m\right) z\left(q^m\right), \ \ z\left(q^{m_0}\right) = z_0, \tag{3.15}$$

has not a periodic solution for any nonzero initial state $z\left(q^{m_0}\right) = z_0$.

Proof. In [13], the solution of

$$x^{\Delta}(t) = A(t)x(t) + f(t) \tag{3.16}$$

is given by

$$x\left(t\right) = X\left(t\right) X^{-1}\left(\tau\right) x_0 + \int_{\tau}^{t} X\left(t\right) X^{-1}\left(\sigma\left(s\right)\right) f\left(s\right) \Delta s$$

on an arbitrary time scale, where $X\left(t\right)$ is a fundamental matrix solution of the homogenous part of the system (3.16) with the initial condition $x\left(\tau\right) = x_0$. As it is shown in [13], one can express the solution $x\left(t\right)$ of (3.14) as follows:

$$x\left(q^m\right) = \Phi_A\left(q^m, q^{m_0}\right) x_0 + \int_{q^{m_0}}^{q^m} \Phi_A\left(q^m, qs\right) f\left(s\right) d_q s, \tag{3.17}$$

where s is of the form q^{τ} for $\tau \in \mathbb{Z}$.

By the previous lemma it is known that $x\left(q^m\right)$ is q^T-periodic if and only if

$x\left(q^{m_0+T}\right)=x_0$ or equivalently

$$\left[I-\Phi_A\left(q^{m_0+T},q^{m_0}\right)\right]x_0=\int_{q^{m_0}}^{q^{m_0+T}}\Phi_A\left(q^{m_0+T},qs\right)f\left(s\right)d_qs, \tag{3.18}$$

where s is in the same form as in (3.17). By guidance of Theorem 3.7, we have to show that (3.14) has a solution if and only if $e_R\left(q^{m_0+T},q^{m_0}\right)$ has no eigenvalues equal to 1.

Let $e_R\left(q^{\varrho+T},q^{\varrho}\right)=\Phi_A\left(q^{\varrho+T},q^{\varrho}\right)$, for some $\varrho\in\mathbb{Z}$, has no eigenvalues equal to 1. That is,

$$\det\left[I-\Phi_A\left(q^{\varrho+T},q^{\varrho}\right)\right]\neq 0.$$

Invertibility and periodicity of Φ_A imply that

$$\begin{aligned}0&\neq\det\left[\Phi_A\left(q^{m_0+T},q^{\varrho+T}\right)\left(I-\Phi_A\left(q^{\varrho+T},q^{\varrho}\right)\right)\Phi_A\left(q^{\varrho},q^{m_0}\right)\right]\\&=\det\left[\Phi_A\left(q^{m_0+T},q^{\varrho+T}\right)\Phi_A\left(q^{\varrho},q^{m_0}\right)-\Phi_A\left(q^{m_0+T},q^{m_0}\right)\right].\end{aligned} \tag{3.19}$$

By periodicity of Φ_A, the invertibility of $\left[I-\Phi_A\left(q^{m_0+T},q^{m_0}\right)\right]$ is equivalent to (3.19) for any $q^{m_0},m_0\in\mathbb{Z}$. Thus, (3.18) has a solution in the following form

$$x_0=\left[I-\Phi_A\left(q^{m_0+T},q^{m_0}\right)\right]^{-1}\int_{q^{m_0}}^{q^{m_0+T}}\Phi_A\left(q^{m_0+T},qs\right)f\left(s\right)d_qs$$

for any initial point q^{m_0} and for any multiplicatively periodic function f with period q^T. Let $\xi\left(q^m\right):=q^{1-m+m_0}$. From the definition of ξ, we have $\xi\left(q^{m+T}\right)=\frac{1}{q^T}\xi\left(q^m\right)$. This shows that ξ is multiplicatively periodic with period q^T. For an arbitrary initial point q^{m_0} and corresponding $f_0:=f\left(q^{m_0}\right)$, one can define a regressive and multiplicatively periodic function f as

$$f\left(q^m\right):=\Phi_A\left(q^{m+1},q^{m_0+T}\right)\xi\left(q^m\right)f_0,\quad m\in[m_0,m_0+T)_{\mathbb{Z}}. \tag{3.20}$$

Then,

$$\int_{q^{m_0}}^{q^{m_0+T}}\Phi_A\left(q^{m_0+T},qs\right)f\left(s\right)d_qs=f_0\int_{q^{m_0}}^{q^{m_0+T}}\xi\left(s\right)d_qs. \tag{3.21}$$

Thus, (3.18) can be rewritten as

$$\left[I - \Phi_A\left(q^{m_0+T}, q^{m_0}\right)\right] x_0 = f_0 \int_{q^{m_0}}^{q^{m_0+T}} \xi(s)\, d_q s. \tag{3.22}$$

For any f given in (3.20), and hence for any corresponding f_0, (3.22) has a solution for x_0 by assumption. Therefore,

$$\det\left[I - \Phi_A\left(q^{m_0+T}, q^{m_0}\right)\right] \neq 0.$$

Consequently, $e_R\left(q^{m_0+T}, q^{m_0}\right) = \Phi_A\left(q^{m_0+T}, q^{m_0}\right)$ has no eigenvalue 1. Then, one can conclude by Theorem 3.7 that (3.15) has no periodic solution. □

3.3 q-Floquet multipliers and q-Floquet exponents

In this section, Floquet multipliers and exponents of the q-Floquet system (3.1) and their properties are investigated. Let $\Phi(q^m)$ be the fundamental matrix solution at q^τ (i.e. $\Phi(q^\tau) = I$) for the system (3.1). Then, any fundamental matrix solution $\Psi(q^m)$ can be written as follows

$$\Psi(q^m) = \Phi(q^m)\Psi(q^\tau) \text{ or } \Psi(q^m) = \Phi_A(q^m, q^{m_0})\Psi(q^{m_0}), \tag{3.23}$$

where Φ_A is the transition matrix of the system (3.1).

Definition 27. For a nonzero initial state $x_0 \in \mathbb{R}^n$, the monodromy operator $M : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$M(x_0) := \Phi_A\left(q^{m_0+T}, q^{m_0}\right) x_0 = \Psi\left(q^{m_0+T}\right)\Psi^{-1}(q^{m_0})\, x_0, \tag{3.24}$$

where Φ_A is the transition matrix and Ψ is any fundamental matrix solution of (3.1). The eigenvalues of the monodromy operator are called Floquet (characteristic) multipliers of the system (3.1).

The following result can be presented in a similar way as in Theorem 7.1 in [37].

Lemma 3.10 *The monodromy operator is an invertible matrix and as a consequence, every Floquet multiplier is nonzero.*

Theorem 3.11 *The monodromy operator M corresponding to different fundamental matrices of the system (3.1) is unique.*

Proof. Suppose that M_1 and M_2 are monodromy operators corresponding to fundamental matrices $\Psi_1(q^m)$ and $\Psi_2(q^m)$ for $m \in \mathbb{Z}$, respectively. By using Definition 27, one can obtain the monodromy operator $M_2(x_0)$ corresponding to $\Psi_2(q^m)$ as follows

$$M_2(x_0) = \Psi_2\left(q^{m_0+T}\right)\Psi_2^{-1}(q^{m_0})\,x_0.$$

Using (3.23), we get

$$\begin{aligned} M_2(x_0) &= \Psi_2\left(q^{m_0+T}\right)\Psi_2^{-1}(q^{m_0})\,x_0 \\ &= \Psi_1\left(q^{m_0+T}\right)\Psi_2(q^{\tau})\Psi_2^{-1}(q^{\tau})\Psi_1^{-1}(q^{m_0})\,x_0 \\ &= \Psi_1\left(q^{m_0+T}\right)\Psi_1^{-1}(q^{m_0})\,x_0 \\ &= M_1(x_0). \end{aligned}$$

□

By using Theorem 3.5, (3.23), and (3.24), we obtain

$$\Phi_A(q^m, q^{m_0}) = \Psi_1(q^m)\Psi_1^{-1}(q^{m_0}) = L(q^m)\,e_R(q^m, q^{m_0})\,L^{-1}(q^{m_0}) \tag{3.25}$$

and

$$M(x_0) = \Phi_A\left(q^{m_0+T}, q^{m_0}\right)x_0 = \Psi_1\left(q^{m_0+T}\right)\Psi_1^{-1}(q^{m_0})\,x_0. \tag{3.26}$$

Combining (3.25) and (3.26) yields

$$\Phi_A\left(q^{m_0+T}, q^{m_0}\right) = \Psi_1\left(q^{m_0+T}\right)\Psi_1^{-1}(q^{m_0}) = L\left(q^{m_0+T}\right)e_R\left(q^{m_0+T}, q^{m_0}\right)L^{-1}\left(q^{m_0+T}\right).$$

The periodicity of L gives

$$\Phi_A\left(q^{m_0+T},q^{m_0}\right)=L\left(q^{m_0}\right)e_R\left(q^{m_0+T},q^{m_0}\right)L^{-1}\left(q^{m_0}\right),\ m_0\in\mathbb{Z}. \tag{3.27}$$

Hence, it is observed that Floquet multipliers of the system (3.1) are the eigenvalues of the matrix $e_R\left(q^{m_0+T},q^{m_0}\right)$.

Definition 28 (Floquet exponent)**.** The Floquet exponent of the system (3.1) is defined as to be the function satisfying the equation

$$e_\gamma\left(q^{m_0+T},q^{m_0}\right)=\lambda,$$

where λ is the Floquet multiplier of (3.1).

The next result is the q-analogue of the spectral mapping theorem and it can be proven by following the same way of the proof of [38, Theorem 5.3].

Theorem 3.12 *Let $R\left(q^m\right)$ be a matrix function as in Theorem 3.1, with eigenvalues $\gamma_1\left(q^m\right),\ldots,\gamma_n\left(q^m\right)$ repeated according to multiplicities. Then $\gamma_1^p\left(q^m\right),\ldots,\gamma_n^p\left(q^m\right)$ are the eigenvalues of $R^p\left(t\right)$ and the eigenvalues of e_R are $e_{\gamma_1},\ldots,e_{\gamma_n}$. As a consequence, Floquet exponents are the eigenvalues of the matrix R.*

Next theorem shows that Floquet exponents of the system (3.1) are not unique.

Theorem 3.13 *Let γ be a Floquet exponent of the system and λ be the corresponding Floquet multiplier such that $e_\gamma\left(q^{m_0+T},q^{m_0}\right)=\lambda$. Then $\gamma\left(q^m\right)\oplus\mathring{\imath}\frac{2\pi\tau}{(q-1)q^{m_0}}$ is also a Floquet exponent for (3.1) for all $\tau\in\mathbb{Z}$.*

Proof. For all $\tau\in\mathbb{Z}$ and any initial point q^{m_0} for $m_0\in\mathbb{Z}$ we have

$$e_{\gamma\oplus\mathring{\imath}\frac{2\pi\tau}{(q-1)q^{m_0}}}\left(q^{m_0+T},q^{m_0}\right)=e_\gamma\left(q^{m_0+T},q^{m_0}\right)e_{\mathring{\imath}\frac{2\pi\tau}{(q-1)q^{m_0}}}\left(q^{m_0+T},q^{m_0}\right),$$

where

$$\mathring{\imath}\frac{2\pi\tau}{\left(q-1\right)q^{m_0}}=\frac{e^{\frac{i2\pi\tau q^m}{q^{m_0}}}-1}{\left(q-1\right)q^m}.$$

Then by using explicit form of q-exponential function, we obtain

$$\begin{aligned} e_{\gamma\oplus\overset{\circ}{i}\frac{2\pi\tau}{(q-1)q^{m_0}}}\left(q^{m_0+T},q^{m_0}\right) &= e_\gamma\left(q^{m_0+T},q^{m_0}\right)\prod_{\varrho\in[m_0,m_0+T)_{\mathbb{Z}}}\left[1+(q-1)\,q^{\varrho}\left[\frac{e^{\frac{i2\pi\tau q^{\varrho}}{q^{m_0}}}-1}{(q-1)\,q^{\varrho}}\right]\right] \\ &= e_\gamma\left(q^{m_0+T},q^{m_0}\right)\prod_{\varrho\in[m_0,m_0+T)_{\mathbb{Z}}} e^{\frac{i2\pi\tau q^{\varrho}}{q^{m_0}}} \\ &= e_\gamma\left(q^{m_0+T},q^{m_0}\right), \end{aligned}$$

which gives the desired result. □

Lemma 3.14 *Let $\tau\in\mathbb{Z}$. Then the exponential functions $e_{\overset{\circ}{i}\frac{2\pi\tau}{q^{m_0}(q-1)}}$ and $e_{\ominus\overset{\circ}{i}\frac{2\pi\tau}{q^{m_0}(q-1)}}$ are q periodic.*

Proof. Consider the explicit form of $\overset{\circ}{i}\frac{2\pi\tau}{q^{m_0}(q-1)}$ on q-calculus

$$\overset{\circ}{i}\frac{2\pi\tau}{q^{m_0}(q-1)}=\frac{\exp(i2\pi\tau q^{m-m_0})-1}{(q-1)\,q^m}$$

which enables to write

$$\begin{aligned} e_{\overset{\circ}{i}\frac{2\pi\tau}{q^{m_0}(q-1)}}\left(q^{m+1},q^{m_0}\right) &= \prod_{\varrho\in[m_0,m+1)_{\mathbb{Z}}} 1+(q-1)\,q^{\varrho}\frac{\exp(i2\pi\tau q^{\varrho-m_0})-1}{(q-1)\,q^{\varrho}} \\ &= \exp(i2\pi\tau q^{m-m_0})\prod_{\varrho\in[m_0,m)_{\mathbb{Z}}} 1+(q-1)\,q^{\varrho}\frac{\exp(i2\pi\tau q^{\varrho-m_0})-1}{(q-1)\,q^{\varrho}} \\ &= \prod_{\varrho\in[m_0,m)_{\mathbb{Z}}} 1+(q-1)\,q^{\varrho}\frac{\exp(i2\pi\tau q^{\varrho-m_0})-1}{(q-1)\,q^{\varrho}} \\ &= e_{\overset{\circ}{i}\frac{2\pi\tau}{q^{m_0}(q-1)}}\left(q^{m},q^{m_0}\right). \end{aligned}$$

The periodicity of $e_{\ominus\overset{\circ}{i}\frac{2\pi\tau}{q^{m_0}(q-1)}}$ can be shown by repeating the same procedure, hence we omit it. □

Theorem 3.15 *If the q-Floquet system (3.1) has a Floquet exponent $\gamma(q^m)$, then the transition matrix $\Phi_A(q^m,q^{m_0})$ has the following decomposition*

$$\Phi_A(q^m,q^{m_0})=L(q^m)\,e_R(q^m,q^{m_0}),$$

where $\gamma(q^m)$ is an eigenvalue of $R(q^m)$.

Proof. Consider the Floquet decomposition $\Phi_A(q^m, q^{m_0}) = \widetilde{L}(q^m)\, e_{\widetilde{R}}(q^m, q^{m_0})$ and let γ be the Floquet exponent according to Floquet multiplier λ. The matrix function $\tilde{R}(q^m)$ has an eigenvalue $\tilde{\gamma}(q^m)$ such that $e_{\tilde{\gamma}}\left(q^{m_0+T}, q^{m_0}\right) = \lambda$, where $\tilde{\gamma}(q^m)$ can be generated from Theorem 3.13 for $\tau \in \mathbb{Z}$ as

$$\tilde{\gamma}(q^m) := \gamma(q^m) \oplus \mathring{i}\frac{2\pi\tau}{(q-1)\, q^{m_0}}.$$

Then, we set

$$R(q^m) := \widetilde{R}(q^m) \ominus \mathring{i}\frac{2\pi\tau}{(q-1)\, q^{m_0}} I,$$

and

$$L(q^m) := \tilde{L}(q^m)\, e_{\mathring{i}\frac{2\pi\tau}{(q-1)q^{m_0}} I}(q^m, q^{m_0}),$$

gives

$$\widetilde{R}(q^m) := R(q^m) \oplus \mathring{i}\frac{2\pi\tau}{(q-1)\, q^{m_0}} I.$$

Hence,

$$\begin{aligned} L(q^m)\, e_R(q^m, q^{m_0}) &= \tilde{L}(q^m)\, e_{\mathring{i}\frac{2\pi\tau}{(q-1)q^{m_0}} I}(q^m, q^{m_0})\, e_R(q^m, q^{m_0}) \\ &= \tilde{L}(q^m)\, e_{\mathring{i}\frac{2\pi\tau}{(q-1)q^{m_0}} I \oplus R}(q^m, q^{m_0}) = \tilde{L}(q^m)\, e_{\tilde{R}}(q^m, q^{m_0}). \end{aligned}$$

This shows that $\Phi_A(q^m, q^{m_0}) = L(q^m)\, e_R(q^m, q^{m_0})$ is another Floquet decomposition of (3.1) where $\gamma(q^m)$ is an eigenvalue of $R(q^m)$. $\square$

Theorem 3.16 *Let λ be a Floquet multiplier of the q-Floquet system (3.1) and $\gamma(q^m)$ be the corresponding Floquet exponent. Then, (3.1) has a nontrivial solution of the form*

$$x(q^m) = e_\gamma(q^m, q^{m_0})\, \kappa(q^m) \tag{3.28}$$

satisfying

$$x\left(q^{m+T}\right) = \lambda x(q^m),$$

where κ is a q^T-periodic function.

Proof. Let $\Phi_A(q^m, q^{m_0})$ be the transition matrix of (3.1) and $\Phi_A(q^m, q^{m_0}) = L(q^m)\, e_R(q^m, q^{m_0})$ is Floquet decomposition such that $\gamma(q^m)$ is an eigenvalue of $R(q^m)$ for $m, m_0 \in \mathbb{Z}$. There exists a nonzero vector $u \neq 0$ such that $R(q^m)\, u =$

$\gamma\left(q^m\right)u$, and therefore, $e_R\left(q^m, q^{m_0}\right)u = e_\gamma\left(q^m, q^{m_0}\right)u$ by spectral mapping theorem stated in Theorem 3.12. Then, the solution $x\left(q^m\right) := \Phi_A\left(q^m, q^{m_0}\right)u$ can be represented as follows

$$x\left(q^m\right) = L\left(q^m\right)e_R\left(q^m, q^{m_0}\right)u = e_\gamma\left(q^m, q^{m_0}\right)L\left(q^m\right)u.$$

The above equation implies (3.28) when $\kappa\left(q^m\right) = L\left(q^m\right)u$.

In order to prove the second part of the theorem consider the following equation:

$$\begin{aligned} x\left(q^{m+T}\right) &= e_\gamma\left(q^{m+T}, q^{m_0}\right)\kappa\left(q^{m+T}\right) \\ &= e_\gamma\left(q^{m+T}, q^{m_0+T}\right)e_\gamma\left(q^{m_0+T}, q^{m_0}\right)\kappa\left(q^m\right) \\ &= e_\gamma\left(q^{m_0+T}, q^{m_0}\right)e_\gamma\left(q^m, q^{m_0}\right)L\left(q^m\right)u \\ &= e_\gamma\left(q^{m_0+T}, q^{m_0}\right)x\left(q^m\right) \\ &= \lambda x\left(q^m\right), \end{aligned}$$

which completes the proof. □

This theorem provides a procedure for the construction of a solution to the system (3.1) when a Floquet multiplier is given, where the next one shows the linear independence of two solutions corresponding to two distinct characteristic multipliers.

Theorem 3.17 *Let λ_1 and λ_2 be the characteristic multipliers of the system (3.1) and γ_1, γ_2 are Floquet exponents such that*

$$e_{\gamma_i}(q^{m_0+T}, q^{m_0}) = \lambda_i, \quad i = 1, 2.$$

If $\lambda_1 \neq \lambda_2$, then there exist q^T-periodic functions χ_1 and χ_2 such that

$$x_i(q^m) = e_{\gamma_i}(q^m, q^{m_0})\chi_i(q^m), \;\; i = 1, 2$$

are linearly independent solutions of the system(3.1).

Proof. Let $\Phi_A\left(q^m, q^{m_0}\right) = L\left(q^m\right)e_R\left(q^m, q^{m_0}\right)$ and $\gamma_1\left(q^m\right)$ be an eigenvalue of $R(q^m)$ corresponding to eigenvector v_1. Since λ_2 is an eigenvalue of $\Phi_A\left(q^{m_0+T}, q^{m_0}\right)$, by Theorem 3.12 there is an eigenvalue $\gamma(q^m)$ of $R(q^m)$ satisfying

$$e_\gamma\left(q^{m_0+T}, q^{m_0}\right) = \lambda_2 = e_{\gamma_2}\left(q^{m_0+T}, q^{m_0}\right).$$

Hence, for some $\tau \in \mathbb{Z}$ we have $\gamma_2(q^m) = \gamma(q^m) \oplus \mathring{i}\frac{2\pi\tau}{(q-1)q^{m_0}}$. Furthermore, $\lambda_1 \neq \lambda_2$ implies that $\gamma(q^m) \neq \gamma_1(q^m)$. If v_2 is a nonzero eigenvector of $R(q^m)$ corresponding to the eigenvalue $\gamma(q^m)$, then the eigenvectors v_1 and v_2 are linearly independent. Similar to the related part in the proof of Theorem 3.16, we can state the solutions of the system (3.1) can be written as

$$x_1(q^m) = e_{\gamma_1}(q^m, q^{m_0}) L(q^m) v_1 \tag{3.29}$$

and

$$x_2(q^m) = e_\gamma(q^m, q^{m_0}) L(q^m) v_2.$$

Since $x_1(q^{m_0}) = v_1$ and $x_2(q^{m_0}) = v_2$, the solutions $x_1(q^m)$ and $x_2(q^m)$ are linearly independent. Moreover, the solution x_2 can be rewritten in the following form

$$\begin{aligned} x_2(q^m) &= e_{\gamma_2}(q^m, q^{m_0})\, e_{\gamma\ominus\gamma_2}(q^m, q^{m_0}) L(q^m)\nu_2 \\ &= e_{\gamma_2}(q^m, q^{m_0})\, e_{\ominus\mathring{i}\frac{2\pi\tau}{(q-1)q^{m_0}}}(q^m, q^{m_0})\, L(q^m)\nu_2. \end{aligned} \tag{3.30}$$

The conclusion of proof follows by substituting $\chi_1(q^m) = L(q^m) v_1$ and $\chi_2(q^m) = e_{\ominus\mathring{i}\frac{2\pi\tau}{(q-1)q^{m_0}}}(q^m, q^{m_0}) L(q^m)\nu_2$ in (3.29) and (3.30), respectively. □

3.4 Stability properties of q-Floquet systems

In this section, q-Floquet theory established in previous sections is employed to investigate the stability characteristics of the regressive multiplicatively periodic system

$$D_q x(q^m) = A(q^m) x(q^m), \ x(q^{m_0}) = x_0, \text{ for } m, m_0 \in \mathbb{Z}, \ m \geq m_0. \tag{3.31}$$

By Theorem 3.1, the matrix function R is given by

$$R(q^m) = \frac{\left[\Phi_A\left(q^{m_0+T}, q^{m_0}\right)\right]^{(q^{-T+1})} - I}{(q-1)\, q^m}, \tag{3.32}$$

where it is used in the Floquet decomposition theorem (see Theorem 3.5). Also, Theorem 3.6 concludes that the solution $z(q^m)$ of the uniformly regressive system

$$D_q z(q^m) = R(q^m)\, z(q^m),\ \ z(q^{m_0}) = x_0 \tag{3.33}$$

can be expressed in terms of the solution $x(q^m)$ of the system (3.31) as follows

$$z(q^m) = L^{-1}(q^m)x(q^m) \tag{3.34}$$

where $L(q^m)$ is the Lyapunov transformation given by (3.10).

In preparation for the main result, the following definitions and results regarding stability properties of homogeneous systems are presented. Notice that the literature provided below is just q-analogues of definitions and results established on time scales. We address [38] for stability and asymptotical stability properties of the solution. Moreover, we refer to [30] for exponential stability criteria for the solution of the homogeneous system (3.31).

Definition 29 (Stability)**.** The q-Floquet system (3.31) is uniformly stable if there exists a positive constant α such that for any $m_0 \in \mathbb{Z}$ the corresponding solution $x(q^m)$ satisfies

$$\|x(q^m)\| \leq \alpha \|x(q^{m_0})\|,\ \ m \geq m_0.$$

Theorem 3.18 *Let Φ_A be the transition matrix of the q-Floquet system (3.31). Then, (3.31) is uniformly stable if and only if the inequality*

$$\|\Phi_A(q^m, q^{m_0})\| \leq \alpha$$

holds for a positive constant $\alpha > 0$ and for all $m \geq m_0$.

Definition 30 (Asymptotical stability)**.** In addition to uniform stability condition, if for any given $c > 0$, there exists a $K > 0$ such that the inequality

$$\|x(q^m)\| \leq c \|x(q^{m_0})\|, \; q^m \geq q^{m_0} + K.$$

holds, then the system (3.31) is uniformly asymptotically stable.

Definition 31 (Exponential stability)**.** The q-Floquet system (3.31) is uniformly exponentially stable if there exist $\alpha, \beta > 0$ such that the inequality

$$\|x(q^m)\| \leq \|x(q^{m_0})\| \, \alpha e_{\ominus\beta}(q^m, q^{m_0}), \; m \geq m_0,$$

holds for any initial state and associated solution.

We provide necessary and sufficient conditions for exponential stability in the next result.

Theorem 3.19 *The system (3.31) is uniformly exponentially stable if and only if there exist α, $\beta > 0$ such that the inequality*

$$\|\Phi_A(q^m, q^{m_0})\| \leq \alpha e_{\ominus\beta}(q^m, q^{m_0})$$

is satisfied for the transition matrix Φ_A for all $m \geq m_0$.

Given a constant $n \times n$ matrix M, let S be a nonsingular matrix that transforms M into its Jordan canonical form

$$J := S^{-1} M S = diag \left[J_{m_1}(\lambda_1), \ldots, J_{m_k}(\lambda_k) \right]$$

where $k \leq n$, $\sum_{i=1}^{k} m_i = n$, λ_i are the eigenvalues of M, and $J_m(\lambda)$ is an $m \times m$

Jordan block given by

$$J_m(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}.$$

The uniform regressivity notion for functions defined on $q^{\mathbb{Z}}$, $q > 1$ is given in the next definition:

Definition 32. ([67], See also [38, Definition 7.1]) The scalar function $\gamma : q^{\mathbb{Z}} \to \mathbb{C}$ is uniformly regressive if there exists a constant $\theta > 0$ such that $0 < \theta^{-1} \leq |1 + (q-1) q^m \gamma(q^m)|$, for all $m \in \mathbb{Z}$.

Lemma 3.20 *The q-Floquet system (3.31) has uniformly regressive Floquet exponents i.e. all eigenvalues of the matrix function $R(q^m)$ in (3.33) are uniformly regressive.*

Proof. Similar to Corollary 3.2, let

$$\gamma_i(q^m) = \frac{\lambda_i^{(q^{-T+1})} - 1}{(q-1) q^m}, \; i = 1, 2, ..., k \tag{3.35}$$

be any of the $k \leq n$ distinct eigenvalues of $R(q^m)$. Then $|1 + (q-1) q^m \gamma_i(q^m)| = \left|\lambda_i^{(q^{-T+1})}\right|$ and setting $\theta^{-1} := \min\{\left|\lambda_1^{(q^{-T+1})}\right|, \ldots, \left|\lambda_k^{(q^{-T+1})}\right|\}$, gives

$$0 < \theta^{-1} \leq |1 + (q-1) q^m \gamma_i(q^m)|,$$

where $0 < \theta^{-1}$ is obtained from Remark 4.3. □

The following definition is the q-analogue of the Definition 7.3 given in [38].

Definition 33. A matrix function $H(q^m)$ is said to have a dynamic eigenvector $w(q^m)$ with the dynamic eigenvalue $\xi(q^m)$ if the pair $\{\xi(q^m), w(q^m)\}$ satisfies

$$D_q w(q^m) = H(q^m) w(q^m) - \xi(q^m) w\left(q^{m+1}\right), \; m \in \mathbb{Z}. \tag{3.36}$$

Then the pair $\{\xi(q^m), w(q^m)\}$ is called a dynamic eigenpair. Moreover, the vector function

$$\chi_i := e_{\xi_i}(q^m, q^{m_0})\, w_i(q^m) \tag{3.37}$$

is called the mode vector of the matrix function $H(q^m)$ associated with the pair $\{\xi(q^m), w(q^m)\}$.

The following result can be given in a similar way in Lemma 7.4 in [38].

Lemma 3.21 *Any regressive matrix function H has n dynamic eigenpairs with linearly independent eigenvectors. Moreover, if the dynamic eigenvectors form the columns of a matrix function $W(q^m)$, then W satisfies the matrix dynamic eigenvalue problem*

$$D_q W(q^m) = H(q^m)\, W(q^m) - W\left(q^{m+1}\right) \Xi(q^m), \tag{3.38}$$

where $\Xi(q^m) := diag\left[\xi_1(q^m), \ldots, \xi_n(q^m)\right].$

The following theorem basically shows that mode vectors can be used for stability analysis.

Theorem 3.22 *Solutions to the uniformly regressive (but not necessarily periodic) time varying linear q-difference system (3.31) are*

1. *stable if and only if there exists a $\gamma > 0$ such that every mode vector $\chi_i(q^m)$ of $A(q^m)$ satisfies $\|\chi_i(q^m)\| \leq \gamma < \infty$ for all $1 \leq i \leq n$;*

2. *asymptotically stable if and only if, in addition to (1), $\|\chi_i(q^m)\| \to 0$, for all $1 \leq i \leq n$,*

3. *exponentially stable if and only if there exists $\gamma, \lambda > 0$ such that $\|\chi_i(q^m)\| \leq \gamma e_{\ominus\lambda}(q^m, q^{m_0})$, for all $1 \leq i \leq n$ and $m \geq m_0$.*

Proof. Proof can be done by using exactly the same procedure in [38, Theorem 7.5]. □

Definition 34. For each $y \in \mathbb{N}_0$ the mappings $h_y : q^{\mathbb{Z}} \times q^{\mathbb{Z}} \to \mathbb{R}^+$, recursively defined by

$$h_0(q^m, q^{m_0}) :\equiv 1, \quad h_{y+1}(q^m, q^{m_0}) = \int_{q^{m_0}}^{q^m} \frac{q}{(q-1)\,\tau} h_y(\tau, q^{m_0}) d_q\tau \tag{3.39}$$

are called monomials.

Lemma 3.23 *Let $\gamma(q^m)$ be an eigenvalue of $R(q^m)$ and λ be the corresponding Floquet multiplier. If*

$$-Re_\mu \gamma(q^m) > 0 \tag{3.40}$$

holds, then

$$\lim_{m\to\infty} h_y(q^m, q^{m_0}) e_\gamma(q^m, q^{m_0}) = 0$$

for $y \in \mathbb{N}_{\not{0}}$ and initial point q^{m_0}.

Proof. Inspired by [47, Theorem 7.4], it suffices to show that $\lim_{m\to\infty} h_y(q^m, q^{m_0}) e_{Re_\mu \gamma(q^m)}(q^m, q^{m_0}) = 0$, where

$$Re_\mu \gamma(q^m) := \frac{|\gamma(q^m)\,(q-1)\,q^m + 1| - 1}{(q-1)\,q^m}.$$

We proceed by mathematical induction. Taking $y = 0$ gives $h_0(q^m, q^{m_0}) \equiv 1$ and by [67, Lemma 17], we have

$$\lim_{m\to\infty} e_{Re_\mu \gamma(q^m)}(q^m, q^{m_0}) = 0 \text{ for fixed } m_0.$$

Suppose that it is true for a fixed $y \in \mathbb{N}$ and consider the $(y+1)^{th}$ step

$$\begin{aligned}
&\lim_{m\to\infty} h_{y+1}(q^m, q^{m_0})\, e_{Re_\mu \gamma(q^m)}(q^m, q^{m_0}) \\
&= \lim_{m\to\infty} \left[\int_{q^{m_0}}^{q^m} \frac{q}{(q-1)\,\tau} h_y(\tau, q^{m_0})\, d_q\tau\right] \left(e_{\ominus Re_\mu \gamma(q^m)}(q^m, q^{m_0})\right)^{-1} \\
&= \lim_{m\to\infty} \left[\frac{q}{(q-1)\,q^m} h_y(q^m, q^{m_0})\right] \frac{e_{Re_\mu \gamma(q^m)}(q^{m+1}, q^{m_0})}{-Re_\mu \gamma(q^m)} \\
&= \lim_{m\to\infty} \left[\frac{\frac{q}{(q-1)q^m} h_y(q^m, q^{m_0})\, e_{Re_\mu \gamma(t)}(q^{m+1}, q^{m_0})}{-Re_\mu \gamma(q^m)}\right],
\end{aligned} \tag{3.41}$$

where q-analogue of l'Hospital theorem is used [24]. Using (3.35) we obtain

$$-\,Re_{\mu}\gamma_i(q^m) = \frac{1-\left|\lambda^{(q^{-T+1})}\right|}{(q-1)\,q^m}. \tag{3.42}$$

Substituting (3.42) into (3.41), one can obtain

$$\begin{aligned}\lim_{m\to\infty} h_{y+1}\left(q^m,q^{m_0}\right) e_{Re_{\mu}\gamma(q^m)}\left(q^m,q^{m_0}\right) &= \lim_{m\to\infty}\left[\frac{q\left|\lambda^{(q^{-T+1})}\right| h_y\left(q^m,q^{m_0}\right) e_{Re_{\mu}\gamma(q^m)}\left(q^m,q^{m_0}\right)}{1-\left|\lambda^{(q^{-T+1})}\right|}\right]\\ &= 0.\end{aligned}$$

□

Theorem 3.24 *Let $\{\gamma_i\left(q^m\right)\}_{i=1}^{n}$ be the set of conventional eigenvalues of the matrix $R(q^m)$ given in (3.32) and $\{w_i\left(q^m\right)\}_{i=1}^{n}$ be the set of corresponding linearly independent dynamic eigenvectors as defined by Lemma 3.21. Then, $\{\gamma_i\left(q^m\right), w_i\left(q^m\right)\}_{i=1}^{n}$ is a set of dynamic eigenpairs of $R(q^m)$ with the property that for each $1\le i\le n$ there are positive constants $D_i>0$ such that*

$$\left\|w_i\left(q^m\right)\right\| \le D_i \sum_{y=0}^{m_i-1} h_y\left(q^m,q^{m_0}\right), \tag{3.43}$$

holds where $h_y\left(q^m,q^{m_0}\right)$, $y=0,1,...,m_i-1$, are the monomials defined as in (3.39) and m_i is the dimension of the Jordan block which contains the i^{th} eigenvalue, for all $1\le i\le n$.

Proof. There exists an appropriate $n\times n$ constant, nonsingular matrix S which transforms $\Phi_A\left(q^{m_0+T},q^{m_0}\right)$ to its Jordan canonical form given by

$$\begin{aligned} J &:= S^{-1}\Phi_A\left(q^{m_0+T},q^{m_0}\right)S\\ &= \begin{bmatrix} J_{m_1}\left(\lambda_1\right) & & & \\ & J_{m_2}\left(\lambda_2\right) & & \\ & & \ddots & \\ & & & J_{m_d}\left(\lambda_d\right)\end{bmatrix}_{n\times n}, \end{aligned} \tag{3.44}$$

where $d \leq n$, $\sum_{i=1}^{d} m_i = n$, λ_i are the eigenvalues of $\Phi_A\left(q^{m_0+T}, q^{m_0}\right)$. By utilizing above determined matrix S, one can define the following:

$$\begin{aligned} K\left(q^m\right) &:= S^{-1} R\left(q^m\right) S \\ &= S^{-1}\left(\frac{[\Phi_A\left(q^{m_0+T}, q^{m_0}\right)]^{\left(q^{-T+1}\right)} - I}{\left(q-1\right) q^m}\right) S \\ &= \frac{S^{-1}[\Phi_A\left(q^{m_0+T}, q^{m_0}\right)]^{\left(q^{-T+1}\right)} S - I}{\left(q-1\right) q^m}. \end{aligned}$$

This along with [38, Theorem A.6] yields $K\left(q^m\right) = \frac{J^{\left(q^{-T+1}\right)} - I}{(q-1)q^m}$.

Note that, $K\left(q^m\right)$ has the block diagonal form

$$K\left(t\right) = diag\left[K_1\left(q^m\right), \ldots, K_d\left(q^m\right)\right]$$

in which each $K_i\left(q^m\right)$ given by

$$K_i\left(q^m\right) := \begin{bmatrix} \frac{\lambda_i^{\left(q^{-T+1}\right)}-1}{(q-1)q^m} & \frac{q^{-T+1}\lambda_i^{\left(q^{-T+1}-1\right)}}{(q-1)q^m} & \cdots & \frac{\left(\prod_{\tau=0}^{n-2}[q^{-T+1}-\tau]\right)\lambda_i^{\left(q^{-T+1}-n+1\right)}}{(q-1)q^m(n-1)!} \\ & \frac{\lambda_i^{\left(q^{-T+1}\right)}-1}{(q-1)q^m} & \cdots & \frac{\left(\prod_{\tau=0}^{n-3}[q^{-T+1}-\tau]\right)\lambda_i^{\left(q^{-T+1}-n+2\right)}}{(q-1)q^m(n-2)!} \\ & & \ddots & \\ & & & \frac{\lambda_i^{\left(q^{-T+1}\right)}-1}{(q-1)q^m} \end{bmatrix}_{m_i \times m_i}.$$

It should be mentioned that, since $R\left(q^m\right)$ and $K\left(q^m\right)$ are similar, they have the same conventional eigenvalues

$$\gamma_i\left(q^m\right) = \frac{\lambda_i^{\left(q^{-T+1}\right)} - 1}{\left(q-1\right) q^m},\ i = 1, 2, ..., n$$

with corresponding multiplicities. Furthermore, if the dynamic eigenvalues of $K(q^m)$ are taken as the conventional eigenvalues $\gamma_i\left(q^m\right)$, then the corresponding dynamic eigenvectors $\left\{u_i\left(q^m\right)\right\}_{i=1}^{n}$ of $K\left(q^m\right)$ can be given by $u_i\left(q^m\right) = S^{-1} w_i\left(q^m\right)$.

In order to prove this claim let's show that $\left\{\gamma_i\left(q^m\right), u_i\left(q^m\right)\right\}_{i=1}^{n}$ is a set of dynamic eigenpairs of $K(q^m)$. By Definition 33,

$$\begin{aligned} D_q u_i(q^m) &= S^{-1} D_q w_i(q^m) \\ &= S^{-1} R(q^m) w_i(q^m) - S^{-1}\gamma_i(q^m) w_i\left(q^{m+1}\right) \\ &= K(q^m) S^{-1} w_i(q^m) - \gamma_i(q^m) S^{-1} w_i\left(q^{m+1}\right) \\ &= K(q^m) u_i(q^m) - \gamma_i(q^m) u_i\left(q^{m+1}\right), \end{aligned} \tag{3.45}$$

for all $1 \le i \le n$. The next step is to show that $u_i(q^m)$ satisfies (3.43). Since $\{\gamma_i(q^m), u_i(q^m)\}_{i=1}^n$ is the set of dynamic eigenpairs of $K(q^m)$, it satisfies (3.45) for all $1 \le i \le n$. By choosing the i^{th} block of $K(q^m)$ with dimension $m_i \times m_i$, one can construct the following linear dynamic system

$$D_q v(q^m) = \tilde{K}_i(q^m) v(q^m) = \begin{bmatrix} 0 & \frac{q^{-T+1}}{(q-1)q^m\lambda_i} & \frac{q^{-T+1}(q^{-T+1}-1)}{2(q-1)q^m\lambda_i^2} & \cdots & \frac{\left(\prod_{\tau=0}^{n-2}\left[q^{-T+1}-\tau\right]\right)}{(n-1)!(q-1)q^m\lambda_i^{n-1}} \\ & 0 & \frac{q^{-T+1}}{(q-1)q^m\lambda_i} & & \frac{\left(\prod_{\tau=0}^{n-3}\left[q^{-T+1}-\tau\right]\right)}{(n-2)!(q-1)q^m\lambda_i^{n-2}} \\ & & 0 & \ddots & \vdots \\ & & & \ddots & \frac{q^{-T+1}}{(q-1)q^m\lambda_i} \\ & & & & 0 \end{bmatrix} v(q^m) \tag{3.46}$$

where $\tilde{K}_i(q^m) := K_i(q^m) \ominus \gamma_i(q^m) I$. There are m_i linearly independent solutions of (3.46). Let us denote these solutions by $v_{i,j}(q^m)$, where i corresponds to the i^{th} block matrix $K_i(q^m)$ and $j = 1, \ldots, m_i$. As it is done in the proof of [38, Theorem 7.6], we have the following construction. For $1 \le i \le d$, we define $l_i = \sum_{s=0}^{i-1} m_s$, with $m_0 = 0$. Then, the form of an arbitrary $n \times 1$ column vector u_{l_i+j} for $i \le j \le m_i$ can be given as

$$u_{l_i+j} = [\underbrace{0, \ldots, 0}_{m_1+\ldots+m_{i-1}}, \underbrace{v_{i,j}^T(q^m)}_{m_i}, \underbrace{0, \ldots, 0}_{m_{i+1},\ldots,m_d}]_{1\times n}. \tag{3.47}$$

Considering all vector solutions of (3.45), the solution of the $n \times n$ matrix dynamic equation can be written as

$$D_qU(q^m) = K(q^m)U(q^m) - U\left(q^{m+1}\right)\Gamma(q^m),$$

where $\Gamma(q^m) := diag\left[\gamma_1(q^m), \ldots, \gamma_n(q^m)\right]$, and hence

$$U(q^m) := \left[u_1, \ldots, u_{m_1}, \ldots, u_{\left(\sum_{k=1}^{i-1} m_k\right)}, \ldots, u_{\left(\sum_{k=1}^{i} m_k\right)}, \ldots, u_{\left(\sum_{k=1}^{d} m_k\right)-1}, u_n\right]$$

$$= \begin{bmatrix} \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,m_1} \\ & v_{1,1} & \ddots & v_{1,m_1-1} \\ & & \ddots & \cdots \\ & & & v_{1,1} \end{bmatrix}_{m_1\times m_1} & & \\ & \ddots & \\ & & \begin{bmatrix} v_{d,1} & v_{d,2} & \cdots & v_{d,m_d} \\ & v_{d,1} & \ddots & v_{d,m_d-1} \\ & & \ddots & \cdots \\ & & & v_{d,1} \end{bmatrix}_{m_d\times m_d} \end{bmatrix}_{n\times n}.$$

The m_i linearly independent solutions of (3.46) have the form

$$\begin{aligned} v_{i,1}(q^m) &:= \left[v_{i,m_i}(q^m), 0, \ldots, 0\right]^T_{m_i\times 1}, \\ v_{i,2}(q^m) &:= \left[v_{i,m_i-1}(q^m), v_{i,m_i}(q^m), 0, \ldots, 0\right]^T_{m_i\times 1}, \\ &\vdots \\ v_{i,m_i}(q^m) &:= \left[v_{i,1}(q^m), v_{i,2}(q^m), \ldots, v_{i,m_i-1}(q^m), v_{i,m_i}(q^m)\right]^T_{m_i\times 1}. \end{aligned}$$

Then, the dynamic equations are in the form

$$\begin{aligned} D_qv_{i,m_i}(q^m) &= 0, \\ D_qv_{i,m_i-1}(q^m) &= \frac{q^{-T+1}}{(q-1)q^m\lambda_i}v_{i,m_i}(q^m) \end{aligned}$$

$$\begin{aligned}
D_q v_{i,m_i-2}\left(q^m\right) &= \frac{q^{-T+1}\left(q^{-T+1}-1\right)}{2\left(q-1\right)q^m\lambda_i^2} v_{i,m_i}\left(q^m\right) + \frac{q^{-T+1}}{\left(q-1\right)q^m\lambda_i} v_{i,m_i-1}\left(q^m\right) \\
&\vdots \\
D_q v_{i,1}\left(q^m\right) &= \frac{\prod_{\tau=0}^{m_i-2}\left[q^{-T+1}-\tau\right]}{\left(m_i-1\right)!\left(q-1\right)q^m\lambda_i^{m_i-1}} v_{i,m_i}\left(q^m\right) \\
&\quad + \frac{\prod_{\tau=0}^{m_i-3}\left[q^{-T+1}-\tau\right]}{\left(m_i-2\right)!\left(q-1\right)q^m\lambda_i^{m_i-2}} v_{i,m_i-1}\left(q^m\right) \\
&\quad + \cdots + \frac{q^{-T+1}}{\left(q-1\right)q^m\lambda_i} v_{i,2}\left(q^m\right).
\end{aligned}$$

Moreover, one can get the following solutions

$$v_{i,m_i}\left(q^m\right) = 1, v_{i,m_i-1}\left(q^m\right) = \int_{q^{m_0}}^{q^m} \frac{q^{-T+1}}{\left(q-1\right)\tau\lambda_i} v_{i,m_i}\left(\tau\right) d_q\tau,$$

$$\begin{aligned}
v_{i,m_i-2}\left(q^m\right) &= \int_{q^{m_0}}^{q^m} \frac{q^{-T+1}\left(q^{-T+1}-1\right)}{2\left(q-1\right)\tau\lambda_i^2} v_{i,m_i}\left(\tau\right) d_q\tau + \int_{q^{m_0}}^{q^m} \frac{q^{-T+1}}{\left(q-1\right)\tau\lambda_i} v_{i,m_i-1}\left(\tau\right) d_q\tau, \\
&\vdots \\
v_{i,1}\left(q^m\right) &= \int_{q^{m_0}}^{q^m} \frac{\prod_{\varrho=0}^{m_i-2}\left[q^{-T+1}-\varrho\right]}{\left(m_i-1\right)!\left(q-1\right)\tau\lambda_i^{m_i-1}} v_{i,m_i}\left(\tau\right) d_q\tau \\
&\quad + \int_{q^{m_0}}^{q^m} \frac{\prod_{\varrho=0}^{m_i-3}\left[q^{-T+1}-\varrho\right]}{\left(m_i-2\right)!\left(q-1\right)\tau\lambda_i^{m_i-2}} v_{i,m_i-1}\left(\tau\right) d_q\tau + \\
&\quad \cdots + \int_{q^{m_0}}^{q^m} \frac{q^{-T+1}}{\left(q-1\right)\tau\lambda_i} v_{i,2}\left(\tau\right) d_q\tau.
\end{aligned}$$

It can be shown that each v is bounded. There exist constants $B_{i,j}$, $i = 1, \ldots, d$ and $j = 1, \ldots, m_i$, such that

$$\begin{aligned}
|v_{i,m_i}(q^m)| &= 1 \leq B_{i,m_i} h_0\left(q^m, q^{m_0}\right) = B_{i,m_i}, \\
|v_{i,m_i-1}(q^m)| &\leq \int_{q^{m_0}}^{q^m} \left| \frac{q^{-T+1}}{\left(q-1\right)\tau\lambda_i} v_{i,m_i}(\tau) \right| d_q\tau \\
&= \frac{q^{-T}}{|\lambda_i|} \int_{q^{m_0}}^{q^m} \frac{q}{\left(q-1\right)\tau} h_0\left(\tau, q^{m_0}\right) d_q\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^{-T}}{|\lambda_i|} h_1\left(q^m, q^{m_0}\right) \leq B_{i,m_i-1} h_1\left(q^m, q^{m_0}\right), \\
|v_{i,m_i-2}(q^m)| &\leq \int_{q^{m_0}}^{q^m} \left| \frac{q^{-T+1}\left(q^{-T+1}-1\right)}{2(q-1)\tau\lambda_i^2} v_{i,m_i}(\tau) \right| d_q\tau \\
&\quad + \int_{q^{m_0}}^{q^m} \left| \frac{q^{-T+1}}{(q-1)\tau\lambda_i} v_{i,m_i-1}(\tau) \right| d_q\tau \\
&\leq \frac{q^{-T}\left(q^{-T+1}-1\right)}{2\lambda_i^2} \int_{q^{m_0}}^{q^m} \frac{q}{(q-1)\tau} h_0\left(\tau, q^{m_0}\right) d_q\tau \\
&\quad + \frac{q^{-2T}}{\lambda_i^2} \int_{q^{m_0}}^{q^m} \frac{q}{(q-1)\tau} h_1\left(\tau, q^{m_0}\right) d_q\tau \\
&\leq B_{i,m_i-2}\left(h_1\left(q^m, q^{m_0}\right) + h_2\left(q^m, q^{m_0}\right)\right), \\
&\vdots \\
|v_{i,1}(q^m)| &\leq B_{i,1} \sum_{y=0}^{m_i-1} h_y\left(q^m, q^{m_0}\right).
\end{aligned}
$$

Setting $\beta_i := \max_{j=1,\ldots,m_i}\{B_{i,j}\}$ for each $1 \leq i \leq d$ gives

$$
\left\| u_{l_i+j}\left(q^m\right) \right\| \leq \beta_i \sum_{y=0}^{m_i-1} h_y\left(q^m, q^{m_0}\right)
$$

for $1 \leq i \leq d$ and $j = 1, 2, \ldots, m_i$. Since $w_i = Su_i$

$$
\begin{aligned}
\left\| w_i\left(q^m\right) \right\| = \left\| Su_i\left(q^m\right) \right\| &\leq \|S\| \, \beta_i \sum_{y=0}^{m_i-1} h_y\left(q^m, q^{m_0}\right) \\
&= D_i \sum_{y=0}^{m_i-1} h_y\left(q^m, q^{m_0}\right),
\end{aligned}
$$

where $D_i := \|S\| \, \beta_i$, for all $1 \leq i \leq n$. $\square$

The following definition is the special case of Definition 7.8 in [38] when $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$.

Definition 35. Let $\mathbb{C}_\mu := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{(q-1)q^m} \right\}$. Given an element $q^m, m \in \mathbb{Z}$, the Hilger circle is defined by

$$
\mathcal{H}_{q^m} := \left\{ z \in \mathbb{C}_\mu : Re_\mu(z) < 0 \right\}.
$$

Next, we present Floquet stability theorem which shows the strong relationship between the stability results of the q-Floquet system (3.31) and the eigenvalues of the corresponding nonautonomous linear dynamic system (3.33).

Theorem 3.25 (Floquet stability theorem) *The stability results for the solutions of the system (3.31) are given as follows*

1. *If*
$$-Re_{\mu}\gamma_i(q^m) > 0 \tag{3.48}$$
for all $i = 1, \ldots, n$, *then the system (3.31) is asymptotically stable. Moreover, if there is a positive constant* ε *such that (3.48) and*
$$-Re_{\mu}\gamma_i\left(q^m\right) \geq \varepsilon \tag{3.49}$$
for all $i = 1, \ldots, n$, *then the system (3.31) is exponentially stable.*

2. *If*
$$-Re_{\mu}\gamma_i(q^m) \geq 0$$
for all $i = 1, \ldots, n$, *and if, for each characteristic exponent with*
$$Re_{\mu}\left(\gamma_i\left(q^m\right)\right) = 0$$
the algebraic multiplicity equals the geometric multiplicity, then the system (3.31) is stable; otherwise the system (3.31) is unstable.

3. *If*
$$Re_{\mu}\left(\gamma_i\left(q^m\right)\right) > 0$$
for some $i = 1, \ldots, n$, *then the system (3.31) is unstable.*

Proof. Let $e_R\left(q^m, q^{m_0}\right)$ be the transition matrix of the system (3.33) and $R\left(q^m\right)$ be defined as in (3.32). Given the eigenvalues $\left\{\gamma_i\left(q^m\right)\right\}_{i=1}^{n}$ of $R\left(q^m\right)$, one can define the set of dynamic eigenpairs $\left\{\gamma_i\left(q^m\right), w_i\left(q^m\right)\right\}_{i=1}^{n}$ and from Theorem 3.24, the dynamic eigenvector $w_i\left(q^m\right)$ satisfies (3.43). Moreover, let us define $W(q^m)$

by

$$W(q^m) = e_R(q^m, q^\tau) e_{\ominus\Xi}(q^m, q^\tau). \tag{3.50}$$

Then we have

$$e_R(q^m, q^\tau) = W(q^m) e_\Xi(q^m, q^\tau), \tag{3.51}$$

where $\tau \in \mathbb{Z}$ and $\Xi(q^m)$ is given as in Lemma 3.21. Employing (3.51), one can write that

$$e_R(q^\tau, q^{m_0}) = e_\Xi(q^\tau, q^{m_0}) W^{-1}(q^{m_0}). \tag{3.52}$$

Combining (3.51) and (3.52), the transition matrix of the system (3.33) can be represented by

$$e_R(q^m, q^{m_0}) = W(q^m) e_\Xi(q^m, q^{m_0}) W^{-1}(q^{m_0}), \tag{3.53}$$

where $W(q^m) := [w_1(q^m), w_2(q^m), \dots, w_n(q^m)]$. The matrix $W^{-1}(q^{m_0})$ can be denoted as follows:

$$W^{-1}(q^{m_0}) = \begin{bmatrix} v_1^T(q^{m_0}) \\ v_2^T(q^{m_0}) \\ \vdots \\ v_n^T(q^{m_0}) \end{bmatrix}.$$

Since $\Xi(q^m)$ is a diagonal matrix, (3.53) can be written as follows

$$e_R(q^m, q^{m_0}) = \sum_{i=1}^{n} e_{\gamma_i}(q^m, q^{m_0}) W(q^m) F_i W^{-1}(q^{m_0}) \tag{3.54}$$

where $F_i := \delta_{i,j}$ is $n \times n$ matrix. Using $v_i^T(q^m) w_j(q^m) = \delta_{i,j}$ for all $m \in \mathbb{Z}$, F_i can be written as

$$F_i = W^{-1}(q^m) [0, \dots, 0, w_i(q^m), 0, \dots, 0]. \tag{3.55}$$

By means of (3.54) and (3.55) we have

$$e_R(q^m, q^{m_0}) = \sum_{i=1}^{n} e_{\gamma_i}(q^m, q^{m_0}) w_i(q^m) v_i^T(q^{m_0}) = \sum_{i=1}^{n} \chi_i(q^m) v_i^T(q^{m_0}),$$

where $\chi_i(q^m)$ is mode vector of system (3.33).

Case 1 By (3.37), for each $1 \le i \le n$, we can write that

$$\begin{aligned}\|\chi_i(q^m)\| &\le D_i \sum_{y=0}^{d_i-1} h_y(q^m, q^{m_0}) \left|e_{\gamma_i}(q^m, q^{m_0})\right| \\ &\le D_i \sum_{y=0}^{d_i-1} h_y(q^m, q^{m_0}) e_{Re_\mu(\gamma_i)}(q^m, q^{m_0})\end{aligned}$$

where D_i is as in Theorem 3.24, d_i represents the dimension of the Jordan block which contains i^{th} eigenvalue of $R(q^m)$.Using Lemma 3.23 we get

$$\lim_{m\to\infty} h_y(q^m, q^{m_0}) e_{Re_\mu(\gamma_i)}(q^m, q^{m_0}) = 0$$

for each $1 \le i \le n$ and all $y = 0, 1, ..., d_i - 1$. This along with Theorem 3.22 implies that (3.33) is asymptotically stable. By Theorem 3.6, since the solutions of (3.31) and (3.33) are related by Lyapunov transformation, we can state that solution of (3.31) is asymptotically stable. For the second part, consider

$$\begin{aligned}\|\chi_i(q^m)\| &\le D_i \sum_{y=0}^{d_i-1} h_y(q^m, q^{m_0}) \left|e_{\gamma_i}(q^m, q^{m_0})\right| \\ &\le D_i \sum_{y=0}^{d_i-1} h_y(q^m, q^{m_0}) e_{Re_\mu(\gamma_i)\oplus\varepsilon}(q^m, q^{m_0}) e_{\ominus\varepsilon}(q^m, q^{m_0}). \end{aligned} \tag{3.56}$$

If (3.49) holds, then $Re_\mu(\gamma_i) \oplus \varepsilon$ satisfies (3.40). Hence, by Lemma 3.23 the term $h_k(q^m, q^{m_0}) e_{Re_\mu(\gamma_i)\oplus\varepsilon}(q^m, q^{m_0})$ converges to zero as $m \to \infty$. That is, there is an upper bound C_ε for the sum $\sum_{y=0}^{d_i-1} h_y(q^m, q^{m_0}) e_{Re_\mu(\gamma_i)\oplus\varepsilon}(q^m, q^{m_0})$. This along with (3.56) yields

$$\|\chi_i(q^m)\| \le D_i C_\varepsilon e_{\ominus\varepsilon}(q^m, q^{m_0}).$$

Hence, Theorem 3.22 implies that (3.33) is exponentially stable. Using the above given argument (3.31) is exponentially stable.

Case 2 Assume that $Re_\mu[\gamma_c(q^m)] = 0$ for some $1 \le c \le n$ with equal algebraic and geometric multiplicities corresponding to $\gamma_c(q^m)$. Then the Jordan block of

$\gamma_c(q^m)$ is 1×1. Hence,

$$\begin{aligned}\lim_{m\to\infty} \|\chi_c(q^m)\| &\le \lim_{m\to\infty} D_c \left|e_{\gamma_c}(q^m, q^{m_0})\right| \\ &\le \lim_{m\to\infty} D_c e_{Re_\mu(\gamma_c)}(q^m, q^{m_0}) \\ &= D_c.\end{aligned}$$

By Theorem 3.22, the system (3.33) is stable. By Theorem 3.6, the solutions of (3.31) and (3.33) are related by Lyapunov transformation. This implies that the system (3.31) is stable.

Case 3 Suppose that $Re_\mu(\gamma_i(q^m)) > 0$ for some $i = 1, \ldots, n$. Then, we have

$$\lim_{m\to\infty} \|e_R(q^m, q^{m_0})\| = \infty,$$

and by the relationship between solutions of (3.31) and (3.33), one can write that

$$\lim_{m\to\infty} \|\Phi_A(q^m, q^{m_0})\| = \infty.$$

Therefore, (3.31) is unstable.

The proof is complete. □

Example 3.2. Let us reconsider the Example 3.1 given in Chapter 3:

$$D_q x(q^m) = \begin{bmatrix} c_1 q^{-m} & 0 \\ 0 & c_2 q^{-m} \end{bmatrix} x(q^m), \ \ x(1) = x_0, \ \ m \in \mathbb{Z} \text{ and } c_1, c_2 \in \mathbb{R}_+. \tag{3.57}$$

As in Example 3.1, $R(q^m)$ can be obtained as follows:

$$R(q^m) = \begin{bmatrix} c_1 q^{-m} & 0 \\ 0 & c_2 q^{-m} \end{bmatrix}.$$

Then $R(q^m)$ has eigenvalues $\gamma_1(q^m) = c_1 q^{-m}$ and $\gamma_2(q^m) = c_2 q^{-m}$. As a consequence, $Re_\mu(\gamma_{1,2}(t)) > 0$. Thus, it can be seen by the last theorem that the system (3.57) is unstable.

One state the following corollary as a consequence of Theorem 3.25.

Corollary 3.26 *Let λ_i be a Floquet multiplier of the q-Floquet system (3.31) for $i = 1, \ldots, n$. Then, we have*

1. *If $|\lambda_i| < 1$ for all $i = 1, \ldots, n.$, then the system (3.31) is exponentially stable;*
2. *If $|\lambda_i| \leq 1$ for all $i = 1, \ldots, n$ and if, for each $|\lambda_i| = 1$ for some $i = 1, \ldots, n$, the algebraic multiplicity equals to geometric multiplicity, then the system (3.31) is stable.*
3. *$|\lambda_i| > 1$ for some $i = 1, \ldots, n$, then the system (3.31) is unstable.*

Chapter 4

Extension of Floquet theory to time scales periodic in shifts $\delta_\pm$

In this chapter, Floquet theory is reconstructed on more general domains including both additively and nonadditively periodic time scales. Hereafter, $\mathbb{T}$ is supposed to be a T-periodic time scale in shifts $\delta_\pm$ and that the shift operators $\delta_\pm$ are Δ-differentiable with rd-continuous derivatives. For brevity, the term "periodic in shifts" is used to mean periodicity in shifts $\delta_\pm$. Throughout the chapter, the notation $\delta_\pm^T(t)$ is employed to indicate the shifts $\delta_\pm(T,t)$. Furthermore, the notation $\delta_\pm^{(n)}(T,t)$, $n \in \mathbb{N}$, represents the n-times composition of shifts, $\delta_\pm^T$, by itself, namely,

$$\delta_\pm^{(n)}(T,t) := \underbrace{\delta_\pm^T \circ \delta_\pm^T \circ \ldots \circ \delta_\pm^T}_{n-times}(t)\,.$$

Observe that, the period of a function f does not have to be equal to period of the time scale on which f is determined. However, for simplicity of the results the period of time scale $\mathbb{T}$ is assumed to be equal to period of the all functions defined on $\mathbb{T}$. The following definition plays a key role in the following analysis:

Definition 36. [38, Definition 2.1]A Lyapunov transformation is an invertible matrix $L(t) \in C_{rd}^1(\mathbb{T}, \mathbb{R}^{n\times n})$ satisfying

$$\|L(t)\| \leq \rho \text{ and } |\det L(t)| \geq \eta \text{ for all } t \in \mathbb{T}$$

where ρ and η are arbitrary positive reals.

4.1 Floquet theory based on new periodicity concept: Homogeneous case

Consider the following regressive nonautonomous linear dynamic system

$$x^{\Delta}(t) = A(t)\, x(t), \ \ x(t_0) = x_0, \tag{4.1}$$

where $A : \mathbb{T}^{*} \rightarrow \mathbb{R}^{n\times n}$ is Δ-periodic in shifts with period T. Observe that if the time scale is additively periodic, then $\delta_{\pm}^{\Delta}(T,t) = 1$ and Δ-periodicity in shifts becomes the same as the periodicity in shifts. Hence, the homogeneous system focused in this section is more general than that of [37] and [38].
In preparation for the next result, the following set

$$P(t_0) := \left\{\delta_{+}^{(k)}(T,t_0), \ \ k = 0,1,2,\dots\right\} \tag{4.2}$$

and the function

$$\Theta(t) := \sum_{j=1}^{m(t)} \delta_{-}\left(\delta_{+}^{(j-1)}(T,t_0), \delta_{+}^{(j)}(T,t_0)\right) + G(t), \tag{4.3}$$

where

$$m(t) := \min\left\{k \in \mathbb{N}_0 : \delta_{+}^{(k)}(T,t_0) \geq t\right\} \tag{4.4}$$

and

$$G(t) := \begin{cases} 0 & \text{if } t \in P(t_0) \\ -\delta_{-}\left(t, \delta_{+}^{(m(t))}(T,t_0)\right) & \text{if } t \notin P(t_0) \end{cases} \tag{4.5}$$

are defined.

Remark. For an additive periodic time scale one always has $\Theta(t) = t - t_0$.

Theorem 4.1 *For a nonsingular, $n\times n$ constant matrix M a solution R :*

$\mathbb{T} \to \mathbb{C}^{n\times n}$ *of matrix exponential equation*

$$e_R\left(\delta_+^T(t_0), t_0\right) = M$$

can be given by

$$R(t) = \lim_{s\to t} \frac{M^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]} - I}{\sigma(t) - s}, \tag{4.6}$$

where I is the $n\times n$ identity matrix and Θ is as in (4.3).

Proof. Let's construct the matrix exponential function $e_R(t,t_0)$ as follows

$$e_R(t,t_0) := M^{\frac{1}{T}\Theta(t)} \text{ for } t \geq t_0, \tag{4.7}$$

where Θ is given by (4.3) and real power of a nonsingular matrix M is given by Definition 26. To show that the function $e_R(t,t_0)$ constructed in (4.7) is the matrix exponential we first observe that

$$e_R(t_0,t_0) = M^{\frac{1}{T}\Theta(t_0)} = I,$$

where we use (4.7) along with $\Theta(t_0) = G(t_0) = 0$. Second, differentiating (4.7) yields

$$e_R^{\Delta}(t,t_0) = R(t)\, e_R(t,t_0).$$

To see this, first suppose that t is right-scattered. Then,

$$\begin{aligned} e_R^{\Delta}(t,t_0) &= \frac{e_R(\sigma(t),t_0) - e_R(t,t_0)}{\sigma(t)-t} \\ &= \frac{M^{\frac{1}{T}\Theta(\sigma(t))} - M^{\frac{1}{T}\Theta(t)}}{\sigma(t)-t} \\ &= \frac{M^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(t)]} - I}{\sigma(t)-t} M^{\frac{1}{T}\Theta(t)} \\ &= R(t)\, e_R(t,t_0). \end{aligned}$$

If t is right dense, then $\sigma(t) = t$. Setting $s = t+h$ in (4.3) and using (4.7) one can obtain

$$\begin{aligned} e_R^{\Delta}(t,t_0) &= \lim_{h\to 0}\frac{e_R(t+h,t_0)-e_R(t,t_0)}{h} \\ &= \lim_{h\to 0}\frac{M^{\frac{1}{T}\Theta(t+h)}-M^{\frac{1}{T}\Theta(t)}}{h} \\ &= \lim_{h\to 0}\frac{M^{\frac{1}{T}[\Theta(t+h)-\Theta(t)]}-I}{h}M^{\frac{1}{T}\Theta(t)} \\ &= R(t)\,e_R(t,t_0). \end{aligned}$$

In any case, we have $e_R^{\Delta}(t,t_0)=R(t)\,e_R(t,t_0)$. Finally, we obtain

$$\Theta\left(\delta_+^T(t_0)\right)=\delta_-\left(t_0,\delta_+^T(t_0)\right)=\delta_+^T(t_0)=T,$$

and therefore,

$$e_R\left(\delta_+^T(t_0),t_0\right)=M^{\frac{1}{T}\Theta\left(\delta_+^T(t_0)\right)}=M.$$

□

The following result is generalization of Corollary (3.2).

Corollary 4.2 *The eigenvectors of the matrices $R(t)$ and M are same.*

Proof. For any eigenpairs $\{\lambda_i,v_i\}$, $i=1,2,...,n$ of M, using $Mv_i=\lambda_i v_i$ one can write that

$$\lim_{s\to t}M^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]}v_i=\lim_{s\to t}\lambda_i^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]}v_i.$$

This implies

$$R(t)v_i=\lim_{s\to t}\left(\frac{\lambda_i^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]}-1}{\sigma(t)-s}\right)v_i. \tag{4.8}$$

Substituting $\gamma_i(t)=\lim_{s\to t}\left(\frac{\lambda_i^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]}-1}{\sigma(t)-s}\right)$ into (4.8) one can conclude that $R(t)$ has the eigenpairs $\{\gamma_i(t),v_i\}_{i=1}^n$. □

The following two results can be proven similar to the Lemma (3.3).

Lemma 4.3 *Let* $\mathbb{T}$ *be a time scale and* $P \in \mathcal{R}(\mathbb{T}^*, \mathbb{R}^{n\times n})$ *be a* $\Delta-$*periodic matrix valued function in shifts with period* T*, i.e.*

$$P(t) = P\left(\delta_{\pm}^{T}(t)\right)\delta_{\pm}^{\Delta T}(t)$$

Then the solution of the dynamic matrix initial value problem

$$Y^{\Delta}(t) = P(t)\,Y(t)\,,\ Y(t_0) = Y_0, \tag{4.9}$$

is unique up to a period T *in shifts. That is*

$$\Phi_P(t, t_0) = \Phi_P\left(\delta_{+}^{T}(t), \delta_{+}^{T}(t_0)\right) \tag{4.10}$$

for all $t \in \mathbb{T}^*$.

Corollary 4.4 *Let* $\mathbb{T}$ *be a time scale and* $P \in \mathcal{R}(\mathbb{T}^*, \mathbb{R}^{n\times n})$ *be a* $\Delta-$*periodic matrix valued function in shifts. Then*

$$e_P(t, t_0) = e_P\left(\delta_{+}^{T}(t), \delta_{+}^{T}(t_0)\right). \tag{4.11}$$

Theorem 4.5 (Floquet decomposition) *Let* A *be a matrix valued function that is* Δ*-periodic in shifts with period* T*. The transition matrix for* A *can be given in the form*

$$\Phi_A(t, \tau) = L(t)\,e_R(t, \tau)\,L^{-1}(\tau)\,,\ \text{for all } t, \tau \in \mathbb{T}^*, \tag{4.12}$$

where $R : \mathbb{T} \to \mathbb{C}^{n\times n}$ *is* Δ*-periodic function in shifts and* $L(t) \in C_{rd}^{1}(\mathbb{T}^*, \mathbb{R}^{n\times n})$ *is periodic in shifts with the period* T*.*

Proof. Let the matrix function R is defined as in Theorem 4.1 with a constant $n \times n$ matrix $M := \Phi_A\left(\delta_{+}^{T}(t_0), t_0\right)$. Then,

$$e_R\left(\delta_{+}^{T}(t_0), t_0\right) = \Phi_A\left(\delta_{+}^{T}(t_0), t_0\right),$$

and the Lyapunov transformation $L(t)$ is defined as

$$L(t) := \Phi_A(t, t_0)\,e_R^{-1}(t, t_0)\,. \tag{4.13}$$

Obviously, $L(t) \in C^1_{rd}(\mathbb{T}^*, \mathbb{R}^{n\times n})$ and L is invertible. The equality

$$\Phi_A(t, t_0) = L(t)\, e_R(t, t_0)\,. \tag{4.14}$$

along with (4.13) implies

$$\begin{aligned}\Phi_A(t_0, t) &= e_R^{-1}(t, t_0)\, L^{-1}(t)\\ &= e_R(t_0, t)\, L^{-1}(t)\,.\end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15), one can obtain (4.12). The periodicity in shifts of L is shown by using (4.10-4.11) as follows

$$\begin{aligned}L\left(\delta_+^T(t)\right) &= \Phi_A\left(\delta_+^T(t), t_0\right) e_R^{-1}\left(\delta_+^T(t), t_0\right)\\ &= \Phi_A\left(\delta_+^T(t), \delta_+^T(t_0)\right) \Phi_A\left(\delta_+^T(t_0), t_0\right) e_R\left(t_0, \delta_+^T(t)\right)\\ &= \Phi_A\left(\delta_+^T(t), \delta_+^T(t_0)\right) \Phi_A\left(\delta_+^T(t_0), t_0\right) e_R\left(t_0, \delta_+^T(t_0)\right) e_R\left(\delta_+^T(t_0), \delta_+^T(t)\right)\\ &= \Phi_A\left(\delta_+^T(t), \delta_+^T(t_0)\right) e_R\left(\delta_+^T(t_0), \delta_+^T(t)\right)\\ &= \Phi_A\left(\delta_+^T(t), \delta_+^T(t_0)\right) e_R^{-1}\left(\delta_+^T(t), \delta_+^T(t_0)\right)\\ &= \Phi_A(t, t_0)\, e_R^{-1}(t, t_0)\\ &= L(t)\,.\end{aligned}$$

□

The following result is the extension of Theorem (3.6) and [38, Theorem 3.7] and it can be proved similar to [38, Theorem 3.7].

Theorem 4.6 *Assume that the transition matrix Φ_A of the unified Floquet system (4.1) has the decomposition of the form $\Phi_A(t, t_0) = L(t)\, e_R(t, t_0)$. Then, $x(t) = \Phi_A(t, t_0)\, x_0$ is a solution of (4.1) if and only if the linear dynamic equation*

$$z^{\Delta}(t) = R(t)\, z(t)\,, \quad z(t_0) = x_0.$$

has a solution of the form $z(t) = L^{-1}(t)\, x(t)$.

Next, the necessary and sufficient condition for the existence of solution of

Floquet system (4.1) which is periodic in shifts is given.

Theorem 4.7 *The solution of the unified Floquet system (4.1) has a T-periodic solution in shifts with an initial state $x(t_0) = x_0 \neq 0$ if and only if at least one of the eigenvalues of*

$$e_R\left(\delta_+^T(t_0), t_0\right) = \Phi_A\left(\delta_+^T(t_0), t_0\right)$$

is 1.

Proof. Let $x(t)$ be a solution of the periodic system (4.1) which is T-periodic in shifts corresponding with the nonzero initial state $x(t_0) = x_0$. Then according to Theorem 4.5, the Floquet decomposition of x can be written as

$$x(t) = \Phi_A(t, t_0)\, x_0 = L(t)\, e_R(t, t_0)\, L^{-1}(t_0)\, x_0,$$

which also yields

$$x\left(\delta_+^T(t)\right) = L\left(\delta_+^T(t)\right) e_R\left(\delta_+^T(t), t_0\right) L^{-1}(t_0)\, x_0.$$

By T-periodicity of x and L in shifts, one can obtain

$$e_R(t, t_0)\, L^{-1}(t_0)\, x_0 = e_R\left(\delta_+^T(t), t_0\right) L^{-1}(t_0)\, x_0,$$

and therefore,

$$e_R(t, t_0)\, L^{-1}(t_0)\, x_0 = e_R\left(\delta_+^T(t), \delta_+^T(t_0)\right) e_R\left(\delta_+^T(t_0), t_0\right) L^{-1}(t_0)\, x_0.$$

Since $e_R\left(\delta_+^T(t), \delta_+^T(t_0)\right) = e_R(t, t_0)$ the last equality implies

$$e_R(t, t_0)\, L^{-1}(t_0)\, x_0 = e_R(t, t_0)\, e_R\left(\delta_+^T(t_0), t_0\right) L^{-1}(t_0)\, x_0$$

and thus

$$L^{-1}(t_0)\, x_0 = e_R\left(\delta_+^T(t_0), t_0\right) L^{-1}(t_0)\, x_0.$$

Hence, the matrix $e_R\left(\delta_+^T(t_0), t_0\right)$ has an eigenvector $L^{-1}(t_0)\, x_0$ corresponding to the eigenvalue 1.

Conversely, assume that 1 is an eigenvalue of $e_R\left(\delta_{+}^{T}(t_0), t_0\right)$ with corresponding eigenvector z_0. This means z_0 is real valued and nonzero. Using $e_R(t, t_0) = e_R\left(\delta_{+}^{T}(t), \delta_{+}^{T}(t_0)\right)$, one can arrive at the following equality

$$\begin{aligned} z\left(\delta_{+}^{T}(t)\right) &= e_R\left(\delta_{+}^{T}(t), t_0\right) z_0 \\ &= e_R\left(\delta_{+}^{T}(t), \delta_{+}^{T}(t_0)\right) e_R\left(\delta_{+}^{T}(t_0), t_0\right) z_0 \\ &= e_R\left(\delta_{+}^{T}(t), \delta_{+}^{T}(t_0)\right) z_0 \\ &= e_R(t, t_0) z_0 \\ &= z(t), \end{aligned}$$

which shows that $z(t) = e_R(t, t_0) z_0$ is T-periodic in shifts. Applying the Floquet decomposition and setting $x_0 := L(t_0) z_0$, the nontrivial solution x of (4.1) is obtained as follows

$$x(t) = \Phi_A(t, t_0) x_0 = L(t) e_R(t, t_0) L^{-1}(t_0) x_0 = L(t) e_R(t, t_0) z_0 = L(t) z(t),$$

which is T-periodic in shifts since L and z are T-periodic in shifts. This completes the proof. □

Example 4.1. Suppose that $\mathbb{T} = \cup_{k=0}^{\infty}\left[3^{\pm k}, 2.3^{\pm k}\right] \cup \{0\}$. Then, $\mathbb{T}$ is 3-periodic in shifts $\delta_{\pm}(s, t) = s^{\pm 1} t$. Setting $A(t) = \frac{1}{t} I_{2\times 2}$, one may get

$$A(\delta_{\pm}(3, t))\, \delta_{\pm}^{\Delta}(3, t) = A(3t)\, 3 = A(t)$$

which shows that A is Δ-periodic in shifts with the period 3. Consider the system

$$x^{\Delta}(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{bmatrix} x(t), x(1) = x_0$$

whose transition matrix is given by

$$\Phi_A(t, 1) = \begin{bmatrix} e_{1/t}(t, 1) & 0 \\ 0 & e_{1/t}(t, 1) \end{bmatrix}.$$

Then

$$\Phi_A\left(\delta_+^3(1), 1\right) = \Phi_A(3,1) = \begin{bmatrix} e_{1/t}(3,1) & 0 \\ 0 & e_{1/t}(3,1) \end{bmatrix}.$$

As in Theorem 4.1, one can write that

$$e_R(3,1) = \Phi_A(3,1) = \begin{bmatrix} e_{1/t}(3,1) & 0 \\ 0 & e_{1/t}(3,1) \end{bmatrix} = M.$$

On the other hand, by (4.6) and (4.7) one can obtain

$$\begin{aligned} e_R(t,1) &= M^{\frac{1}{3}\Theta(t)} \\ &= \begin{cases} M^{\frac{1}{3}\left[3m(t) - 3^{m(t)}/t\right]} & \text{if } t \notin P(1) \\ M^{m(t)} & \text{if } t \in P(1) \end{cases}, \end{aligned}$$

and

$$\begin{aligned} R(t) &= \lim_{s \to t} \frac{M^{\frac{1}{3}[\Theta(\sigma(t)) - \Theta(s)]} - I}{\sigma(t) - s} \\ &= \begin{cases} \frac{2}{t}\left(M^{\frac{1}{3}\left[\Theta\left(\frac{3}{2}t\right) - \Theta(t)\right]} - I\right) & \text{if } \sigma(t) > t \\ \left[\frac{1}{3}\Theta(t)\right]^{\Delta} Log\left[M\right] & \text{if } \sigma(t) = t \end{cases}, \end{aligned}$$

where $P(t)$ and $m(t)$ are defined by (4.2) and (4.4), respectively. Then the matrix function $L(t)$ which is 3-periodic in shifts is given as

$$\begin{aligned} L(t) &= \Phi_A(t,1)\, e_R^{-1}(t,1) \\ &= \begin{bmatrix} e_{1/t}(t,1) & 0 \\ 0 & e_{1/t}(t,1) \end{bmatrix} \begin{bmatrix} e_{1/t}(3,1) & 0 \\ 0 & e_{1/t}(3,1) \end{bmatrix}^{-\frac{1}{3}\Theta(t)}. \end{aligned}$$

Example 4.2. Consider the time scale $\mathbb{T} = \mathbb{R}$ that is periodic in shifts $\delta_{\pm}(s,t) = s^{\pm 1}t$ associated with the initial point $t_0 = 1$. Let us define the matrix function $A(t) : \mathbb{T}^* \to \mathbb{R}^{n \times n}$ as follows

$$A(t) = \begin{bmatrix} \frac{1}{t}\sin\left(\pi\frac{\ln|t|}{\ln 2}\right) & 0 \\ 0 & \frac{1}{t}\sin\left(\pi\frac{\ln|t|}{\ln 2}\right) \end{bmatrix}.$$

Then $A(t)$ is Δ-periodic in shifts with the period 4. The following system

$$x^{\Delta}(t) = \begin{bmatrix} \frac{1}{t}\sin\left(\pi\frac{\ln t}{\ln 2}\right) & 0 \\ 0 & \frac{1}{t}\sin\left(\pi\frac{\ln t}{\ln 2}\right) \end{bmatrix} x(t), x(1) = x_0$$

has the transition matrix

$$\Phi_A(t,1) = \begin{bmatrix} e_{u(t)}(t,1) & 0 \\ 0 & e_{u(t)}(t,1) \end{bmatrix},$$

where $u(t) = \frac{1}{t}\sin\left(\pi\frac{\ln t}{\ln 2}\right)$. Moreover,

$$\Phi_A\left(\delta_+^4(1),1\right) = \Phi_A(4,1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M.$$

Thus, $R(t)$ is 2×2 zero matrix, and hence, $e_R(t,1) = I$. Finally, the matrix function $L(t)$ which is 4-periodic in shifts is obtained as follows:

$$\begin{aligned} L(t) &= \Phi_A(t,1)\, e_R^{-1}(t,1) \\ &= \Phi_A(t,1). \end{aligned}$$

4.2 Floquet theory based on new periodicity concept: Nonhomogeneous case

Let us focus on the nonhomogeneous regressive time varying linear dynamic initial value problem

$$x^{\Delta}(t) = A(t)x(t) + F(t), \;\; x(t_0) = x_0, \tag{4.16}$$

where $A : \mathbb{T}^* \to \mathbb{R}^{n\times n}$, $F \in \mathcal{R}(\mathbb{T}^*, \mathbb{R}^n)$. Hereafter, both A and F are supposed to be Δ-periodic in shifts with the period T. Similar to Lemma 3.8, one can prove the following result.

Lemma 4.8 *A solution $x(t)$ of (3.14) is T-periodic in shifts if and only if $x\left(\delta_+^T(t_0)\right) = x(t_0)$ for all $t \in \mathbb{T}^*$.*

Theorem 4.9 *The solution of (4.16)is T-periodic in shifts $\delta_{\pm}$ for any inital point t_0, and corresponding initial state $x(t_0) = x_0$ if and only if the T-periodic homogeneous initial value problem*

$$z^{\Delta}(t) = A(t) z(t), \; z(t_0) = z_0, \tag{4.17}$$

has not a T-periodic solution in shifts for any nonzero initial state $z(t_0) = z_0$.

Proof. In [13], the following representation for the solution of (4.16) is given

$$x(t) = X(t) X^{-1}(\tau) x_0 + \int_{\tau}^{t} X(t) X^{-1}(\sigma(s)) F(s) \Delta s,$$

where $X(t)$ is a fundamental matrix solution of the homogenous system (4.1) with respect to initial condition $x(\tau) = x_0$. As it is done in [13], $x(t)$ can be expressed as

$$x(t) = \Phi_A(t, t_0) x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s)) F(s) \Delta s.$$

By the previous lemma $x(t)$ is T-periodic in shifts if and only if $x\left(\delta_+^T(t_0)\right) = x_0$ or equivalently

$$\left[I - \Phi_A\left(\delta_+^T(t_0), t_0\right)\right] x_0 = \int_{t_0}^{\delta_+^T(t_0)} \Phi_A\left(\delta_+^T(t_0), \sigma(s)\right) F(s) \Delta s. \tag{4.18}$$

By guidance of Theorem 4.7, it should be shown that (4.16) has a solution with respect to initial condition $x(t_0) = x_0$ if and only if $e_R\left(\delta_+^T(t_0), t_0\right)$ has no eigenvalues equal to 1.

Let $e_R\left(\delta_+^T(\eta), \eta\right) = \Phi_A\left(\delta_+^T(\eta), \eta\right)$, for some $\eta \in \mathbb{T}^*$, has no eigenvalues equal to 1. That is,

$$\det\left[I - \Phi_A\left(\delta_+^T(\eta), \eta\right)\right] \neq 0.$$

Invertibility and periodicity of Φ_A imply

$$\begin{aligned} 0 &\neq \det\left[\Phi_A\left(\delta_+^T(t_0),\delta_+^T(\eta)\right)\left(I-\Phi_A\left(\delta_+^T(\eta),\eta\right)\right)\Phi_A(\eta,t_0)\right] \\ &= \det\left[\Phi_A\left(\delta_+^T(t_0),\delta_+^T(\eta)\right)\Phi_A(\eta,t_0)-\Phi_A\left(\delta_+^T(t_0),t_0\right)\right]. \end{aligned} \tag{4.19}$$

By periodicity of Φ_A, the invertibility of $\left[I-\Phi_A\left(\delta_+^T(t_0),t_0\right)\right]$ is equivalent to (3.19) for any $t_0 \in \mathbb{T}^*$. Thus, (4.18) has a solution

$$x_0 = \left[I-\Phi_A\left(\delta_+^T(t_0),t_0\right)\right]^{-1}\int_{t_0}^{\delta_+^T(t_0)}\Phi_A\left(\delta_+^T(t_0),\sigma(s)\right)F(s)\Delta s$$

for any $t_0 \in \mathbb{T}^*$ and for any Δ-periodic function F in shifts with period T.

Suppose that (4.18) has a solution for every $t_0 \in \mathbb{T}^*$ and every Δ-periodic function F in shifts with period T. Let us define the set $P_-(t)$ as

$$P_-(t)=\left\{k\in\mathbb{Z}:\delta_-^{(k)}(T,t)\right\}.$$

It is clear that, $P_-(t)=P_-\left(\delta_+^T(t)\right)$. Additionally, let the function ξ be defined by

$$\begin{aligned} \xi(t) &:= \prod_{s\in P_-(t)\cap[t_0,t)}\left(\delta_+^{\Delta T}(s)\right)^{-1} \\ &= \left(\delta_+^{\Delta T}(\delta_-(T,t))\right)^{-1}\times\left(\delta_+^{\Delta T}\left(\delta_-^{(2)}(T,t)\right)\right)^{-1}\times\ldots\times\left(\delta_+^{\Delta T}\left(\delta_-^{\left(m^-(t)\right)}(T,t)\right)\right)^{-1}, \end{aligned}$$

where $m^-(t)=\max\left\{k\in\mathbb{Z}:\delta_-^{(k)}(T,t)\geq t_0\right\}$. By definition of ξ, one can write

$$\begin{aligned} \xi\left(\delta_+^T(t)\right) &= \prod_{s\in P_-\left(\delta_+^T(t)\right)\cap\left[t_0,\delta_+^T(t)\right)}\left(\delta_+^{\Delta T}(s)\right)^{-1} \\ &= \prod_{s\in P_-(t)\cap\left[t_0,\delta_+^T(t)\right)}\left(\delta_+^{\Delta T}(s)\right)^{-1} \end{aligned}$$

$$= \left(\delta_{+}^{\Delta T}(t)\right)^{-1} \prod_{s \in P_{-}(t) \cap [t_0, t)} \left(\delta_{+}^{\Delta T}(s)\right)^{-1}$$
$$= \left(\delta_{+}^{\Delta T}(t)\right)^{-1} \xi(t),$$

which shows that ξ is Δ-periodic in shifts with period T. For an arbitrary t_0 and corresponding F_0, define a regressive and Δ-periodic function F in shifts as follows

$$F(t) := \Phi_A\left(\sigma(t), \delta_{+}^{T}(t_0)\right) \xi(t) F_0, \quad t \in \left[t_0, \delta_{+}^{T}(t_0)\right) \cap \mathbb{T}. \tag{4.20}$$

Then,

$$\int_{t_0}^{\delta_{+}^{T}(t_0)} \Phi_A\left(\delta_{+}^{T}(t_0), \sigma(s)\right) F(s) \Delta s = F_0 \int_{t_0}^{\delta_{+}^{T}(t_0)} \xi(s) \Delta s. \tag{4.21}$$

Thus, (4.18) can be rewritten as follows

$$\left[I - \Phi_A\left(\delta_{+}^{T}(t_0), t_0\right)\right] x_0 = \int_{t_0}^{\delta_{+}^{T}(t_0)} \xi(s) \Delta s. \tag{4.22}$$

For any F which is defined in (4.20), and hence for any corresponding F_0, (4.22) has a solution for x_0 by assumption. Thus,

$$\det\left[I - \Phi_A\left(\delta_{+}^{T}(t_0), t_0\right)\right] \neq 0.$$

Consequently, $e_R\left(\delta_{+}^{T}(t_0), t_0\right) = \Phi_A\left(\delta_{+}^{T}(t_0), t_0\right)$ has no eigenvalue 1. Then, one can conclude by Theorem 4.7, (4.17) has no periodic solution in shifts. $\square$

4.3 Floquet Multipliers and Floquet exponents of unified Floquet systems

This part of the thesis is devoted to Floquet multipliers and Floquet exponents of systems periodic with respect to new periodicity perception on time scales. Similar to the q-Floquet theory, let $\Phi_A(t, t_0)$ be the transition matrix and $\Phi(t)$ the fundamental matrix at $t = \tau$ of the system (4.1). Then any fundamental matrix $\Psi(t)$ can be represented as follows:

$$\Psi(t) = \Phi(t)\Psi(\tau) \text{ or } \Psi(t) = \Phi_A(t, t_0)\Psi(t_0). \tag{4.23}$$

Additionally, for a nonzero initial vector $x_0 \in \mathbb{R}^n$, the monodromy operator $M : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$M(x_0) := \Phi_A\left(\delta_+^T(t_0), t_0\right)x_0 = \Psi\left(\delta_+^T(t_0)\right)\Psi^{-1}(t_0)x_{0.} \tag{4.24}$$

In parallel to preceding chapter, the eigenvalues of monodromy operator M are called Floquet (characteristic) multipliers of the system (4.1).

Similar to [38, Theorem 5.2 (i)], the following remark can be given:

Remark. The monodromy operator of the linear system (4.1) is invertible and consequently, every Floquet (characteristic) multiplier is nonzero.

Theorem 4.10 *The monodromy operator M corresponding to different fundamental matrices of the system (4.1) is unique.*

Proof. Suppose that M_1 and M_2 are the monodromy operators corresponding to fundamental matrices $\Psi_1(t)$ and $\Psi_2(t)$, respectively. One can write the monodromy operator $M_2(x_0)$ corresponding to $\Psi_2(t)$ as

$$M_2(x_0) = \Psi_2\left(\delta_+^T(t_0)\right)\Psi_2^{-1}(t_0)x_0.$$

Using (4.23) yields

$$\begin{aligned} M_2(x_0) &= \Psi_2\left(\delta_+^T(t_0)\right)\Psi_2^{-1}(t_0)x_0 \\ &= \Psi_1\left(\delta_+^T(t_0)\right)\Psi_2(\tau)\Psi_2^{-1}(\tau)\Psi_1^{-1}(t_0)x_0 \\ &= \Psi_1\left(\delta_+^T(t_0)\right)\Psi_1^{-1}(t_0)x_0 \\ &= M_1(x_0). \end{aligned}$$

□

By using Theorem 4.5, (4.23) and (4.24), one can obtain

$$\Phi_A(t,t_0) = \Psi_1(t)\,\Psi_1^{-1}(t_0) = L(t)\,e_R(t,t_0)\,L^{-1}(t_0) \tag{4.25}$$

and

$$M(x_0) = \Phi_A\left(\delta_+^T(t_0),t_0\right)x_0 = \Psi_1\left(\delta_+^T(t_0)\right)\Psi_1^{-1}(t_0)\,x_0. \tag{4.26}$$

The following equation is obtained by combining (4.25) and (4.26)

$$\Phi_A\left(\delta_+^T(t_0),t_0\right) = \Psi_1\left(\delta_+^T(t_0)\right)\Psi_1^{-1}(t_0) = L\left(\delta_+^T(t_0)\right)e_R\left(\delta_+^T(t_0),t_0\right)L^{-1}\left(\delta_+^T(t_0)\right).$$

By using the periodicity in shifts of L, the following equality

$$\Phi_A\left(\delta_+^T(t_0),t_0\right) = L(t_0)\,e_R\left(\delta_+^T(t_0),t_0\right)L^{-1}(t_0). \tag{4.27}$$

is obtained. Hence, the Floquet multipliers of the unified Floquet system (4.1) are the eigenvalues of the matrix $e_R\left(\delta_+^T(t_0),t_0\right)$.

Definition 37. The Floquet exponent of the system (4.1) is the function $\gamma(t)$ satisfying the equation

$$e_\gamma\left(\delta_+^T(t_0),t_0\right) = \lambda,$$

where λ is the Floquet multiplier of the system.

The next result can be proven similar to [38, Theorem 5.3].

Theorem 4.11 *Let $R(t)$ be a matrix function as in Theorem 3.1, with eigenvalues $\gamma_1(t),\ldots,\gamma_n(t)$ repeated according to multiplicities. Then $\gamma_1^k(t),\ldots,\gamma_n^k(t)$ are the eigenvalues of $R^k(t)$ and eigenvalues of e_R are $e_{\gamma_1},\ldots,e_{\gamma_n}$.*

Theorem 4.12 *The Floquet exponent γ of (4.1) with corresponding Floquet multiplier λ is not unique. That is, $\gamma(t)\oplus \mathring{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}$ is also a Floquet exponent for (4.1) for all $k\in\mathbb{Z}$.*

Proof. For all $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
e_{\gamma \oplus \mathring{\imath} \frac{2\pi k}{\delta_+^T(t_0)-t_0}}\left(\delta_+^T(t_0), t_0\right) &= e_\gamma\left(\delta_+^T(t_0), t_0\right) e_{\mathring{\imath} \frac{2\pi k}{\delta_+^T(t_0)-t_0}}\left(\delta_+^T(t_0), t_0\right) \\
&= e_\gamma\left(\delta_+^T(t_0), t_0\right) \exp\left(\int_{t_0}^{\delta_+^T(t_0)} \frac{\log\left(1+\mu(\tau)\, \mathring{\imath} \frac{2\pi k}{\delta_+^T(t_0)-t_0}\right)}{\mu(\tau)} \Delta\tau\right) \\
&= e_\gamma\left(\delta_+^T(t_0), t_0\right) \exp\left(\int_{t_0}^{\delta_+^T(t_0)} \frac{\log\left(\exp\left(i \frac{2\pi k \mu(\tau)}{\delta_+^T(t_0)-t_0}\right)\right)}{\mu(\tau)} \Delta\tau\right) \\
&= e_\gamma\left(\delta_+^T(t_0), t_0\right) \exp\left(\int_{t_0}^{\delta_+^T(t_0)} \frac{i2\pi k}{\delta_+^T(t_0)-t_0} \Delta\tau\right) \\
&= e_\gamma\left(\delta_+^T(t_0), t_0\right) e^{i2\pi k} \\
&= e_\gamma\left(\delta_+^T(t_0), t_0\right),
\end{aligned}
$$

which gives the desired result. □

Lemma 4.13 *Let $\mathbb{T}$ be a time scale that is p-periodic in shifts $\delta_{\pm}$ associated with the initial point t_0 and $k \in \mathbb{Z}$. If $\frac{\delta_+^p(t)-t}{\delta_+^p(t_0)-t_0} \in \mathbb{Z}$, then the functions $e_{\mathring{\imath} \frac{2\pi k}{\delta_+^p(t_0)-t_0}}$ and $e_{\ominus \mathring{\imath} \frac{2\pi k}{\delta_+^p(t_0)-t_0}}$ are p periodic in shifts.*

Proof. If $\frac{\delta_+^p(t)-t}{\delta_+^p(t_0)-t_0} \in \mathbb{Z}$, then

$$
\begin{aligned}
e_{\mathring{\imath} \frac{2\pi k}{\delta_+^p(t_0)-t_0}}\left(\delta_+^p(t), t_0\right) &= \exp\left(\int_{t_0}^{\delta_+^p(t)} \frac{i2\pi k}{\delta_+^p(t_0)-t_0} \Delta\tau\right) \\
&= \exp\left(\int_{t}^{\delta_+^p(t)} \frac{i2\pi k}{\delta_+^p(t_0)-t_0} \Delta\tau\right) \exp\left(\int_{t_0}^{t} \frac{i2\pi k}{\delta_+^p(t_0)-t_0} \Delta\tau\right) \\
&= \exp\left(i2\pi k \frac{\delta_+^p(t)-t}{\delta_+^p(t_0)-t_0}\right) \exp\left(\int_{t_0}^{t} \frac{i2\pi k}{\delta_+^p(t_0)-t_0} \Delta\tau\right)
\end{aligned}
$$

$$= \exp\left(\int_{t_0}^{t} \frac{i2\pi k}{\delta_+^p(t_0) - t_0}\Delta\tau\right) = e_{\overset{\circ}{i}\frac{2\pi k}{\delta_+^p(t_0)-t_0}}(t,t_0)$$

which proves the periodicity of $e_{\overset{\circ}{i}\frac{2\pi k}{\delta_+^p(t_0)-t_0}}$. The periodicity of $e_{\ominus\overset{\circ}{i}\frac{2\pi k}{\delta_+^p(t_0)-t_0}}$ can be proven by using the periodicity of $e_{\overset{\circ}{i}\frac{2\pi k}{\delta_+^p(t_0)-t_0}}$ and the identity $e_{\ominus\alpha} = 1/e_\alpha$. □

Remark. Notice that the condition $\frac{\delta_+^p(t)-t}{\delta_+^p(t_0)-t_0} \in \mathbb{Z}$ holds not only for all additive periodic time scales but also for the many time scales that are periodic in shifts. For example for the time scales $\overline{2^{\mathbb{Z}}}$ and $\cup_{k=0}^{\infty}\left[3^{\pm k}, 2.3^{\pm(k+1)}\right]\cup\{0\}$ periodic in shifts $\delta_\pm(s,t) = s^{\pm1}t$ associated with the initial point $t_0 = 1$, the condition $\frac{\delta_+^p(t)-t}{\delta_+^p(t_0)-t_0} \in \mathbb{Z}$ is always satisfied for $p = 2$ and $p = 3$, respectively.

Theorem 4.14 *If the unified Floquet system (4.1) has a Floquet exponent $\gamma(t)$, then the corresponding transition matrix Φ_A can be decomposed as*

$$\Phi_A(t,t_0) = L(t)\, e_R(t,t_0),$$

where $\gamma(t)$ is an eigenvalue of $R(t)$.

Proof. Consider the Floquet decomposition $\Phi_A(t,t_0) = \widetilde{L}(t)\, e_{\widetilde{R}}(t,t_0)$ and let γ be a Floquet exponent of (4.1) with corresponding Floquet multiplier λ. Moreover, there is an eigenvalue $\tilde{\gamma}(t)$ of $\tilde{R}(t)$ so that $e_{\tilde{\gamma}}\left(\delta_+^T(t_0), t_0\right) = \lambda$, where $\tilde{\gamma}(t)$ can be defined as

$$\tilde{\gamma}(t) := \gamma(t) \oplus \overset{\circ}{i}\frac{2\pi k}{\delta_+^T(t_0) - t_0}$$

by Theorem 4.12. Setting

$$R(t) := \widetilde{R}(t) \ominus \overset{\circ}{i}\frac{2\pi k}{\delta_+^T(t_0) - t_0} I$$

and

$$L(t) := \tilde{L}(t)\, e_{\overset{\circ}{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0} I}(t,t_0),$$

then one can write

$$\widetilde{R}(t) := R(t) \oplus \overset{\circ}{i}\frac{2\pi k}{\delta_+^T(t_0) - t_0} I,$$

and hence,

$$L(t)\, e_R(t,t_0) = \tilde{L}(t)\, e_{\overset{\circ}{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}I}(t,t_0)\, e_R(t,t_0) = \tilde{L}(t)\, e_{\overset{\circ}{i}\frac{2\pi k}{\delta_+^T(t_0)-t_0}I\oplus R}(t,t_0) = \tilde{L}(t)\, e_{\tilde{R}}(t,t_0).$$

This shows that $\Phi_A(t,t_0) = L(t)\, e_R(t,t_0)$ is an alternative Floquet decomposition where $\gamma(t)$ is an eigenvalue of $R(t)$. □

Theorem 4.15 *Let $\gamma(t)$ be a Floquet exponent of the system (4.1) and λ be the corresponding Floquet multiplier. Then, the unified Floquet system (4.1) has a nontrivial solution of the form*

$$x(t) = e_\gamma(t,t_0)\, \kappa(t) \tag{4.28}$$

satisfying

$$x\left(\delta_+^T(t)\right) = \lambda x(t),$$

where κ is a T-periodic function in shifts.

Proof. Let $\Phi_A(t,t_0)$ be the transition matrix of (4.1) and $\Phi_A(t,t_0) = L(t)\, e_R(t,t_0)$ is Floquet decomposition such that $\gamma(t)$ is an eigenvalue of $R(t)$. There exists a nonzero vector $u \neq 0$ such that $R(t)\, u = \gamma(t)\, u$, and therefore, $e_R(t,t_0)\, u = e_\gamma(t,t_0)\, u$. Then, the solution $x(t) := \Phi_A(t,t_0)\, u$ can be represented as follows

$$x(t) = L(t)\, e_R(t,t_0)\, u = e_\gamma(t,t_0)\, L(t)\, u.$$

Setting $\kappa(t) = L(t)\, u$, the last equality implies (4.28). Thus, the first part of the theorem is proven. The second part is proven by the following equality.

$$\begin{aligned} x\left(\delta_+^T(t)\right) &= e_\gamma\left(\delta_+^T(t), t_0\right) q\left(\delta_+^T(t)\right) \\ &= e_\gamma\left(\delta_+^T(t), \delta_+^T(t_0)\right) e_\gamma\left(\delta_+^T(t_0), t_0\right) q(t) \\ &= e_\gamma\left(\delta_+^T(t_0), t_0\right) e_\gamma(t,t_0)\, L(t)\, u \\ &= e_\gamma\left(\delta_+^T(t_0), t_0\right) x(t) \\ &= \lambda x(t). \end{aligned}$$

□

Next result shows that two solutions of (4.1) according to two distinct Floquet multipliers are linearly independent.

Theorem 4.16 *Let λ_1 and λ_2 be the characteristic multipliers of the system (4.1) and γ_1 and γ_2 are Floquet exponents such that*

$$e_{\gamma_i}(\delta_+^T(t_0), t_0) = \lambda_i, \quad i = 1, 2.$$

If $\lambda_1 \neq \lambda_2$, then there exist T-periodic functions κ_1 and κ_2 in shifts such that

$$x_i(t) = e_{\gamma_i}(t, t_0)\kappa_i(t), \ i = 1, 2$$

are linearly independent solutions of the system (4.1).

Proof. Let $\Phi_A(t, t_0) = L(t)\, e_R(t, t_0)$ and $\gamma_1(t)$ be an eigenvalue of $R(t)$ corresponding to nonzero eigenvector v_1. Since λ_2 is an eigenvalue of $\Phi_A\left(\delta_+^T(t_0), t_0\right)$, by Theorem 3.12 there is an eigenvalue $\gamma(t)$ of $R(t)$ satisfying

$$e_\gamma\left(\delta_+^T(t_0), t_0\right) = \lambda_2 = e_{\gamma_2}\left(\delta_+^T(t_0), t_0\right).$$

Hence, for some $k \in \mathbb{Z}$ we have $\gamma_2(t) = \gamma(t) \oplus \overset{\circ}{i} \frac{2\pi k}{\delta_+^T(t_0) - t_0}$. Furthermore, $\lambda_1 \neq \lambda_2$ implies that $\gamma(t) \neq \gamma_1(t)$. If v_2 is a nonzero eigenvector of $R(t)$ corresponding to eigenvalue $\gamma(t)$, then the eigenvectors v_1 and v_2 are linearly independent. Similar to the related part in the proof of Theorem 4.15, one can state the solutions of the system (4.1) as follows:

$$x_1(t) = e_{\gamma_1}(t, t_0)\, L(t)\, v_1 \tag{4.29}$$

and

$$x_2(t) = e_\gamma(t, t_0)\, L(t)\, v_2.$$

Since $x_1(t_0) = L(t_0)v_1$ and $x_2(t_0) = L(t_0)v_2$, the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Moreover, the solution x_2 can be rewritten in the following form

$$\begin{aligned} x_2(t) &= e_{\gamma_2}(t,t_0)\, e_{\gamma\ominus\gamma_2}(t,t_0) L(t)\nu_2 \\ &= e_{\gamma_2}(t,t_0)\, e_{\ominus \mathring{\iota}\frac{2\pi k}{\delta_+^T(t_0)-t_0}}(t,t_0)\, L(t)\nu_2. \end{aligned} \tag{4.30}$$

Letting $\kappa_1(t) = L(t)\, v_1$ and $\kappa_2(t) = e_{\ominus \mathring{\iota}\frac{2\pi k}{\delta_+^T(t_0)-t_0}}(t,t_0)\, L(t)\nu_2$ in (4.29) and (4.30), respectively, we complete the proof. □

4.4 Stability properties of unified Floquet systems

In this section, the unified Floquet theory established in previous sections is employed to investigate the stability characteristics of the regressive periodic system

$$x^{\Delta}(t) = A(t)\, x(t),\ \ x(t_0) = x_0. \tag{4.31}$$

By Theorem 4.1, the matrix R in the Floquet decomposition of Φ_A is given by

$$R(t) = \lim_{s\to t} \frac{\Phi_A\left(\delta_+^T(t_0), t_0\right)^{\frac{1}{T}[\Theta(\sigma(t))-\Theta(s)]} - I}{\sigma(t) - s}. \tag{4.32}$$

Also, Theorem 4.6 concludes that the solution $z(t)$ of the regressive system

$$z^{\Delta}(t) = R(t)\, z(t),\ \ z(t_0) = x_0 \tag{4.33}$$

can be expressed in terms of the solution $x(t)$ of the system (4.31) as follows: $z(t) = L^{-1}(t)x(t)$, where $L(t)$ is the Lyapunov transformation given by (4.13). In preparation for the main result, the following definitions and results which can be found in [37] and [38] are presented for stability and asymptotical stability properties of the solution of (4.31). Furthermore, the exponential stability definition is given according to [30].

Definition 38 (Stability)**.** The unified Floquet system (4.31) is uniformly stable if there exists a constant $\alpha > 0$ such that the following inequality

$$\|x(t)\| \leq \alpha \|x(t_0)\|, \ t \geq t_0$$

holds for any initial state and corresponding solution.

Theorem 4.17 *Let Φ_A be the transition matrix of the system (4.31). Then, (4.31) is uniformly stable if and only if there exists a $\alpha > 0$ such that the inequality*

$$\|\Phi_A(t, t_0)\| \leq \alpha, \ t \geq t_0$$

satisfied.

Definition 39 (Asymptotical stability)**.** In addition to uniform stability condition, if for any given $c > 0$ there exists a $K > 0$ such that the inequality

$$\|x(t)\| \leq c \|x(t_0)\|, \ t \geq t_0 + K$$

holds, then the unified Floquet system (4.31) is uniformly asymptotically stable.

Definition 40 (Exponential stability)**.** The unified Floquet system (4.31) is uniformly exponentially stable if there exist $\alpha, \beta > 0$ such that the inequality

$$\|x(t)\| \leq \|x(t_0)\| \alpha e_{\ominus\beta}(t, t_0), \ t \geq t_0$$

holds for any initial state and corresponding solution.

Moreover, necessary and sufficient conditions for exponential stability can be stated as the following:

Theorem 4.18 *The system (4.31) is uniformly exponentially stable if and only if there exist α, $\beta > 0$ with such that the inequality*

$$\|\Phi_A(t, t_0)\| \leq \alpha e_{\ominus\beta}(t, t_0), \ t \geq t_0$$

is satisfied for the transition matrix Φ_A.

Definition 41. ([67] See also [38, Definition 7.1]) The scalar function $\gamma : \mathbb{T}^* \to \mathbb{C}$ is uniformly regressive if there exists a constant $\theta > 0$ such that $0 < \theta^{-1} \leq |1 + \mu(t)\gamma(t)|$, for all $t \in \mathbb{T}^\kappa$.

Lemma 4.19 *Each eigenvalue of the matrix $R(t)$ in (4.33) is uniformly regressive.*

Proof. Define $\Lambda(t, s)$ by

$$\Lambda(t, s) := \Theta(\sigma(t)) - \Theta(s). \tag{4.34}$$

As it is done in Corollary 4.2, let

$$\gamma_i(t) = \lim_{s \to t}\left(\frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)} - 1}{\sigma(t) - s}\right), \ i = 1, 2, ..., k$$

be any of the $k \leq n$ distinct eigenvalues of $R(t)$. Now, there are two cases:

1. If $|\lambda_i| \geq 1$, then

$$|1 + \mu(t)\gamma_i(t)| = \lim_{s \to t}\left|1 + \mu(s)\frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)} - 1}{\sigma(t) - s}\right| = \lim_{s \to t}\left|\lambda_i^{\frac{1}{T}\Lambda(t,s)}\right| > 1.$$

2. If $0 \leq |\lambda_i| < 1$, then,

$$|1 + \mu(t)\gamma_i(t)| = \lim_{s \to t}\left|1 + \mu(s)\frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)} - 1}{\sigma(t) - s}\right| = \lim_{s \to t}\left|\lambda_i^{\frac{1}{T}\Lambda(t,s)}\right| \geq |\lambda_i|.$$

Setting $\theta^{-1} := \min\{1, |\lambda_1|, \ldots, |\lambda_k|\}$, then one can obtain

$$0 < \theta^{-1} < |1 + \mu(t)\gamma_i(t)|.$$

□

Definition 42. [38, Definition 7.3] A matrix function $H(t)$ is said to admit a dynamic eigenvector $w(t)$, where $w(t)$ is a Δ-differentiable nonzero vector function, if the following equality is satisfied

$$w^{\Delta}(t) = H(t)\,w(t) - \xi(t)\,w^{\sigma}(t)\,,\ \ t \in \mathbb{T}^{k} \tag{4.35}$$

for corresponding dynamic eigenvalue $\xi(t)$. Then the pair $\{\xi(t), w(t)\}$ is called a dynamic eigenpair. Additionally, the mode vector χ for a function $H(t)$ is given by

$$\chi_i := e_{\xi_i}(t, t_0)\,w_i(t)\,, \tag{4.36}$$

where $\{\xi_i(t), w_i(t)\}$ is associated dynamic eigenpair.

The following results can be proven similar to [38, Lemma 7.4, Theorem 7.5]:

Lemma 4.20 *Any regressive matrix function H has n dynamic eigenpairs with linearly independent eigenvectors. Moreover, if the dynamic eigenvectors form the columns of a matrix function $W(t)$, then W satisfies the matrix dynamic eigenvalue problem*

$$W^{\Delta}(t) = H(t)\,W(t) - W^{\sigma}(t)\,\Xi(t)\,,\ \ \textit{where}\ \Xi(t) := diag\,[\xi_1(t), \ldots, \xi_n(t)]\,. \tag{4.37}$$

where $\Xi(t) := diag\,[\xi_1(t), \ldots, \xi_n(t)]\,.$

Theorem 4.21 *Solutions to the uniformly regressive (but not necessarily periodic) time varying linear dynamic system (4.31) are:*

1. *stable if and only if there exists a $\gamma > 0$ such that every mode vector $\chi_i(t)$ of $A(t)$ satisfies $\|\chi_i(t)\| \leq \gamma < \infty$, $t > t_0$, for all $1 \leq i \leq n$;*
2. *asymptotically stable if and only if, in addition to (1), $\|\chi_i(t)\| \to 0$, $t > t_0$, for all $1 \leq i \leq n$,*
3. *exponentially stable if and only if there exists $\gamma, \lambda > 0$ such that $\|\chi_i(t)\| \leq \gamma e_{\ominus\lambda}(t, t_0)$, $t > t_0$, for all $1 \leq i \leq n$.*

Definition 43. For each $k \in \mathbb{N}_0$ the mappings $h_k : \mathbb{T}^{*} \times \mathbb{T}^{*} \to \mathbb{R}^{+}$, recursively defined by

$$h_0(t,t_0) :\equiv 1, \quad h_{k+1}(t,t_0) = \int_{t_0}^{t} \left(\lim_{s\to\tau} \frac{\Lambda(\tau,s)}{\sigma(\tau)-s} \right) h_k(\tau,t_0)\Delta\tau \text{ for } n \in \mathbb{N}_0, \tag{4.38}$$

are called monomials, where $\Lambda(t,s)$ is given by (4.34).

Remark. For an additive periodic time scale we always have $\Theta(t) = t - t_0$, and hence, $\Lambda(t,s) = \sigma(t) - s$.

Lemma 4.22 *Let $\mathbb{T}$ be a time scale which is unbounded above and $\gamma(t)$ be an eigenvalue of $R(t)$. If there exists a constant $H \geq t_0$ such that*

$$\inf_{t\in[H,\infty)_{\mathbb{T}}} \left[-\left(\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) \right)^{-1} \operatorname{Re}_{\mu} \gamma(t) \right] > 0 \tag{4.39}$$

holds, then

$$\lim_{t\to\infty} h_k(t,t_0)\, e_{\gamma}(t,t_0) = 0, \ \ k \in \mathbb{N}_0. \tag{4.40}$$

Proof. It suffices to show that $\lim_{t\to\infty} h_k(t,t_0)\, e_{\operatorname{Re}_{\mu}\gamma(t)}(t,t_0) = 0$ (see [47, Theorem 7.4]). Let's proceed by mathematical induction. For $k = 0$, it is known that $h_0(t,t_0) = 1$ and by [67], we have

$$\lim_{t\to\infty} e_{\operatorname{Re}_{\mu}\gamma_i(t)}(t,t_0) = 0 \text{ for } t_0 \in \mathbb{T}.$$

Suppose that it is true for a fixed $k \in \mathbb{N}$ and consider the $(k+1)^{th}$ step.

$$\begin{aligned}
&\lim_{t\to\infty} h_{k+1}(t,t_0)\, e_{\operatorname{Re}_{\mu}\gamma(t)}(t,t_0) \\
&= \lim_{t\to\infty} \left[\int_{t_0}^{t} \lim_{s\to\tau} \left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s} \right) h_k(\tau,t_0)\,\Delta\tau \right] e_{\ominus \operatorname{Re}_{\mu}\gamma(t)}(t,t_0)^{-1} \\
&= \lim_{t\to\infty} \left[\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) h_k(t,t_0) \right] \frac{e_{\operatorname{Re}_{\mu}\gamma(t)}(\sigma(t),t_0)}{-\operatorname{Re}_{\mu}\gamma(t)} \\
&= \lim_{t\to\infty} \left[\frac{\lim_{s\to t} \left(\frac{\Lambda(t,s)}{\sigma(t)-s} \right) h_k(t,t_0)\, e_{\operatorname{Re}_{\mu}\gamma(t)}(\sigma(t),t_0)}{-\operatorname{Re}_{\mu}\gamma(t)} \right],
\end{aligned} \tag{4.41}$$

where (4.39) is used together with [24, Theorem 1.120] and [27, Theorem 3.4] to obtain the second equality. The last term in (4.41) can be written as

$$\begin{aligned}
&\lim_{t\to\infty}\left[\frac{\lim_{s\to t}\left(\frac{\Lambda(t,s)}{\sigma(t)-s}\right)h_k\left(t,t_0\right)e_{\mathrm{Re}_\mu\gamma(t)}\left(\sigma(t),t_0\right)}{-\mathrm{Re}_\mu\gamma(t)}\right]\\
&=\lim_{t\to\infty}\left[\frac{\left(1+\mu(t)\mathrm{Re}_\mu\gamma(t)\right)h_k\left(t,t_0\right)e_{\mathrm{Re}_\mu\gamma(t)}\left(t,t_0\right)}{-\left(\lim_{s\to t}\left(\frac{\Lambda(t,s)}{\sigma(t)-s}\right)\right)^{-1}\mathrm{Re}_\mu\left(\gamma\left(t\right)\right)}\right]\\
&\leq\lim_{t\to\infty}\left[\frac{\left(1+\mu(t)\mathrm{Re}_\mu\gamma(t)\right)h_k\left(t,t_0\right)e_{\mathrm{Re}_\mu\gamma(t)}\left(t,t_0\right)}{\inf_{t\in[H,\infty)_{\mathbb{T}}}\left[-\left(\lim_{s\to t}\left(\frac{\Lambda(t,s)}{\sigma(t)-s}\right)\right)^{-1}\mathrm{Re}_\mu\left(\gamma\left(t\right)\right)\right]}\right].
\end{aligned}\tag{4.42}$$

Now, one may use (4.3) and (4.34) to get the inequality

$$1+\mu(t)\mathrm{Re}_\mu\gamma(t)=\left|1+\mu(t)\lim_{s\to t}\left(\frac{\lambda^{\frac{1}{T}\Lambda(t,s)}-1}{\sigma\left(t\right)-s}\right)\right|\leq\max\left\{1,|\lambda|\right\}$$

which along with (4.42) implies

$$\lim_{t\to\infty}h_{k+1}\left(t,t_0\right)e_{\mathrm{Re}_\mu\gamma(t)}\left(t,t_0\right)=0$$

as desired. □

Theorem 4.23 *Let $\{\gamma_i\left(t\right)\}_{i=1}^n$ be the set of conventional eigenvalues of the matrix $R(t)$ given in (4.32) and $\{w_i\left(t\right)\}_{i=1}^n$ be the set of corresponding linearly independent dynamic eigenvectors as defined by Lemma 4.20. Then, $\{\gamma_i\left(t\right),w_i\left(t\right)\}_{i=1}^n$ is a set of dynamic eigenpairs of $R(t)$ with the property that for each $1\leq i\leq n$ there are positive constants $D_i>0$ such that*

$$\left\|w_i\left(t\right)\right\|\leq D_i\sum_{k=0}^{m_i-1}h_k\left(t,t_0\right),\tag{4.43}$$

holds where $h_k\left(t,t_0\right)$, $k=0,1,...,m_i-1$, are the monomials defined as in (4.38) and m_i is the dimension of the Jordan block which contains the i^{th} eigenvalue, for all $1\leq i\leq n$.

Proof. By Lemma 4.20, it is obvious that, $\{\gamma_i(t), w_i(t)\}_{i=1}^n$ is the set of eigenpairs of $R(t)$. First, there exists an appropriate $n \times n$ constant, nonsingular matrix S which transforms $\Phi_A\left(\delta_+^T(t_0), t_0\right)$ to its Jordan canonical form given by

$$\begin{aligned} J &:= S^{-1}\Phi_A\left(\delta_+^T(t_0), t_0\right) S \\ &= \begin{bmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{m_d}(\lambda_d) \end{bmatrix}_{n\times n}, \end{aligned} \tag{4.44}$$

where $d \leq n$, $\sum_{i=1}^d m_i = n$, λ_i are the eigenvalues of $\Phi_A\left(\delta_+^T(t_0), t_0\right)$. By utilizing above determined matrix S, define the following:

$$\begin{aligned} K(t) &:= S^{-1}R(t)S \\ &= S^{-1}\left(\lim_{s\to t}\frac{\Phi_A\left(\delta_+^T(t_0), t_0\right)^{\frac{1}{T}\Lambda(t,s)} - I}{\sigma(t) - s}\right)S \\ &= \lim_{s\to t}\frac{S^{-1}\Phi_A\left(\delta_+^T(t_0), t_0\right)^{\frac{1}{T}\Lambda(t,s)} S - I}{\sigma(t) - s}. \end{aligned}$$

This along with [38, Theorem A.6] yields

$$K(t) = \lim_{s\to t}\frac{J^{\frac{1}{T}\Lambda(t,s)} - I}{\sigma(t) - s}.$$

Note that, $K(t)$ has the block diagonal form

$$K(t) = diag\left[K_1(t), \ldots, K_d(t)\right]$$

in which each $K_i(t)$ given by

$$K_i(t) := \lim_{s\to t} K_i(t) := \lim_{s\to t} \begin{bmatrix} \frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)}-1}{\sigma(t)-s} & \frac{\frac{1}{T}\Lambda(t,s)\lambda_i^{\frac{1}{T}\Lambda(t,s)-1}}{(\sigma(t)-s)2!} & \cdots & \frac{\left(\prod_{k=0}^{n-2}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)\lambda_i^{\frac{1}{T}\Lambda(t,s)-n+1}}{(n-1)!(\sigma(t)-s)} \\ & \frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)}-1}{\sigma(t)-s} & \cdots & \frac{\left(\prod_{k=0}^{n-3}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)\lambda_i^{\frac{1}{T}\Lambda(t,s)-n+2}}{(n-2)!(\sigma(t)-s)} \\ & & \ddots & \vdots \\ & & & \frac{\lambda_i^{\frac{1}{T}\Lambda(t,s)}-1}{\sigma(t)-s} \end{bmatrix}_{m_i\times m_i}.$$

It should be mentioned that, since $R(t)$ and $K(t)$ are similar, they have the same conventional eigenvalues

$$\gamma_i(t) = \lim_{s\to t}\left(\frac{\lambda_i^{\frac{1}{T}[\Lambda(t,s)]}-1}{\sigma(t)-s}\right),\ i = 1, 2, ..., n,$$

with corresponding multiplicities. Moreover, if we set the dynamic eigenvalues of $K(t)$ to be same as conventional eigenvalues $\gamma_i(t)$, then the corresponding dynamic eigenvectors $\{u_i(t)\}_{i=1}^n$ of $K(t)$ can be given by $u_i(t) = S^{-1}w_i(t)$.

This claim can be proven by showing that $\{\gamma_i(t), u_i(t)\}_{i=1}^n$ is a set of dynamic eigenpairs of $K(t)$. By Definition 42, one can write that

$$\begin{aligned} u_i^{\Delta}(t) &= S^{-1}w_i^{\Delta}(t) \\ &= S^{-1}R(t)\,w_i(t) - S^{-1}\gamma_i(t)\,w_i^{\sigma}(t) \\ &= K(t)\,S^{-1}w_i(t) - \gamma_i(t)\,S^{-1}w_i^{\sigma}(t) \\ &= K(t)\,u_i(t) - \gamma_i(t)\,u_i^{\sigma}(t), \end{aligned} \tag{4.45}$$

for all $1 \le i \le n$ and this proves our claim. Now, it should be shown that $u_i(t)$ satisfies (4.43). Since $\{\gamma_i(t), u_i(t)\}_{i=1}^n$ is the set of dynamic eigenpairs of $K(t)$, it satisfies (4.45) for all $1 \le i \le n$. By choosing the i^{th} block of $K(t)$ with dimension

$m_i \times m_i$, we can construct the following linear dynamic system:

$$v^{\Delta}(t) = \tilde{K}_i(t)\, v(t) = \lim_{s\to t}\begin{bmatrix} 0 & \frac{\frac{1}{T}\Lambda(t,s)}{(\sigma(t)-s)\lambda_i} & \frac{\left(\frac{1}{T}\Lambda(t,s)\right)\left(\frac{1}{T}\Lambda(t,s)-1\right)}{(\sigma(t)-s)\lambda_i 2!} & \cdots & \frac{\left(\prod_{k=0}^{n-2}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)}{(n-1)!(\sigma(t)-s)\lambda_i^{n-1}} \\ & 0 & \frac{\frac{1}{T}\Lambda(t,s)}{(\sigma(t)-s)\lambda_i} & & \frac{\left(\prod_{k=0}^{n-3}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)}{(n-2)!(\sigma(t)-s)\lambda_i^{n-2}} \\ & & 0 & \ddots & \vdots \\ & & & \ddots & \frac{\frac{1}{T}\Lambda(t,s)}{(\sigma(t)-s)\lambda_i} \\ & & & & 0 \end{bmatrix} v(t), \tag{4.46}$$

where $\tilde{K}_i(t)(t) := K_i(t) \ominus \gamma_i(t) I$. There are m_i linearly independent solutions of (4.46). Let us denote these solutions by $v_{i,j}(t)$, where i corresponds to the i^{th} block matrix $K_i(t)$ and $j = 1, \ldots, m_i$. For $1 \le i \le d$, define $l_i = \sum_{s=0}^{i-1} m_s$, with $m_0 = 0$. Then, the form of an arbitrary $n \times 1$ column vector u_{l_i+j} for $i \le j \le m$ can be given as

$$u_{l_i+j} = [\underbrace{0,\ldots,0}_{m_1+\ldots+m_{i-1}}, \underbrace{v_{i,j}^T(t)}_{m_i}, \underbrace{0,\ldots,0}_{m_{i+1},\ldots,m_d}]_{1\times n}. \tag{4.47}$$

Considering the all vector solutions of (4.45), the solution of the $n \times n$ matrix dynamic equation

$$U^{\Delta}(t) = K(t)\, U(t) - U^{\sigma}(t)\, \Gamma(t),$$

where $\Gamma(t) := diag[\gamma_1(t), \ldots, \gamma_n(t)]$, can be written as

$$U(t) := \left[u_1, \ldots, u_{m_1}, \ldots, u_{\left(\sum_{k=1}^{i-1} m_k\right)}, \ldots, u_{\left(\sum_{k=1}^{i} m_k\right)}, \ldots, u_{\left(\sum_{k=1}^{d} m_k\right)-1}, u_n\right]$$

$$= \begin{bmatrix} \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,m_1} \\ & v_{1,1} & \ddots & v_{1,m_1-1} \\ & & \ddots & \vdots \\ & & & v_{1,1} \end{bmatrix}_{m_1\times m_1} & & \\ & \ddots & \\ & & \begin{bmatrix} v_{d,1} & v_{d,2} & \cdots & v_{d,m_d} \\ & v_{d,1} & \ddots & v_{d,m_d-1} \\ & & \ddots & \vdots \\ & & & v_{d,1} \end{bmatrix}_{m_d\times m_d} \end{bmatrix}_{n\times n}.$$

The m_i linearly independent solutions of (4.46) have the form

$$\begin{aligned} v_{i,1}(t) &:= [v_{i,m_i}(t), 0, \ldots, 0]^T_{m_i\times 1}, \\ v_{i,2}(t) &:= [v_{i,m_i-1}(t), v_{i,m_i}(t), 0, \ldots, 0]^T_{m_i\times 1}, \\ &\vdots \\ v_{i,m_i}(t) &:= [v_{i,1}(t), v_{i,2}(t), \ldots, v_{i,m_i-1}(t), v_{i,m_i}(t)]^T_{m_i\times 1}. \end{aligned}$$

Then, we have the dynamic equations

$$\begin{aligned} v^{\Delta}_{i,m_i}(t) &= 0, \\ v^{\Delta}_{i,m_i-1}(t) &= \lim_{s\to t} \frac{[\Lambda(t,s)]}{T(\sigma(t)-s)\lambda_i} v_{i,m_i}(t), \\ v^{\Delta}_{i,m_i-2}(t) &= \lim_{s\to t} \frac{\left(\prod_{k=0}^{1}[\frac{1}{T}\Lambda(t,s)-k]\right)}{2(\sigma(t)-s)\lambda_i^2} v_{i,m_i}(t) + \lim_{s\to t} \frac{\Lambda(t,s)}{T(\sigma(t)-s)\lambda_i} v_{i,m_i-1}(t), \\ &\vdots \\ v^{\Delta}_{i,1}(t) &= \lim_{s\to t} \frac{\left(\prod_{k=0}^{m_i-2}[\frac{1}{T}\Lambda(t,s)-k]\right)}{(m_i-1)!(\sigma(t)-s)\lambda_i^{m_i-1}} v_{i,m_i}(t) \end{aligned}$$

$$+\lim_{s\to t}\frac{\left(\prod_{k=0}^{m_i-3}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)}{(m_i-2)!(\sigma(t)-s)\lambda^{m_i-2}}v_{i,m_i-1}\left(t\right)+$$
$$\ldots+\lim_{s\to t}\frac{\left(\prod_{k=0}^{1}\left[\frac{1}{T}\Lambda(t,s)-k\right]\right)}{2(\sigma(t)-s)\lambda_i^2}v_{i,3}\left(t\right)+\lim_{s\to t}\frac{\Lambda(t,s)}{T(\sigma(t)-s)\lambda_i}v_{i,2}\left(t\right).$$

Moreover, we have the following solutions:

$$v_{i,m_i}\left(t\right)=1,\quad v_{i,m_i-1}\left(t\right)=\int_{t_0}^{t}\lim_{s\to\tau}\frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i}v_{i,m_i}\left(\tau\right)\Delta\tau,$$

$$v_{i,m_i-2}\left(t\right)=\int_{t_0}^{t}\lim_{s\to\tau}\frac{\left(\prod_{k=0}^{1}\left[\frac{1}{T}\Lambda(\tau,s)-k\right]\right)}{2(\sigma(\tau)-s)\lambda_i^2}v_{i,m_i}\left(\tau\right)\Delta\tau+\int_{t_0}^{t}\lim_{s\to\tau}\frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i}v_{i,m_i-1}\left(\tau\right)\Delta\tau,$$
$$\vdots$$
$$\begin{aligned}v_{i,1}(t)=&\int_{t_0}^{t}\lim_{s\to\tau}\frac{\left(\prod_{k=0}^{m_i-2}\frac{1}{T}\Lambda(\tau,s)-k]\right)}{(m_i-1)!(\sigma(\tau)-s)\lambda_i^{m_i-1}}v_{i,m_i}\left(\tau\right)\Delta\tau\\&+\int_{t_0}^{t}\lim_{s\to\tau}\frac{\left(\prod_{k=0}^{m_i-3}\frac{1}{T}\Lambda(\tau,s)-k]\right)}{(m_i-2)!(\sigma(\tau)-s)\lambda_i^{m_i-2}}v_{i,m_i-1}(\tau)\Delta\tau+\\&\ldots+\int_{t_0}^{t}\lim_{s\to\tau}\frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i}v_{i,2}\left(\tau\right)\Delta\tau.\end{aligned}$$

Then we can show that each $v_{i,j}$ is bounded. There exist constants $B_{i,j}$, $i=1,\ldots,d$ and $j=1,\ldots,m_i$, such that

$$\left|v_{i,m_i}\left(t\right)\right|=1\leq B_{i,m_i}h_0\left(t,t_0\right)=B_{i,m_i},$$

$$
\begin{aligned}
|v_{i,m_i-1}(t)| &\leq \int_{t_0}^{t} \left| \lim_{s\to\tau} \left(\frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i} \right) v_{i,m_i}(\tau) \right| \Delta\tau \leq \frac{1}{T|\lambda_i|} \int_{t_0}^{t} \lim_{s\to\tau} \left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s} \right) h_0(\tau,t_0)\,\Delta\tau \\
&\leq \frac{h_1(t,t_0)}{T|\lambda_i|} \leq B_{i,m_i-1} h_1(t,t_0), \\
|v_{i,m_i-2}(t)| &\leq \int_{t_0}^{t} \left| \lim_{s\to\tau} \frac{\left(\prod_{k=0}^{1} [\frac{1}{T}\Lambda(\tau,s)-k] \right)}{2(\sigma(\tau)-s)\lambda_i^2} \right| v_{i,m_i}(\tau)\,\Delta\tau \\
&\quad + \int_{t_0}^{t} \lim_{s\to\tau} \left(\frac{\Lambda(\tau,s)}{T(\sigma(\tau)-s)\lambda_i} \right) v_{i,m_i-1}(\tau)\,\Delta\tau.
\end{aligned}
$$

Since

$$0 \leq \Theta(\sigma(\tau)) - \Theta(s) \leq T \text{ as } s \to \tau,$$

we get

$$\left| \frac{1}{T}\Lambda(\tau,s) - k \right| \leq k \text{ as } s \to \tau \text{ for } k = 1, 2, \ldots.$$

Then

$$
\begin{aligned}
|v_{i,m_i-2}(t)| &\leq \frac{1}{2T\lambda_i^2} \int_{t_0}^{t} \lim_{s\to\tau} \left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s} \right) h_0(\tau,t_0)\,\Delta\tau \\
&\quad + \frac{1}{T^2\lambda_i^2} \int_{t_0}^{t} \lim_{s\to\tau} \left(\frac{\Lambda(\tau,s)}{\sigma(\tau)-s} \right) h_1(\tau,t_0)\,\Delta\tau \\
&= \frac{h_1(t,t_0)}{2T\lambda_i^2} + \frac{h_2(t,t_0)}{T^2\lambda_i^2} \\
&\leq B_{i,m_i-2} \sum_{j=1}^{2} h_j(t,t_0) \\
&\vdots \\
|v_{i,1}| &\leq B_{i,1} \sum_{j=1}^{m_i-1} h_j(t,t_0).
\end{aligned}
$$

Setting $\beta_i := \max_{j=1,...,m_i}\{B_{i,j}\}$ for each $1 \leq i \leq d$, we obtain

$$\|u_{l_i+j}(t)\| \leq \beta_i \sum_{k=0}^{m_i-1} h_k(t, t_0)$$

for $1 \leq i \leq d$ and $j = 1, 2, ..., m_i$. Since $w_i = Su_i$ we have

$$\begin{aligned}\|w_i(t)\| = \|Su_i(t)\| &\leq \|S\| \beta_i \sum_{k=0}^{m_i-1} h_k(t, t_0) \\ &= D_i \sum_{k=0}^{m_i-1} h_k(t, t_0),\end{aligned}$$

where $D_i := \|S\| \beta_i$, for all $1 \leq i \leq n$. The proof is complete. □

Theorem 4.24 (Unified Floquet stability theorem) *Let $\mathbb{T}$ be a periodic time scale in shifts that is unbounded above. We get the following stability results for the solutions of the system (4.31) based on the eigenvalues $\{\gamma_i(t)\}_{i=1}^n$ of system (4.33):*

1. *If there is a positive constant H such that*

$$\inf_{t\in[H,\infty)_{\mathbb{T}}}\left[-\left(\lim_{s\to t}\left(\frac{\Lambda(t,s)}{\sigma(t)-s}\right)\right)^{-1}\operatorname{Re}_{\mu}\gamma_i(t)\right] > 0 \tag{4.48}$$

   for all $i = 1, \ldots, n$, then the system (4.31) is asymptotically stable. Moreover, if there are positive constants H and ε such that (4.48) and

$$-\operatorname{Re}_{\mu}\gamma_i(t) > \varepsilon \tag{4.49}$$

   for all $t \in [H, \infty)_{\mathbb{T}}$ and all $i = 1, \ldots, n$, then the system (4.31) is exponentially stable.

2. *If there is a positive constant H such that*

$$\inf_{t\in[H,\infty)_{\mathbb{T}}}\left[-\left(\lim_{s\to t}\left(\frac{\Lambda(t,s)}{\sigma(t)-s}\right)\right)^{-1}\operatorname{Re}_{\mu}\gamma_i(t)\right] \geq 0 \tag{4.50}$$

for all $i = 1, \ldots, n$, *and if, for each characteristic exponent with*

$$\mathrm{Re}_{\mu}\left(\gamma_i\left(t\right)\right) = 0 \text{ for all } t \in [H, \infty)_{\mathbb{T}},$$

the algebraic multiplicity equals the geometric multiplicity, then the system (4.31) is stable; otherwise it is unstable.

3. *If* $\mathrm{Re}_{\mu}\left(\gamma_i\left(t\right)\right) > 0$ *for all* $t \in \mathbb{T}$ *and some* $i = 1, \ldots, n$, *then the system (4.31) is unstable.*

Proof. Let $e_R\left(t, t_0\right)$ be the transition matrix of the system (4.33) and $R\left(t\right)$ be defined as in (4.32). Given the conventional eigenvalues $\left\{\gamma_i\left(t\right)\right\}_{i=1}^{n}$ of $R\left(t\right)$, we can define the set of dynamic eigenpairs $\left\{\gamma_i\left(t\right), w_i\left(t\right)\right\}_{i=1}^{n}$ and from Theorem 4.23, the dynamic eigenvector $w_i\left(t\right)$ satisfies (4.43). Moreover, let us define $W(t)$ as the following:

$$W(t) = e_R\left(t, \tau\right) e_{\ominus\Xi}\left(t, \tau\right) \tag{4.51}$$

and we have

$$e_R\left(t, \tau\right) = W(t) e_{\Xi}\left(t, \tau\right), \tag{4.52}$$

where $\tau \in \mathbb{T}$ and $\Xi\left(t\right)$ is given as in Lemma 4.20. Employing (4.52), one can write that

$$e_R\left(\tau, t_0\right) = e_{\Xi}\left(\tau, t_0\right) W^{-1}(t_0). \tag{4.53}$$

By combining (4.52) and (4.53), the transition matrix of the system (4.33) can be represented by

$$e_R\left(t, t_0\right) = W\left(t\right) e_{\Xi}\left(t, t_0\right) W^{-1}\left(t_0\right), \tag{4.54}$$

where $W\left(t\right) := \left[w_1\left(t\right), w_2\left(t\right), \ldots, w_n\left(t\right)\right]$. Furthermore, the matrix $W^{-1}\left(t_0\right)$ can be denoted as follows:

$$W^{-1}\left(t_0\right) = \begin{bmatrix} v_1^T\left(t_0\right) \\ v_2^T\left(t_0\right) \\ \vdots \\ v_n^T\left(t_0\right) \end{bmatrix}.$$

Since $\Xi(t)$ is a diagonal matrix, we can write (4.54) as

$$e_R(t,t_0) = \sum_{i=1}^{n} e_{\gamma_i}(t,t_0) W(t) F_i W^{-1}(t_0), \tag{4.55}$$

where $F_i := \delta_{i,j}$ is $n \times n$ matrix. Using $v_i^T(t) w_j(t) = \delta_{i,j}$ for all $t \in \mathbb{T}$, F_i can be rewritten as follows:

$$F_i = W^{-1}(t)[0,\ldots,0,w_i(t),0,\ldots,0]. \tag{4.56}$$

By means of (4.55) and (4.56) we have

$$e_R(t,t_0) = \sum_{i=1}^{n} e_{\gamma_i}(t,t_0) w_i(t) v_i^T(t_0) = \sum_{i=1}^{n} \chi_i(t) v_i^T(t_0),$$

where $\chi_i(t)$ is mode vector of system (4.33).

Case 1 By (4.36), for each $1 \leq i \leq n$, we can write that

$$\begin{aligned} \|\chi_i(t)\| &\leq D_i \sum_{k=0}^{d_i-1} h_k(t,t_0) |e_{\gamma_i}(t,t_0)| \\ &\leq D_i \sum_{k=0}^{d_i-1} h_k(t,t_0) e_{\mathrm{Re}_\mu(\gamma_i)}(t,t_0) \end{aligned}$$

where D_i is as in Theorem 4.23, d_i represents the dimension of the Jordan block which contains i^{th} eigenvalue of $R(t)$.Using Lemma 4.22 we get

$$\lim_{t\to\infty} h_k(t,t_0) e_{\mathrm{Re}_\mu(\gamma_i)}(t,t_0) = 0$$

for each $1 \leq i \leq n$ and all $k = 1,2,\ldots,d_i - 1$. This along with Theorem 4.21 implies that the system (4.33) is asymptotically stable. By Theorem 4.6, since the solutions of the systems in (4.31) and (4.33) are related by Lyapunov transformation, we can state that solution of the system (4.31) is asymptotically stable.

For the second part, we first write

$$\begin{aligned}\|\chi_i(t)\| &\leq D_i \sum_{k=0}^{d_i-1} h_k(t,t_0)\left|e_{\gamma_i}(t,t_0)\right| \\ &\leq D_i \sum_{k=0}^{d_i-1} h_k(t,t_0)\, e_{\mathrm{Re}_\mu(\gamma_i)\oplus\varepsilon}(t,t_0)\, e_{\ominus\varepsilon}(t,t_0).\end{aligned} \tag{4.57}$$

If (4.49) holds, then $\mathrm{Re}_\mu(\gamma_i)\oplus\varepsilon$ satisfies (4.39). Hence, by Lemma 4.22 the term $h_k(t,t_0)\, e_{\mathrm{Re}_\mu(\gamma_i)\oplus\varepsilon}(t,t_0)$ converges to zero as $t\to\infty$. That is, there is an upper bound C_ε for the sum $\sum_{k=0}^{d_i-1} h_k(t,t_0)\, e_{\mathrm{Re}_\mu(\gamma_i)\oplus\varepsilon}(t,t_0)$. This along with (4.57) yields

$$\|\chi_i(t)\| \leq D_i C_\varepsilon e_{\ominus\varepsilon}(t,t_0).$$

Thus,Theorem 4.21 implies that the system (4.33) is exponentially stable. Using the above given argument the Floquet system (4.31) is exponentially stable.

Case 2 Assume that $\mathrm{Re}_\mu[\gamma_k(t)]=0$ for some $1\leq k\leq n$ with equal algebraic and geometric multiplicities corresponding to $\gamma_k(t)$. Then the Jordan block of $\gamma_k(t)$ is 1×1. Then,

$$\begin{aligned}\lim_{t\to\infty}\|\chi_k(t)\| &\leq \lim_{t\to\infty} D_k\left|e_{\gamma_k}(t,t_0)\right| \\ &\leq \lim_{t\to\infty} D_k e_{\mathrm{Re}_\mu(\gamma_k)}(t,t_0) \\ &= D_k.\end{aligned}$$

By Theorem 4.21, the system (4.33) is stable. By Theorem 4.6, the solutions of the systems in (4.31) and (4.33) are related by Lyapunov transformation. This implies that the system (4.31) is stable.

Case 3 Suppose that $\mathrm{Re}_\mu(\gamma_i(t))>0$ for some $i=1,\ldots,n$. Then, we have

$$\lim_{t\to\infty}\|e_R(t,t_0)\|=\infty,$$

and by the relationship between solutions of the systems in (4.31) and (4.33), one can write that

$$\lim_{t\to\infty} \|\Phi_A(t, t_0)\| = \infty.$$

Therefore, the Floquet system (4.31) is unstable.

□

Remark. In the case when the time scale is additive periodic, Theorem 4.24 gives its additive counterpart [38, Theorem 7.9]. For an additive time scale the graininess function $\mu(t)$ is bounded above by the period of the time scale. However, this is not true in general for the times scales that are periodic in shifts. The highlight of Theorem 4.24 is to rule out strong restriction that obliges the time scale to be additive periodic. Hence, unlike [38, Theorem 7.9] our stability theorem (i.e. Theorem 4.24) is valid for q-difference systems.

Corollary 4.25 *Let λ_i be a Floquet multiplier of the T-periodic linear dynamic system (4.31) for $i = 1, \ldots, n$. Then, we have*

1. *If $|\lambda_i| < 1$ for all $i = 1, \ldots, n.$, then the system (4.31) is exponentially stable;*
2. *If $|\lambda_i| \leq 1$ for all $i = 1, \ldots, n$ and if, for each $|\lambda_i| = 1$ for some $i = 1, \ldots, n$, the algebraic multiplicity equals to geometric multiplicity, then the system (4.31) is stable, otherwise it is unstable.*
3. *$|\lambda_i| > 1$ for some $i = 1, \ldots, n$, then the system (4.31) is unstable.*

Chapter 5

Conclusion

In this thesis, Floquet theory of q-difference systems is established on $q^{\mathbb{Z}}, q > 1$ by using two periodicity notions on quantum calculus (see Chapter 2). At first homogeneous case is analyzed and canonical Floquet decomposition is obtained for the transition matrix of the q-Floquet system by employing Lyapunov transformation and the solution of the matrix exponential equation (see Theorem 3.1 and Theorem 3.5). Then, necessary and sufficient conditions for the existence of periodic solutions of homogeneous and nonhomogeneous systems are provided. Unlike the existing literature (see [28], [32]), we define the periodic solution of systems to be the one repeating its values at each forward and backward step with a fixed period in parallel to the conventional periodicity perception. Thus, the q-Floquet theory constructed in this thesis also serves an alternative approach to [28] and [32]. In the remaining parts of q-Floquet theory, Floquet multipliers, Floquet exponents and their properties are introduced and proved. Moreover, stability properties of q-Floquet system is analyzed via Floquet exponents and Floquet multipliers despite the unbounded step size (graininess) of $q^{\mathbb{Z}}, q > 1$. By doing so, the gap on the stability characteristics of q-Floquet systems is fulfilled. The generalization and extension of q-Floquet theory to more general domains, which are called time scales periodic in shifts, is performed in Chapter 4. Shift operators and the new periodicity concept on time scales are introduced (see Chapter 2) and the extension of q-Floquet theory is established by employing periodicity notion by means of shift operators. The use of the new periodicity concept on the unification of q-Floquet theory enables us to focus on time scales which do not need to be additively periodic. As an alternative the existing literature, the extension of q-Floquet theory is valid on more time scales such as $\overline{q^{\mathbb{Z}}}$, $\cup_{k=0}^{\infty}\left[3^{\pm k}, 2.3^{\pm k}\right] \cup \{0\}$ and $\mathbb{N}^{1/2}$ which can not be covered by [37] and [38]. Moreover without assuming the bounded step size (graininess) of the time scale dissimilar to [38], the stability properties of the Floquet system are analyzed for the time scales unbounded from above.

BIBLIOGRAPHY

[1] Adamec, L. 2005, *Simple Floquet Theory on Time Scales*: In Proc. of the Eight Internat. Conf. on Difference Equations and Applications, Boca Raton-New York-Singapore,1-6.

[2] Adıvar, M. and S. C. Fang. 2012. *Convex Optimization on Mixed Domains.* J. Ind. Manag. Optim., Vol. 8(1): 189–227.

[3] Adıvar, M., H. C. Koyuncuoglu, and Y. N. Raffoul. 2012. *Classification of Positive Solutions of Nonlinear Systems of Volterra Integro-Dynamic Equations on Time Scales*. Commun. Appl. Anal., Vol. 16(3): 359-376.

[4] Adıvar, M., H. C. Koyuncuoglu, and Y. N. Raffoul. 2013. *Periodic and Asymptotically Periodic Solutions of Systems of Nonlinear Difference Equations with Infinite Delay.* J. Difference Equ. Appl., Vol. 19(12): 1927-1939.

[5] Adıvar, M., H. C. Koyuncuoglu, and Y. N. Raffoul. 2014. *Existence of Periodic Solutions in Shifts* $\delta_{\pm}$ *for Neutral Nonlinear Dynamic Systems*. Appl. Math. Comput., Vol. 242: 328-339.

[6] Adıvar, M. and H. C. Koyuncuoglu. 2016. *Floquet Theory Based on New Periodicity Concept for Hybrid Systems Involving q-Difference Equations.* Appl. Math. Comput., Vol. 273: 1208-1233.

[7] Adıvar, M. and H. C. Koyuncuoglu. 2016. *Almost Automorphic Solutions of Discrete Delayed Neutral System.* J. Math. Anal. Appl., Vol. 435: 532-550.

[8] Adıvar, M., H. C. Koyuncuoglu, and Y. N. Raffoul. 2016. *Almost Automorphic Solutions of Delayed Neutral Dynamic Systems on Hybrid Domains.* Appl. Anal. Discrete Math., Vol. 10: 128-151.

[9] Adıvar, M. and Y. N. Raffoul. 2010. *Shift Operators and Stability in Delayed Dynamic Equations*. Rend. Sem. Mat. Politec Torino, Vol. 68(4): 369-396.

[10] Adıvar, M. and Y. N. Raffoul. 2013. *Necessary and Sufficient Conditions for Uniform Stability of Volterra Integro-Dynamic Equations Using New Resolvent Rquation*. Analele Stiintifice Ale Universitatii "Ovidius" Constanta, Vol. 21(3): 17-32.

[11] Adıvar, M. and Y. N. Raffoul. 2010. *Existence of Resolvent for Volterra Integral Equations on Time Scales*. Bull. Aust. Math. Soc., Vol. 82(1): 139–155.

[12] Adıvar, M. 2013. *A New Periodicity Concept for Time Scales*. Math. Slovaca, Vol. 63(4): 817–828.

[13] Adıvar, M. 2011. *Principal Matrix Solutions and Variation of Parameters Formula for Volterra Integro-Dynamic Equations on Time Scales*. Glasg. Math. J, Vol. 53(3): 1-18.

[14] Ahlbrandt, C. D. and J. Ridenhour. 2003. *Floquet Theory for Time Scales and Putzer Representations of Matrix Logarithms*. J. Difference Equ. Appl., Vol. 9(1): 77–92.

[15] Agarwal, R. P. and D. O'Regan. 2001. *Nonlinear Boundary Value Problems on Time Scales*. Nonlinear Anal., Vol. 44: 527-535.

[16] Agarwal, R. P., M. Bohner, and P. J. Y. Wong. 1999. *Sturm - Liouville Eigenvalue Problems on Time Scales*. Appl. Math. Comput., Vol. 2(3): 153-166.

[17] Agarwal, R. P. vd. 2014. *Floquet Theory for a Volterra Integro-Dynamic Systems*, Appl. Anal., Vol. 93(9): 2002–2013.

[18] Agarwal, R. P. vd. 2005. *Floquet Theory and Stability of Nonlinear Integro-Differential Equations*. Acta Math. Hungar., Vol. 109(4): 305-330.

[19] Agarwal, R. P. vd. 2002. *Dynamic Equations on Time Scales: a Survey*. J. Comput. Appl. Math., Vol. 141: 1-26.

[20] Andr´e, Y. and L. Di Vizio. 2004, *q-Difference Equations and p-Adic Local Monodromy*. Ast´erisque, Vol. 296: 55–111.

[21] Anh, P. K. and H. T. N. Yen. 2006. *Floquet Theorem for Linear Implicit Nonautonomous Difference Systems*. J. Math. Anal. Appl., Vol: 321(2): 921-929.

[22] AtıcıF. M., D. C. Biles, and A. Lebedinsky. 2006. *An Application of Time Scales to Economics*. Math. Comput. Modelling, Vol. 43(7-8): 718–726.

[23] Belbas, S. A. 2013. *Floquet Theory for Integral and Integro-Differential Equations*. Appl. Math. Comput., Vol. 223: 327-345.

[24] Bohner, M. and A. Peterson. 2001. *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston.

[25] Bohner, M. and A. Peterson. 2003. *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston.

[26] Agarwal, R. P., M. Bohner, and S. H. Saker. 2005. *Oscillation of Second Order Delay Dynamic Equations*. Can. Appl. Math. Q., Vol. 13(1).

[27] Bohner, M., G. Sh. Guseinov, and B. Karpuz. 2011. *Properties of the Laplace Transform on Time Scales with Arbitrary Graininess*. Integral Transforms Spec. Funct., Vol. 22(11): 785–800.

[28] Bohner, M. and R. Chieochan. 2012. *Floquet Theory for q-Difference Equations*. Sarajevo J. Math., Vol. 8(21): 355–366.

[29] Becker, L. C., T. A. Burton, and T. Krisztin. 1988. *Floquet Theory for a Volterra Equation*. J. London Math. Soc., Vol. 37(2): 141-147.

[30] Braverman, E. and B. Karpuz. 2012. *Uniform Exponential Stability of First-Order Dynamic Equations with Several Delays*. Appl. Math. Comput., Vol. 218: 10468-10485.

[31] Burke, P. G., P. Francken, and C. J. Joachain. 1991. *R-Matrix-Floquet Theory of Multiphoton Processes*. J. Phys. B: At. Mol. Opt. Phys., Vol. 24.

[32] Chieochan, R. 2012. Periodic q-Difference Equations. Ph.D. Thesis, Missouri University of Science and Technology.

[33] Choi, S. K. and N. Koo. 2009. *Stability of Linear Dynamic Equations on Time Scales*. Discrete Contin. Dyn. Syst.: 161-170.

[34] Chu, S. 1981. *Floquet Theory and Complex Quasivibrational Energy Formalism for Intense Field Molecular Photodissociation.*, J. Chem. Phys., Vol. 75(5).

[35] Cushing, J. M. 1982. *Periodic Kolmogorov systems.* SIAM J. Math. Anal., Vol. 13(5): 811-827.

[36] DaCunha, J. J. 2005. *Stability for Time Varying Linear Dynamic Systems on Time Scales*. J. Comput. Appl. Math., Vol. 176: 381-410.

[37] DaCunha, J. J. 2004. Lyapunov Stability and Floquet Theory for Nonautonomous Linear Dynamic Systems on Time Scales. Ph.D. Thesis, Baylor University.

[38] DaCunha J. J. and J. M. Davis. 2011. *A Unified Floquet Theory for Discrete, Continuous, and Hybrid Periodic Linear Systems*. J. Differential Equations, Vol. 251(11): 2987–3027.

[39] DaCunha, J. J. 2005. *Transition Matrix and Generalized Matrix Exponential via the Peano-Baker Series*. J. Difference Equ. Appl., Vol. 11(15): 1245-1264.

[40] Demir, A. 2000. *Floquet Theory and Nonlinear Perturbation Analysis for Oscillators with Differential-Algebraic Equations*. Int. J. Circ. Theor. Appl., Vol. 28: 163–185.

[41] Dobrogovska, A. and A Odzijewicz. 2006. *Second Order q-Difference Equations Solvable by Factorization Method*. J. Comp. Appl. Math., Vol. 193: 319-346.

[42] Erneux, T. and M. Herschkowitz-Kaufman. 1979. *Bifurcation Diagram of a Model Chemical Reaction - I. Stability Changes on Time-Periodic Solutions.* Bull. Math. Biol., Vol. 41(1): 21-38.

[43] Floquet, G. 1883. *Sur les Équations Différentielles Linéaires à Coefficients Périodiques*. Annales de l'École Normale Supérieure, Vol. 12: 47–88.

[44] Hale, J. K. 1980. *Ordinary Differential Equations*, Krieger Publishing Company, Florida.

[45] Hastings, S. P. and J. D. Murray. 1975. *The Existence of Oscillatory Solutions in the Field-Noyes Model for the Belousov-Zhabotinskii Reaction*. SIAM J. Appl. Math., Vol. 28(3): 678-688.

[46] Hilger, S. 1997. *Differential and Difference Calculus - Unified!*. Nonlinear Anal., Vol. 30(5): 2683-2694.

[47] Hilger, S. 1990. *Analysis on Measure Chains - a Unified Approach to Continuous and Discrete Calculus*. Results Math., Vol. 18: 18-56.

[48] Hilger, S. 1988. Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. Thesis, Universität Würzburg.

[49] Islam, M. N. and Y. N. Raffoul. 2007. *Periodic Solutions of Neutral Nonlinear System of Differential Equations with Functional Delay*. J. Math. Anal. Appl., Vol. 331(2): 1175–1186.

[50] Islam, M. N. 2013. *Three Fixed Point Theorems: Periodic Solutions of a Volterra Type Integral Equation with Infinite Heredity*. Canad. Math. Bull., Vol. 56(1): 80–91.

[51] Jackson, B. 2006. *Partial Dynamic Equations on Time Scales*. J. Comput. Appl. Math., Vol. 186(2): 391-415.

[52] Kac, V. and P. Cheung, *Quantum Calculus,* Universitext. Springer-Verlag, New York.

[53] Kaufmann, E. R. and Y. N. Raffoul. 2006. *Periodic Solutions for a Neutral Nonlinear Dynamical Equation on a Time Scale*. J. Math. Anal. Appl, Vol. 319: 315–325.

[54] Klausmeier, C. A. 2008. *Floquet Theory: A Useful Tool for Understanding Nonequilibrium Dynamics*. Theor. Ecol., Vol. 1: 153-161.

[55] Kuchment, P. A. 1993. *Floquet Theory for Partial Differential Equations*, Springer.

[56] Levante, T.O. vd. 1995. *Formalized quantum mechanical Floquet Theory and Its Application to Sample Spinning in Nuclearmagnetic Resonance*. Molecular Physics: An International Journal at the Interface BetweenChemistry and Physics, Vol: 86(5): 1195-1212.

[57] Liu, Y., X. Zhang and T. Zhou. 2014. *Multiple Periodic Solutions of a Delayed Prey-Predator Model with Non-monotonic Functional Response and Stage Structure*. J. Biol. Dyn., Vol. 8(1): 145-160.

[58] Lungu, R. P. 2000. *The Floquet Theory of the Time-Periodic Hamiltonian Systems: The Pseudo Evolution Method*. Romanian Reports in Physics, Vol. 52(3-4): 233-263.

[59] Malkiewicz, P. and M. Nieszporski. 2013. *Darboux Transformations for q-Discretizations of 2D Second Order Differential Equations*. J. Nonlinear Math. Phys., Vol. 12(2): 231-239.

[60] Marva, M. and R. B. Parra. 2014, *Reduction of Nonautonomous Population Dynamics Models with Two Time Scales*. Acta Biotheoretica, 1-19.

[61] Moore, D. J. 1993. *Time Dependence in Quantum Mechanics: Floquet Theory and the Berry Phase*. Helvetica Physica Acta, Vol. 66(1): 53-104.

[62] Ostrovska, S. 2006. *The Approximation by q-Bernstein Polynomials in the case $q \downarrow 1$*, Arch. Math., Vol. 86: 282-288.

[63] Peterson, A. C., Y. N. Raffoul and C. C. Tisdell. 2004. *Three Point Boundary Value Problems on Time Scales*. J. Difference Equ. Appl., Vol. 10(9): 843-849.

[64] Kelley, W. G. and A. C. Peterson. 2004. *The Theory of Differential Equations, Classical and Qualitative*. Prentice Hall, New York.

[65] Kelley, W. G. and A. C. Peterson. 2001. *Difference Equations: An Introduction With Applications*. 2^{nd}*ed.*, Academic Press, New York.

[66] Posny, D. and J. Wang. 2014. *Modelling Cholera in Periodic Environments*. J. Biol. Dyn., Vol. 8(1): 1-19.

[67] Pötzsche, C., S. Siegmund and F. Wirth. 2003. *A Spectral Characterization of Exponential Stability for Linear Time-Invariant Systems on Time Scales*. Discrete Contin. Dyn. Syst., Vol. 9: 1223-1241.

[68] Pulita, A. 2008. *p-Adic Confluence of q-Difference Equations*. Compos. Math., Vol. 144(4): 867–919.

[69] Saker, S. H. 2004. *Oscillation of Nonlinear Dynamic Equations on Time Scales*. Appl. Math. Comput., Vol. 148: 81-91.

[70] Salido, M. A. and F. Barber. 2015. *Mathematical Solutions for Solving Periodic Railway Transportation*. Math. Probl. Eng., Art. ID 728916.

[71] Ufuktepe, Ü. 2012. *Unification of Probability Theory on Time Scales*. Adv. Difference Equ., Vol. 211.

[72] Quian, D. and X. Li. 2005. *Periodic Solutions for Ordinary Differential Equations with Sublinear Impulsive Effects*. J. Math. Anal. Appl., Vol. 303(1): 288-303.

[73] Yu, J., Z. Guo and Z. Zou. 2005. *Periodic Solutions of Second Order Self-Adjoint Difference Equations*. J. London Math. Soc., Vol. 71(2): 146-160.

[74] Zhang, L., H. X. Li and X. B. Zhang. 2009. *Periodic Solutions of Competition Lotka-Volterra Dynamic System on Time Scales*. Comput. Math. Appl., Vol. 57(7): 1204-1211.

Vitae

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